Ore Extensions of Extended Symmetric and Reversible Rings

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Abstract
Let \( \sigma \) be an endomorphism and \( \delta \) an \( \sigma \)-derivation of a ring \( R \). In this paper, we show that if \( R \) is \((\sigma, \delta)\)-skew Armendariz and \( a\sigma(b) = 0 \) implies \( ab = 0 \) for \( a, b \in R \). Then \( R \) is symmetric (respectively, reversible) if and only if \( R \) is \( \sigma \)-symmetric (respectively, \( \sigma \)-reversible) if and only if \( R[x; \sigma, \delta] \) is symmetric (respectively, reversible). As a consequence we obtain a generalization of [3, 9, 14]. Also, a number of related results are shown.

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Introduction
Throughout this paper, \( R \) denotes an associative ring with unity. \( \sigma \) is a ring endomorphism, and \( \delta \) an \( \sigma \)-derivation of \( R \), that is, \( \delta \) is an additive map such that \( \delta(ab) = \sigma(a)\delta(b) + \delta(a)b \) for all \( a, b \in R \). We denote \( R[x; \sigma, \delta] \) the Ore extension whose elements are polynomials over \( R \), the addition is defined as usual and the multiplication subject to the relation \( xa = \sigma(a)x + \delta(a) \) for all \( a \in R \). A ring \( R \) is called symmetric if \( abc = 0 \) implies \( acb = 0 \) for all \( a, b, c \in R \). A ring \( R \) is called reversible if \( ab = 0 \) implies \( ba = 0 \) for all \( a, b \in R \). Reduced rings (i.e., rings with no nonzero nilpotent elements) are symmetric by Anderson and Camillo [1, Theorem 1.3]. Commutative rings are clearly symmetric, symmetric rings are clearly reversible. Polynomial rings over reversible rings need not to be reversible, and polynomial rings over symmetric rings need not to be symmetric (see [11] and [18]). Rege and Chhawcharia [17] called a ring \( R \) an Armendariz if whenever polynomials \( f = \sum_{i=0}^{n} a_ix^i \), \( g = \sum_{j=0}^{m} b_jx^j \in R[x] \) satisfy \( fg = 0 \), then \( a_ib_j = 0 \) for each \( i, j \). The term Armendariz was introduced
by Rege and Chhawchharia [17]. This nomenclature was used by them since it was Armendariz [2, Lemma 1], who initially showed that a reduced ring always satisfies this condition. According to Krempa [13], an endomorphism $\sigma$ of a ring $R$ is called to be rigid if $a \sigma(a) = 0$ implies $a = 0$ for all $a \in R$. A ring $R$ is called $\sigma$-rigid if there exists a rigid endomorphism $\sigma$ of $R$. Note that any rigid endomorphism is injective and $\sigma$-rigid rings are reduced rings by Hong et al. [7]. Properties of $\sigma$-rigid rings have been studied in [5, 7, 8, 13]. In [8], Hong et al. defined a ring $R$ with an endomorphism $\sigma$ to be $\sigma$-skew Armendariz if whenever polynomials $f = \sum_{i=0}^{n} a_i x^i$, $g = \sum_{j=0}^{m} b_j x^j \in R[x; \sigma]$ satisfy $fg = 0$ then $a_i \sigma^i(b_j) = 0$ for each $i, j$. According to Hong et al. [9]. A ring $R$ is said to be $\sigma$-Armendariz, if whenever polynomials $f = \sum_{i=0}^{n} a_i x^i$, $g = \sum_{j=0}^{m} b_j x^j \in R[x; \sigma]$ satisfy $fg = 0$ then $a_i b_j = 0$ for each $i, j$. From Hashemi and Moussavi [6], a ring $R$ is called a $(\sigma, \delta)$-skew Armendariz ring if for $p = \sum_{i=0}^{n} a_i x^i$ and $q = \sum_{j=0}^{m} b_j x^j$ in $R[x; \sigma, \delta]$, $pq = 0$ implies $a_i \sigma^i b_j x^j = 0$ for each $i, j$. By Hashemi and Moussavi [5], a ring $R$ is $\sigma$-compatible if for each $a, b \in R$, $a \sigma(b) = 0$ if and only if $ab = 0$. Moreover, $R$ is said to be $\delta$-compatible if for each $a, b \in R$, $ab = 0$ implies $a \delta(b) = 0$. If $R$ is both $\sigma$-compatible and $\delta$-compatible, we say that $R$ is $(\sigma, \delta)$-compatible. A ring $R$ is $\sigma$-rigid if and only if $R$ is $(\sigma, \delta)$-compatible and reduced [5, Lemma 2.2]. From [3], a ring $R$ is called right (respectively, left) $\sigma$-reversible if whenever $ab = a \sigma(b) = 0$ (respectively, $\sigma(b) a = 0$). Also, by [14], a ring $R$ is called right (respectively, left) $\sigma$-symmetric if whenever $abc = 0$ for $a, b, c \in R$, $a \sigma(b)c = 0$ (respectively, $\sigma(b)ac = 0$). In this paper, we study the transfert of symmetry ($\sigma$-symmetry) and reversibility ($\sigma$-reversibility) from $R$ to $R[x; \sigma, \delta]$ and vice versa. We show, that if $R$ is $(\sigma, \delta)$-skew Armendariz and $a \sigma(b) = 0$ implies $ab = 0$ for $a, b \in R$. Then $R$ is symmetric (respectively reversible) if and only if $R$ is $\sigma$-symmetric (respectively $\sigma$-reversible) if and only if if $R[x; \sigma, \delta]$ is symmetric (respectively, reversible). As a consequence we obtain a generalization of [9, Theorem 3.6], [10, Proposition 3.4], [11, Proposition 2.4], [3, Corollary 2.11] and [14, Theorem 2.10]. A connection between reversibility (respectively, symmetry) and $\sigma$-reversibility (respectively, $\sigma$-symmetry) of a ring is given. We also investigate connections to other related conditions. Examples to illustrate results are included.

1 Preliminaries and Examples

We begin with the following definition.

**Definition 1.1.** Let $\sigma$ be an endomorphism of a ring $R$. We say that $R$ satisfies the condition $(C_{\sigma})$ if whenever $a \sigma(b) = 0$ with $a, b \in R$, then $ab = 0$.

In the Ore extension $R[x; \sigma, \delta]$, we have for $n \geq 0$, $x^n a = \sum_{i=0}^{n} f^n_i(a) x^i$ where $f^n_i \in \text{End}(R, +)$ will denote the map which is the sum of all possible words in
σ, δ built with i letters σ and n − i letters δ. (In particular, \( f_n^m = \sigma^n, f_0^m = \delta^n \)). [15, Lemma 4.1].

**Lemma 1.2.** Let \( R \) be an \((\sigma, \delta)\)-skew Armendariz ring satisfying the condition \((C_\sigma)\) and \( n \geq 1 \) a natural number. If \( ab = 0 \) then \( \sigma^n(a)b = \delta^n(a)b = 0 \) for all \( a, b \in R \).

**Proof.** It suffices to show the result for \( n = 1 \). Let \( f = \sigma(a)x + \delta(a) \) and \( g = b \) such that \( ab = 0 \). Then \( fg = \sigma(a)xb + \delta(a)b = \sigma(ab)x + \sigma(a)\delta(b) + \delta(a)b = \sigma(ab)x + \delta(ab) = 0 \). By Proposition 2.1, we have \( \sigma(a)b = \delta(a)b = 0 \). \( \Box \)

**Lemma 1.3.** Let \( R \) be an \((\sigma, \delta)\)-skew Armendariz reversible ring satisfying the condition \((C_\sigma)\). If \( ab = 0 \), then \( af_i^j(b) = 0 \) for all \( i, j \) \((i \leq j)\).

**Proof.** Let \( a, b \in R \) such that \( ab = 0 \), since \( R \) is reversible we have \( ba = 0 \) and so \( \sigma^n(b)a = \delta^n(b)a = 0 \), by Lemma 1.2. Also, we have \( a\sigma^n(b) = a\delta^n(b) = 0 \) for all \( n \in \mathbb{N} \). Thus \( af_i^j(b) = 0 \) for all \( i, j \). \( \Box \)

In the next, we show some connections between \((\sigma, \delta)\)-skew Armendariz rings, \( \sigma \)-rigid rings and rings with the condition \((C_\sigma)\).

**Example 1.4.** Let \( \mathbb{Z} \) be the ring of integers and \( \mathbb{Z}_4 \) be the ring of integers modulo 4. Consider the ring

\[
R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, b \in \mathbb{Z}_4 \right\}.
\]

Let \( \sigma: R \to R \) be an endomorphism defined by \( \sigma \left( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} \).

Take the ideal \( I = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in 4\mathbb{Z} \right\} \) of \( R \). Consider the factor ring

\[
R/I \cong \left\{ \begin{pmatrix} \pi & \beta \\ 0 & \pi \end{pmatrix} \mid \pi, \beta \in 4\mathbb{Z} \right\}.
\]

(i) \( R/I \) is not \( \sigma \)-skew Armendariz. In fact, \( \left( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x \right)^2 = 0 \in (R/I)[x; \sigma] \), but \( \left( \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \sigma \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \neq 0 \).

(ii) \( R/I \) satisfies the condition \((C_\sigma)\). Let \( A = \begin{pmatrix} \pi & b \\ 0 & \pi \end{pmatrix} \), \( B = \begin{pmatrix} a^\pi & \beta \\ 0 & a^\pi \end{pmatrix} \) \( \in R/I \).

If \( A\sigma(B) = 0 \) then \( \overline{aa^\pi} = 0 \) and \( \overline{ab^7} = \overline{ba^7} = 0 \), so that \( AB = 0 \).
Example 1.5. Consider a ring of polynomials over $\mathbb{Z}_2$, $R = \mathbb{Z}_2[x]$. Let $\sigma: R \rightarrow R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. Then:
(i) $R$ does not satisfy the condition $(C_\sigma)$. Let $f = 1 + x$, $g = x \in R$, we have $fg = (1 + x)x \neq 0$, however $f\sigma(g) = (1 + x)\sigma(x) = 0$.
(ii) $R$ is $\sigma$-skew Armendariz [8, Example 5].

Example 1.6. Consider the ring $R = \left\{ \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, t \in \mathbb{Q} \right\}$, where $\mathbb{Z}$ and $\mathbb{Q}$ are the set of all integers and all rational numbers, respectively. The ring $R$ is commutative, let $\sigma: R \rightarrow R$ be an automorphism defined by $\sigma \left( \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & t/2 \\ 0 & a \end{pmatrix}$.
(i) $R$ is not $\sigma$-rigid. $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \sigma \left( \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \right) = 0$, but $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \neq 0$, if $t \neq 0$.
(ii) $R$ satisfies the condition $(C_\sigma)$. Let $\begin{pmatrix} a & t \\ 0 & a \end{pmatrix}$ and $\begin{pmatrix} b & x \\ 0 & b \end{pmatrix} \in R$ such that $\sigma \left( \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \right) \sigma \left( \begin{pmatrix} b & x \\ 0 & b \end{pmatrix} \right) = 0$, hence $ab = 0 = ax/2 + tb$, so $a = 0$ or $b = 0$. In each case, $ax + tb = 0$, hence $\begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \begin{pmatrix} b & x \\ 0 & b \end{pmatrix} = 0$. We have also the same for the converse.
(iii) $R$ is $\sigma$-skew Armendariz [7, Example 1].

Examples 1.4 and 1.5 shows that the $(\sigma, \delta)$-skew Armendariz property of a ring and the condition $(C_\sigma)$ are independent of each other. From [5, Lemma 2.2], [4, Lemma 2.5] and Example 1.6. The class of $\sigma$-rigid rings is strictly included in the class of $(\sigma, \delta)$-skew Armendariz rings satisfying the condition $(C_\sigma)$.

2 Reversibility and Symmetry of Ore Extensions

From Isfahani and Moussavi [16], a ring $R$ is called skew Armendariz, if for $f = \sum_{i=0}^{n} a_i x^i$, $g = \sum_{j=0}^{m} b_j x^j \in R[x; \sigma, \delta]$, $fg = 0$ implies $a_0 b_j = 0$ for all $j$. Obviously, $(\sigma, \delta)$-skew Armendariz rings satisfying the condition $(C_\sigma)$ are skew Armendariz. But the converse is not true by Example 2.2.

Proposition 2.1. Let $R$ be a ring, $\sigma$ an endomorphism, and $\delta$ an $\sigma$-derivation of $R$. If one of the following conditions is satisfied.
(i) $R$ is $(\sigma, \delta)$-skew Armendariz and satisfies the condition $(C_\sigma)$;
(ii) $R$ is skew Armendariz and $R[x; \sigma, \delta]$ is symmetric;
Then, for \( f = \sum_{i=0}^{n} a_i x^i \) and \( g = \sum_{j=0}^{m} b_j x^j \in R[x; \sigma, \delta] \), \( fg = 0 \) implies \( a_i b_j = 0 \) for all \( i, j \).

Proof. Let \( f = \sum_{i=0}^{n} a_i x^i \), \( g = \sum_{j=0}^{m} b_j x^j \in R[x; \sigma, \delta] \) such that \( fg = 0 \).

(i) Since \( R \) is \((\sigma, \delta)\)-skew Armendariz, then \( a_i x^i b_j x^j = 0 \) for all \( i, j \). Or \( a_i x^i b_j x^j = a_i \sum_{\ell=0}^{\infty} f^\ell(b_j) x^{i+j+\ell} = a_i \sigma^\ell(b_j) x^{i+j} + p(x) = 0 \) where \( p(x) \) is a polynomial of degree strictly less than \( i+j \). Thus \( a_i \sigma^i(b_j) = 0 \), and by the condition \((C_\sigma)\) we have \( a_i b_j = 0 \) for all \( i, j \).

(ii) We have \((a_0 + a_1 x + \cdots + a_n x^n)(b_0 + b_1 x + \cdots + b_m x^m) = a_0(b_0 + b_1 x + \cdots + b_m x^m) + (a_1 x + \cdots + a_n x^n)(b_0 + b_1 x + \cdots + b_m x^m) = 0 \), or \( a_0 b_j = 0 \) for all \( j \), because \( R \) is skew Armendariz. So that \((a_1 + \cdots + a_n x^{n-1}) (b_0 + b_1 x + \cdots + b_m x^m) = (a_1 + \cdots + a_n x^{n-1}) x (b_0 + b_1 x + \cdots + b_m x^m) = 0 \). Since \( R[x; \sigma, \delta] \) is symmetric, then we have \((a_1 + \cdots + a_n x^n) (b_0 + b_1 x + \cdots + b_m x^m) x = 0 \), hence \( a_1 b_j = 0 \) for all \( j \). Continuing this process, we have \( a_i b_j = 0 \) for all \( i, j \).

Proposition 2.1 shows that, \( \sigma \)-skew Armendariz rings satisfying the condition \((C_\sigma)\) are \( \sigma \)-Armendariz.

Example 2.2. Consider the ring \( R = \mathbb{Z}_2[x] \). Let \( \sigma : R \to R \) be an endomorphism defined by \( \sigma(f(x)) = f(0) \). Then
(i) \( R[y; \sigma] \) is not reversible (so is not symmetric): Let \( f = ay \), \( g = b \in R[y; \sigma] \) with \( a = 1 + x \) and \( b = x \), then \( fg = ayb = a\sigma(b)y = 0 \). But, \( gf = bay = x(1 + x) y \neq 0 \).

(ii) \( R \) does not satisfy the condition \((C_\sigma)\) (see Example 1.5).

(iii) \( R \) is skew Armendariz: Since \( R \) is \( \sigma \)-skew Armendariz, by [8, Example 5]. Then [16, Theorem 2.2] implies that \( R \) is skew Armendariz.

(iv) \( R \) is not \( \sigma \)-Armendariz by [9, Example 1.9].

By Example 2.2, the conditions “\( R[x; \sigma, \delta] \) is symmetric” and “the condition \((C_\sigma)\)” in Proposition 2.1 are not superfluous.

Lemma 2.3. Let \( R \) be an \((\sigma, \delta)\)-skew Armendariz ring satisfying the condition \((C_\sigma)\). Then, for \( f = \sum_{i=0}^{n} a_i x^i \), \( g = \sum_{j=0}^{m} b_j x^j \) and \( h = \sum_{k=0}^{p} c_k x^k \in R[x; \sigma, \delta] \), if \( fg h = 0 \) then \( a_i b_j c_k = 0 \) for all \( i, j, k \).

Proof. Let \( f = \sum_{i=0}^{n} a_i x^i \), \( g = \sum_{j=0}^{m} b_j x^j \) and \( h = \sum_{k=0}^{p} c_k x^k \in R[x; \sigma, \delta] \). Note that if \( fg h = 0 \) then \( a_i g h = 0 \) for all \( i \). Suppose that \( fg h = 0 \) then \( a_i (gh) = 0 \) for all \( i \), so \( (a_i g) h = 0 \) for all \( i \). Then by Proposition 2.1, we have \( a_i b_j c_k = 0 \) for all \( i, j, k \).

Theorem 2.4. Let \( R \) be an \((\sigma, \delta)\)-skew Armendariz ring satisfying the condition \((C_\sigma)\). Then
(1) \( R \) is reversible if and only if \( R[x; \sigma, \delta] \) is reversible;
(2) \( R \) is symmetric if and only if \( R[x; \sigma, \delta] \) is symmetric.
Proof. Note that any subring of symmetric (respectively, reversible) ring is again symmetric (respectively, reversible). Conversely, let \( f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{m} b_j x^j \) and \( h = \sum_{k=0}^{p} c_k x^k \in R[x; \sigma, \delta] \).

(1) If \( fg = 0 \) then \( a_i b_j = 0 \) for all \( i, j \) (by Proposition 2.1). Since \( R \) is reversible, we have \( b_j a_i = 0 \) for all \( i, j \), Lemma 1.3 implies that \( b_j f_k^i(a_i) = 0 \) for all \( i, j \). Thus \( gf = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{p} b_j f_k^i(a_i) x^{i+j+k} = 0. \)

(2) If \( fgh = 0 \) then \( a_i b_j c_k = 0 \) for all \( i, j, k \), by Lemma 2.3. Since \( R \) is symmetric, \( a_i c_k b_j = 0 \) for all \( i, j, k \). Also, \( R \) is reversible then by Lemma 1.3, we have \( a_i f_s^t(c_k f_t^k(b_j)) = 0 \) for all \( i, j, k, s, t \) with \( (s \leq i, t \leq k) \). Thus \( fhg = 0. \)

\( \square \)

In the next we give an example of a symmetric (so reversible) ring which satisfies all the hypothesis of Theorem 2.4.

Consider the following subring of the triangular ring \( T_n(R) \). Let \( R \) be a ring and let

\[
T(R, n) := \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_i \in R \right\},
\]

with \( n \geq 2 \), be a subset of the triangular matrix ring \( T_n(R) \). Then \( T(R, n) \) is a ring with addition point-wise and usual matrix multiplication. We can denote elements of \( T(R, n) \) by \( (a_1, a_2, \cdots, a_n) \). For an endomorphism \( \sigma \) and an \( \sigma \)-derivation \( \delta \) of \( R \), the natural extension \( \overline{\sigma} : T(R, n) \to T(R, n) \) defined by \( \overline{\sigma}(a_i) = \sigma((a_i)) \) is an endomorphism of \( T(R, n) \) and \( \overline{\delta} : T(R, n) \to T(R, n) \) defined by \( \overline{\delta}(a_i) = \delta((a_i)) \), is an \( \overline{\sigma} \)-derivation of \( T(R, n) \).

Since \( R[x]/(x^n) \simeq T(R, n), (n \geq 1) \), then by [10, Theorem 2.3], if \( R \) is a reduced ring then \( T(R, n) \) is symmetric.

Example 2.5. Let \( R \) be a ring, \( \sigma \) an endomorphism of \( R \) and \( \delta \) be a \( \sigma \)-derivation of \( R \). Suppose that \( R \) is \( \sigma \)-rigid. Consider the ring

\[
T(R, 3) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in R \right\}.
\]

\( T(R, 3) \) is symmetric (because \( R \) is reduced). We show that \( T(R, 3) \) is \( (\overline{\sigma}, \overline{\delta}) \)-skew Armendariz. Let \( p \in T(R, 3)[x; \overline{\sigma}, \overline{\delta}] \), \( p \) can be expressed by the form of \( (p_1, p_2, p_3) \) for some \( p_i \)’s in \( R[x; \sigma, \delta] \). Let \( p = (p_1, p_2, p_3) \) and \( q = (q_1, q_2, q_3) \) be elements of \( T(R, 3)[x; \overline{\sigma}, \overline{\delta}] \). Assume that \( pq = 0 \) then \( pq = (p_1q_1, p_1q_2, p_1q_3) + \)
\(p_2q_1, p_1q_3 + p_2q_2 + p_3q_1) = 0\). So we have the following system of equations

\[
\begin{align*}
p_1q_1 &= 0; \quad (1) \\
p_1q_2 + p_2q_1 &= 0; \quad (2) \\
p_1q_3 + p_2q_2 + p_3q_1 &= 0. \quad (3)
\end{align*}
\]

Since \(R[x; \sigma, \delta]\) is reduced by [13, Theorem 3.3], we see that \(q_1p_1 = 0\) from Eq.(1). Multiplying \(p_1\) to Eq.(2) from the right hand side, we have \(p_1q_2p_1 + p_2q_1p_1 = 0\). Thus \(p_1q_2p_1 = 0\) and so \(p_1q_2 = 0\). Hence \(p_2q_1 = 0\). Multiplying \(p_1\) to Eq.(3) from the right hand side, then \(p_1q_3p_1 + p_2q_2p_1 + p_3q_1p_1 = 0\). Then \(p_1q_3p_1 = 0\) and so \(p_1q_3 = 0\) and hence Eq.(3) becomes\(p_2q_2 + p_3q_1 = 0\). (4)

Multiplying \(p_2\) to Eq.(4) from the right hand side we have \(p_2q_2p_2 + p_3q_1p_2 = 0\), then we have \(p_2q_2 = 0\), and so \(p_3q_1 = 0\).

Let \(p = \sum_{i=0}^{2} A_i x^i\) and \(q = \sum_{j=0}^{2} B_j x^i \in T(R, 3)[x; \sigma, \delta]\), where \(A_i = \left(\begin{array}{ccc} a_i & b_i & c_i \\ 0 & a_i & b_i \\ 0 & 0 & a_i \end{array}\right)\) and \(B_j = \left(\begin{array}{ccc} a_j' & b_j' & c_j' \\ 0 & a_j' & b_j' \\ 0 & 0 & a_j' \end{array}\right)\) for \(0 \leq i \leq n, 0 \leq j \leq m\).

Assume that \(pq = 0\). We claim that \(A_i x^i B_j x^j = 0\) for \(0 \leq i \leq n, 0 \leq j \leq m\).

By the preceding expressions of \(p\) and \(q\), we can write \(p_1 = \sum_{i=0}^{2} a_i x^i, p_2 = \sum_{i=0}^{2} b_i x^i, p_3 = \sum_{i=0}^{2} c_i x^i, q_1 = \sum_{j=0}^{2} a_j' x^j, q_2 = \sum_{j=0}^{2} b_j' x^j\) and \(q_3 = \sum_{j=0}^{2} c_j' x^j\). Then \(a_i a_j' = 0, a_i b_j' = 0, b_i a_j' = 0, a_i c_j' = 0, b_i b_j' = 0\) and \(c_i a_j' = 0\) for all \(i, j\) by the preceding results and [7, Proposition 6]. Then \(A_i B_j = 0\) for all \(i, j\). Since \(T(R, 3)\) is \((\sigma, \delta)\)-compatible, we have \(A_i f_j^2(B_j) = 0\) for all \(i, j, \ell\) with \(j \geq \ell\) by [4, Lemma 2.4]. Thus \(A_i x^i B_j x^j = \sum_{\ell=0}^{2} A_i f_j^2(B_j) x^i + j = 0\) for all \(i, j\).

Therefore \(T(R, 3)\) is \((\sigma, \delta)\)-skew Armendariz. By Theorem 2.4, \(T(R, 3)[x; \sigma, \delta]\) is symmetric (so reversible).

According to Hong et al. [9, Theorem 1.8], if \(R\) is \(\sigma\)-Armendariz then \(R\) is \(\sigma\)-skew Armendariz. But the converse is not true by [9, Example 1.9].

**Theorem 2.6.** \(R\) is \(\sigma\)-Armendariz if and only if \(R\) is \(\sigma\)-skew Armendariz and satisfies the condition \((C_\sigma)\).

**Proof.** \((\Rightarrow)\. By [9, Proposition 1.3(ii) and Theorem 1.8]. \((\Leftarrow)\. Let \(f = a_0 + a_1 x + \cdots + a_n x^n\) and \(g = b_0 + b_1 x + \cdots + b_m x^m \in R[x; \sigma]\), such that \(fg = 0\), since \(R\) is \(\sigma\)-skew Armendariz then \(a_i \sigma(b_j) = 0\) for all \(i, j\). Since \(R\) satisfies the condition \((C_\sigma)\) then \(a_i b_j = 0\) for all \(i, j\). Therefore \(R\) is \(\sigma\)-Armendariz. \(\square\)

We clearly obtain the following corollaries of Theorem 2.4.

**Corollary 2.7** ([9, Theorem 3.6]). Let \(R\) be an \(\sigma\)-Armendariz ring. Then

1. \(R\) is reversible if and only if \(R[x; \sigma]\) is reversible.
2. \(R\) is symmetric if and only if \(R[x; \sigma]\) is symmetric.
Corollary 2.8 ([10, Proposition 3.4] and [11, Proposition 2.4]). Let $R$ be an Armendariz ring. Then
(1) $R$ is reversible if and only if $R[x]$ is reversible.
(2) $R$ is symmetric if and only if $R[x]$ is symmetric.

3 $\sigma$-Reversibility and $\sigma$-Symmetry of Ore Extensions

In the next Lemma we give a relationship between $\sigma$-reversibility (respectively, $\sigma$-symmetry) and reversibility (respectively, symmetry).

Lemma 3.1. Let $R$ be a ring and $\sigma$ an endomorphism of $R$. If $R$ satisfies the condition $(C_\sigma)$. Then
(1) $R$ is reversible if and only if $R$ is $\sigma$-reversible;
(2) $R$ is symmetric if and only if $R$ is $\sigma$-symmetric.

Proof. (1) Let $a, b \in R$, $ab = 0$ implies $b\sigma(a) = 0$, with the condition $(C_\sigma)$, we have $ba = 0$. So $R$ is reversible. Conversely, let $a, b \in R$, suppose that $ab = 0$. If $b\sigma(a) \neq 0$ (i.e., $R$ is not right $\sigma$-reversible), the reversibility of $R$ gives $\sigma(a)b \neq 0$. Also, $R$ satisfies $(C_\sigma)$, then $\sigma(ab) \neq 0$. Contradiction. Now, if $\sigma(b)a \neq 0$ (i.e., $R$ is not left $\sigma$-reversible), the condition $(C_\sigma)$ gives $\sigma(ba) \neq 0$. Contradiction, because $R$ is reversible.
(2) Let $a, b, c \in R$ such that $abc = 0$. We have $bca = 0$ (by reversibility), so $c\sigma(b) = 0$ (by right $\sigma$-reversibility), then $\sigma(b)ca = 0$ (by reversibility), and so $\sigma(b)ac = 0$. So we have the left $\sigma$-symmetry. With the same method, we obtain the right $\sigma$-symmetry. Conversely, if $abc = 0$, by right $\sigma$-symmetry we have $acs(b) = 0$, then $acb = 0$ by the condition $(C_\sigma)$.

Example 3.2. Let $\mathbb{Z}_2$ is the ring of integers modulo 2, take $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. $R$ is commutative, and so $R$ is symmetric (so reversible). Now, consider $\sigma: R \to R$ defined by $\sigma((a, b)) = (b, a)$.

Then, we have:
(i) $R$ is not $\sigma$-reversible (so not $\sigma$-symmetric): $(1, 0)(0, 1) = 0$, but $(0, 1)\sigma((1, 0)) = (0, 1)(0, 1) = (0, 1) \neq 0$.
(ii) $R$ does not satisfy the condition $(C_\sigma)$: $(1, 0)\sigma((1, 0)) = 0$, but $(1, 0)^2 = (1, 0) \neq 0$.

By Example 3.2, we see that “the condition $(C_\sigma)$” in Lemma 3.1, is not superfluous.

Theorem 3.3. Let $R$ be an $(\sigma, \delta)$-skew Armendariz ring satisfying the condition $(C_\sigma)$. The following statements are equivalent:
(1) $R$ is reversible;
(2) $R$ is $\sigma$-reversible;
(3) $R$ is right $\sigma$-reversible;
(4) $R[x; \sigma, \delta]$ is reversible.

**Proof.** (1) $\Leftrightarrow$ (4). By Theorem 2.4.
(1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). Immediately from Lemma 3.1.
(3) $\Rightarrow$ (1). Let $a, b \in R$, if $ab = 0$ then $b\sigma(a) = 0$ (right $\sigma$-reversibility), so $ba = 0$ (condition $(C_\sigma)$).

**Theorem 3.4.** Let $R$ be an $(\sigma, \delta)$-skew Armendariz ring satisfying the condition $(C_\sigma)$. The following statements are equivalent:
(1) $R$ is symmetric;
(2) $R$ is $\sigma$-symmetric;
(3) $R$ is right $\sigma$-symmetric;
(4) $R[x; \sigma, \delta]$ is symmetric.

**Proof.** As of Theorem 3.3.

The next corollaries are direct consequences of Theorems 2.6, 3.3 and 3.4.

**Corollary 3.5 ([3, Corollary 2.11]).** Let $R$ be an $\sigma$-Armendariz ring. The following statements are equivalent:
(1) $R$ is reversible;
(2) $R$ is $\sigma$-reversible;
(3) $R$ is right $\sigma$-reversible;
(4) $R[x; \sigma]$ is reversible.

**Corollary 3.6 ([14, Theorem 2.10]).** Let $R$ be an $\sigma$-Armendariz ring. The following statements are equivalent:
(1) $R$ is symmetric;
(2) $R$ is $\sigma$-symmetric;
(3) $R$ is right $\sigma$-symmetric;
(4) $R[x; \sigma]$ is symmetric.

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**References**


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