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On the Groups with Isomorphic Integral Group Rings

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Abstract

We prove that for any groups X and Y, the existence of an isomorphism among their integral group rings $\mathbb{Z}(X)$ and $\mathbb{Z}(Y)$ implies an isomorphism of the integral group (co)homologies of X and Y. As a result we get a new proof of the isomorphism problem for integral group rings of abelian groups. The other conclusion is that the Schur multipliers of X and Y are the same.

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We have in mind that the reader is familiar with the elementary notions of homological algebra (see [2], [4]).

Given the groups X and Y, the existence of an isomorphism among their integral group rings $\mathbb{Z}(X)$ and $\mathbb{Z}(Y)$, does not always imply that $X \cong Y$. There is a counterexample even for finite groups (see [1]). This means that, in general, a group is not determined by its integral group ring. We show in this work that the integral group (co)homologies of an arbitrary group are determined by their integral group rings. We prove a more general fact:

Proposition 1. *Let X and Y be groups such that $\mathbb{Z}(X) \cong \mathbb{Z}(Y)$. Assume that M is an abelian group and that X and Y act trivially on M. Then there is an isomorphism of (co)homology groups:*

$$\mathrm{H}_n(X, M) \cong \mathrm{H}_n(Y, M)\ ,\ \mathrm{H}^n(X, M) \cong \mathrm{H}^n(Y, M)\ \ \textit{for all}\ \ n \geq 0\ .$$

Recall that the augmentation map $\varepsilon_X : \mathbb{Z}(X) \to \mathbb{Z}$ is defined by $\varepsilon_X(\sum n_i x_i) = \sum n_i$, where $n_i \in \mathbb{Z}$ and $x_i \in X$.

Let $f : \mathbb{Z}(X) \to \mathbb{Z}(Y)$ be an isomorphism of rings. We demand that f is compatible with the augmentation map, $\varepsilon_X = \varepsilon_Y f$. Such an isomorphism is called normalized. This does not restrict generality thanks to the following:

Lemma 2. *Let $f : \mathbb{Z}(X) \to \mathbb{Z}(Y)$ be an isomorphism of rings. Then there is an isomorphism $\overline{f} : \mathbb{Z}(X) \to \mathbb{Z}(Y)$ such that $\varepsilon_X = \varepsilon_Y \overline{f}$.*

Proof. Define $\overline{f} : \mathbb{Z}(X) \to \mathbb{Z}(Y)$ in the following way: $\overline{f}(x) = \varepsilon_X(x)(\varepsilon_Y f(x)) f(x)$, for all $x \in X$. Since x and $f(x)$ are units in $\mathbb{Z}(X)$ and $\mathbb{Z}(Y)$, respectively, we have $\varepsilon_X(x), \varepsilon_Y f(x) \in \{\pm 1\}$ for all $x \in X$. Using this fact we easily check that $\overline{f}$ is an isomorphism and $\varepsilon_X(x) = \varepsilon_Y \overline{f}(x)$ for all $x \in X$. □

Given an homomorphism $f : \mathbb{Z}(X) \to \mathbb{Z}(Y)$ of rings and a Y-module M, we define on M an X-module structure via f and denote this X-module by M^f.

Lemma 3. *If $f : \mathbb{Z}(X) \to \mathbb{Z}(Y)$ is a normalized isomorphism and M is a trivial Y-module, then M^f is also a trivial X-module.*

Proof. Straightforward. □

Proposition 4. *Let X and Y be groups and $f : \mathbb{Z}(X) \to \mathbb{Z}(Y)$ a normalized isomorphism of rings. Then, for each Y-module M, there are the isomorphisms of (co)homology groups:*

$$\mathrm{H}_n(Y, M) \cong \mathrm{H}_n(X, M^f)\ ,\ \mathrm{H}^n(Y, M) \cong \mathrm{H}^n(X, M^f)\ \ \textit{for all}\ \ n \geq 0\ .$$

Proof. We only show the first isomorphism. The proof of the second one is similar.

Let $\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} \mathbb{Z} \to 0$ be a free Y-resolution of the trivial Y-module $\mathbb{Z}$. It is clear that $\cdots \xrightarrow{d_2} F_1^f \xrightarrow{d_1} F_0^f \xrightarrow{d_0} \mathbb{Z}^f \to 0$ is a free X-resolution of $\mathbb{Z}^f$. But according to the Remark given above, $\mathbb{Z}^f$ is also a trivial X-module. Hence, the homologies of the following complexes

$$\cdots \to F_1 \otimes_Y M \to F_0 \otimes_Y M \qquad \text{and} \qquad \cdots \to F_1^f \otimes_X M^f \to F_0^f \otimes_X M^f$$

are $\mathrm{H}_*(Y, M)$ and $\mathrm{H}_*(X, M^f)$, respectively. Thus, Proposition 4 is proved, since the complexes mentioned above are isomorphic. □

Combining Lemma 2, Lemma 3 and Proposition 4, we get a proof of Proposition 1.

A new proof of the isomorphism problem for integral group rings of abelian groups. Given a group X, one has the following short exact sequence

$$0 \to IX \to \mathbb{Z}(X) \to \mathbb{Z} \to 0$$

where IX is an ideal of $\mathbb{Z}(X)$ generated by $x-1$, for all $x \in X$. If X acts trivially on $\mathbb{Z}$, then it is well known that

$$\mathrm{H}_1(X, \mathbb{Z}) \cong IX \otimes_X \mathbb{Z} \cong X^{\mathrm{ab}} ,$$

where the last isomorphism is given by $(x-1) \otimes 1 \mapsto x$ (see [4]). Hence, as a result of Proposition 1 we have:

Corollary 5. *If X and Y are the groups such that $\mathbb{Z}(X) \cong \mathbb{Z}(Y)$, then $X^{\mathrm{ab}} \cong Y^{\mathrm{ab}}$.*

In this way we get a new proof of the "isomorphism problem for integral group rings" of abelian groups (see [3, Corollary 2.10]).

It is well known that the second integral group homology defines the Schur multiplier. Thus, we have the following:

Corollary 6. *If X and Y are the groups such that $\mathbb{Z}(X) \cong \mathbb{Z}(Y)$, then the Schur multipliers of X and Y are isomorphic.*

Corollary 7. *If X and Y are the groups such that $\mathbb{Z}(X) \cong \mathbb{Z}(Y)$ and X is perfect, then so is Y.*

Proof. If X is perfect, then $X^{\mathrm{ab}} = 0$, and by Corollary 5, $Y^{\mathrm{ab}} = 0$, and so Y is perfect. □

Finally, we show that the opposite version of Proposition 1 need not hold, i.e. if the groups X and Y are given, then the existence of the isomorphisms

$$\mathrm{H}_n(X, M) \cong \mathrm{H}_n(Y, M) \ , \ \mathrm{H}^n(X, M) \cong \mathrm{H}^n(Y, M) \ \text{ for all } \ n \geq 0 \ ,$$

where M is an abelian group with trivial actions of X and Y, does not always imply that $\mathbb{Z}(X) \cong \mathbb{Z}(Y)$ and a fortiori that $X \cong Y$.

Recall that a group G is said to be acyclic if its integral group homologies, $\mathrm{H}_n(G, \mathbb{Z})$, are trivial for all $n > 0$. By the universal coefficient theorem the integral group cohomologies of acyclic groups are also trivial in positive dimensions. Now, using the standard technique we prove the following:

Proposition 8. *Let G be an acyclic group acting trivially on an abelian group M. Then*

$$\mathrm{H}_0(G, M) \cong \mathrm{H}^0(G, M) \cong M \ ,$$

$$\mathrm{H}_n(G, M) = \mathrm{H}^n(G, M) = 0 \ \textit{ for all } \ n > 0 \ .$$

Proof. The first two isomorphisms immediately follow from the definitions and are well known. We will only check that $\mathrm{H}_n(G, M) = 0$ (for $n > 0$), since the same procedure works for $\mathrm{H}^n(G, M)$. First of all we consider the case $M = \mathbb{Z}/m\mathbb{Z}$, for any $m \in \mathbb{N}$, and prove the following:

$$\mathrm{H}_n(G, \mathbb{Z}/m\mathbb{Z}) = 0 \ \text{ for all } \ n > 0 \ . \tag{1}$$

In fact, the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

gives rise to the long exact sequence in homology:

$$\cdots \longrightarrow \mathrm{H}_n(G, \mathbb{Z}) \xrightarrow{m} \mathrm{H}_n(G, \mathbb{Z}) \longrightarrow \mathrm{H}_n(G, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathrm{H}_{n-1}(G, \mathbb{Z}) \xrightarrow{m} \cdots \ .$$

Thus, taking into account that $\mathrm{H}_0(G, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{H}_n(G, \mathbb{Z}) = 0$ for $n > 1$, the above long exact sequence implies (1).

Now assume that M is a finitely generated abelian group. Then $M = \bigoplus_{i \in I} \mathbb{Z}/m\mathbb{Z}_i$, where I is a finite set and $m_i \in \mathbb{Z}^+$, and

$$\mathrm{H}_n(G, M) = \bigoplus_{i \in I} \mathrm{H}_n(G, \mathbb{Z}/m\mathbb{Z}_i) \ \text{ for all } \ n \geq 0 \ .$$

Hence by (1) $\mathrm{H}_n(G, M) = 0$, when M is finitely generated and $n > 0$. In general M can be presented as a direct limit $M = \varinjlim M_t$, where each M_t is a finitely generated abelian sub-group of M. Therefore

$$\mathrm{H}_n(G, M) = \varinjlim \mathrm{H}_n(G, M_t) = 0 \ \text{ for all } \ n > 0 \ .$$

□

Corollary 9. *Let G and H be groups acting trivially on an abelian group M. If G is acyclic, then*

$$\mathrm{H}_n(G \times H, M) \cong \mathrm{H}_n(H, M) \ , \ \mathrm{H}^n(G \times H, M) \cong \mathrm{H}^n(H, M) \ \textit{ for all } \ n \geq 0 \ .$$

Proof. Straightforward from Proposition 8 and the following spectral sequences:

$$\mathrm{H}_p(H, \mathrm{H}_q(G, M)) \Rightarrow \mathrm{H}_{p+q}(G \times H, M) \ ,$$

$$\mathrm{H}^p(H, \mathrm{H}^q(G, M)) \Rightarrow \mathrm{H}^{p+q}(G \times H, M) \ \text{ for all } \ p, q \geq 0 \ .$$

□

Corollary 10. *Let X, Y and H be groups acting trivially on an abelian group M. If X and Y are acyclic, then*

$$\mathrm{H}_n(X \times H, M) \cong \mathrm{H}_n(Y \times H, M) \ , \ \mathrm{H}^n(X \times H, M) \cong \mathrm{H}^n(Y \times H, M) \ \textit{ for all } \ n \geq 0 \ .$$

Corollary 10 shows that the opposite version of Proposition 1 need not hold.

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