On the Maximal Ideals in $L^1$-Algebra of the Heisenberg Group

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Abstract

It is known that the theory of ideals in $L^1$-algebra of a locally compact group played an important role especially in spectral theory. We know the theory is uneasy, except of locally compact abelian groups which was developed by many mathematicians [1], [9], [10], [12] and [13]. The main idea of this paper is the embedding of the $2n+1$-dimensional Heisenberg group and the $2n + 1$-dimensional vector group, into the $3n+1$-dimensional group. Analysis over the vector groups is easier, this should lead us to obtain the maximal ideals(left) in $L^1$-algebra of Heisenberg group.

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1 Preliminary and Results

1.1. Let $H^n$ be the Heisenberg group of dimension $2n + 1$, which consists of the form

$$\begin{pmatrix}
1 & x & z \\
0 & I & y \\
0 & 0 & 1
\end{pmatrix} \quad (1.1)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $I$ is the identity matrix of order $n$.

Let $G = \mathbb{R}^{n+1} \rtimes_{\rho} \mathbb{R}^n$ be the group of the semi-direct product of the group $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, via the group homomorphism $\rho : \mathbb{R}^n \to Aut(\mathbb{R}^{n+1})$, which is defined by:

$$\rho(x)(z, y) = (z + xy, y) \quad (1.2)$$
for any \( x = (x_1, \ldots, x_n) \in \mathcal{R}^n, y = (y_1, \ldots, y_n) \in \mathcal{R}^n, z \in \mathbb{R} \) and \( xy = \sum_{i=1}^{n} x_i y_i \), where \( \text{Aut}(\mathcal{R}^{n+1}) \) is the group of all automorphisms of \( \mathcal{R}^{n+1} \). The multiplication of two elements \( x = ((z, y); x) \) and \( y = ((z', y'); x') \) in \( G \) is given by:

\[
X \cdot Y = ((z, y); x)((z', y'); x') \\
= ((z, y) + x(z', y'), x + x') \\
= ((z + z' + xy', y + y'); x + x')
\]

(1.3)

where \( x(z', y') = \rho(x)(z', y') \).

The inverse of an element \( X \) in \( G \) is:

\[
X^{-1} = ((z, y); x)^{-1} \\
= (-x(-z, -y); -x) \\
= ((-z + x y, -y); -x)
\]

(1.4)

In view of the group isomorphism \( \Psi : G \rightarrow H^n \) defined by

\[
\Psi((z, y); x) = \begin{pmatrix}
1 & x & z \\
0 & I & y \\
0 & 0 & 1
\end{pmatrix}
\]

We can identify the group \( H^n \) with the group \( G \).

1.2. If \( M \) is an unimodular Lie group, we denote by \( L^1(M) \) the Banach algebra that consists of all complex valued functions on the group \( M \), which are integrable with respect to the Haar measure of \( M \) and multiplication is defined by convolution on \( M \). If \( N \) is a subgroup of \( M \) we will denote by \( L^1(M)|_N \) the restriction of \( L^1(M) \) on the subgroup \( N \). That means:

\[
L^1(M)|_N = \{ F|_N; F \in L^1(M) \}
\]

where \( F|_N \) signifies the restriction of the function \( F \) on \( N \).

2 Invariant functions.

2.1. Let \( B = \mathcal{R}^{n+1} \times \mathcal{R}^n \) be the real vector group which is the direct product of \( \mathcal{R}^{n+1} \) and \( \mathcal{R}^n \). Let \( K = \mathcal{R} \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^n \) be the \( 3n+1 \)– dimensional group with law

\[
X \cdot Y = ((z, y); x, t)((z', y'); x', t') \\
= ((z, y) + t(z', y'); x + x', t + t') \\
= ((z + z' + ty', y + y'); x + x', t + t')
\]

(2.1)
for all $x = ((z, y); x, t)$ and $y = ((z', y); x', t')$ where $x(z', y') = \rho(x)(z', y')$.

The inverse of an element $x = ((z, y); x, t)$ in $L$ is given by:

$$X^{-1} = ((z, y); x, t)^{-1} = (-x (-z, -y); -x, -t) = ((-z + ty, -y); -x, -t)$$

(2.2)

In this case we can identify the group $G$ with the closed subgroup $\mathcal{R}^{n+1} \times \{0\} \rtimes \mathcal{R}^n$ of $K$ and $B$ with the closed subgroup $\mathcal{R}^{n+1} \times \mathcal{R}^n \rtimes \{0\}$ of $K$.

**Definition 2.2.** For every $f \in L^1(K)$, one can define a function $\tilde{f}$ as follows:

$$\tilde{f}((z, y); x, t) = f((x(z)); 0, t + x)$$

(2.3)

for all $((z, y); x, t) \in K$.

**Remark 2.3.** The function $\tilde{f}$ is invariant in the following sense

$$\tilde{f}(k(z, y); x - k, t + k) = \tilde{f}((z, y); x, t)$$

(2.4)

for all $((z, y); x, t) \in K$ and $k \in \mathcal{R}^n$.

Note that the restriction function $\tilde{f}|_G$ belongs to $L^1(G)$ and $\tilde{f}|_B$ belongs to $L^1(B)$. Let $L_1^1(K) \rightleftarrows (L^1(K))$ be the image of $L^1(K)$ by the mapping $\sim$, then the space $L^1(G) = L^1(B)$ can be identified with the space $L_1^1(K)$.

**Definition 2.4.** For every $u \in L^1(G)$ or $u \in L^1(B)$ one can define two convolutions product on the group $L$ by:

(i) $u \ast F((z, y); x, t) = \int_G F[((a, b); c)^{-1}((z, y); x, t)] u((a, b); c) da db dc$

(ii) $u \ast_c F((z, y); x, t) = \int_B F[((z - a, y - b); x - c, t)] u((a, b); c) da db dc$

(2.5)

(2.6)

for any $F \in L^1(K)$, where $da db dc$ is the Lebesgue measure on $G$, $\ast$ is the convolution product on $G$ and $\ast_c$ is the convolution product on $B$.

**Corollary 2.5** (i) For each $u \in L^1(G)$ and $F \in L_1^1(K)$ we have:

$$u \ast F((z, y); x, t) = u \ast_c F((z, y); x, t)$$

(2.7)
for all \(((z,y);x,t)\in K\).

\(\text{(ii) The mapping } \lambda \text{ from } L^1(B) \text{ to } L^1(G) \text{ defined by } \lambda(\tilde{f}|_B)((z,y),0,x) = \tilde{f}|_G(x(z,y),0,x)\)

\[\lambda(\tilde{f}|_B)((z,y),0,x) = \tilde{f}|_G(x(z,y),0,x) \tag{2.8}\]

is a topological isomorphism

\(\text{(iii) The mapping } \tau : L^1(G) \longrightarrow L^1(G) \text{ defined by }\)

\[\tau(\tilde{f}|_G)((z,y),0,x) = (\tilde{f}|_G)((x(z,y),0,x) \tag{2.9}\]

is a topological isomorphism

**Proof.** (i) results immediately from (2.4), (2.5) and (2.6).

(ii) It is clear that the mapping \(\lambda\) is continuous and its inverse \(\lambda^{-1}\) defined by

\[\lambda^{-1}(\tilde{f}|_G)((z,y),x,0) = \tilde{f}|_B((x(z,y)),x,0) \tag{2.10}\]

is also continuous

(iii) \(\tau\) is continuous and its inverse is

\[\tau^{-1}(\tilde{f}|_G)((z,y),0,x) = (\tilde{f}|_G)((x(z,y),0,x) \tag{2.11}\]

is also continuous

## 3 Ideals in Group algebra of Heisenberg Group.

### 3.1. If \(\Gamma\) is a subspace of \(L^1(K)\), we denote by \(\tilde{\Gamma}\) its image by the mapping \(\sim\).

Let \(J = \tilde{\Gamma}|_G\) and \(I = \tilde{\Gamma}|_B\).

Our main result is:

**Theorem 3.2.** Let \(\Gamma\) be a subspace of \(L^1(K)\), then the following conditions are equivalents:

\(\text{(i) } I = \tilde{\Gamma}|_B\) is an ideal in the algebra \(L^1(B)\).

\(\text{(ii) } J = \tilde{\Gamma}|_G\) is a left ideal in the algebra \(L^1(G)\).

**Proof:** Let \(\Gamma\) be a subspace of the space \(L^1(K)\) such that \(I = \tilde{\Gamma}|_B\) is an ideal in \(L^1(B)\), then we have:

\[u*e_I ((z,y),x,0) \subseteq I ((z,y),x,0) \tag{3.1}\]

for any \(u \in L^1(B)\), where

\[u*e_I ((z,y),x,0) = \{u*e_f ((z,y),x,0), \ f \in I\}\]

\[= \{u*e(F|_B)((z,y),x,0), F \in \Gamma\} \tag{3.2}\]
Maximal ideals in $L^1$-algebra of the Heisenberg group

Since
\[ I((z, y), x, 0) = I(x(z, y), 0, x) = \tau(J)((z, y), 0, x) \] (3.3)
then we get
\[ u \ast_c I((z, y), x, 0) = u \ast_c I(x(z, y), 0, x) = \tau(u \ast J)((z, y), 0, x) \] (3.4)
and
\[ \tau(u \ast J)((z, y), 0, x) \subseteq I(x(z, y), 0, x) = \tau(J)((z, y), 0, x) \] (3.5)
where
\[ u \ast J((z, y), 0, x) = \{ u \ast f((z, y), 0, x), f \in J \} \]
\[ = \{ u \ast (F|G)((z, y), 0, x), F \in \Gamma \} \] 3.6

Now the inverse of $\tau$ gives
\[ u \ast J \subseteq J \] (3.7)
this proves (i) implies (ii) and (ii) implies (i), whence the theorem.

**Corollary 3.3.** Let $\Gamma$ be a subspace of the space $L^1(K)$ such that $I = \tilde{\Gamma}|_B$ is an ideal in $L^1(B)$, then the following conditions are verified
- (i) $I = \tilde{\Gamma}|_B$ is a maximal ideal in the algebra $L^1(B)$ if and only if $J = \tilde{\Gamma}|_G$ is a left maximal ideal in the algebra $L^1(G)$.
- (ii) $I = \tilde{\Gamma}|_B$ is a closed ideal in the algebra $L^1(B)$ if and only if $J = \tilde{\Gamma}|_G$ is a left closed ideal in the algebra $L^1(G)$.
- (iii) $I = \tilde{\Gamma}|_B$ is a dense ideal in the algebra $L^1(B)$ if and only if $J = \tilde{\Gamma}|_G$ is a left dense ideal in the algebra $L^1(G)$. The proof of this corollary results immediately from theorem 3.2

**Remark 3.4.** If we consider the classical Fourier transform $T_F = e^{-i(\langle \xi, v \rangle, (X, t))}$ on the vector group $K$, then we get
\[ T_F (f \ast \hat{g})(\xi, v) = \mathcal{F}(f)(\xi) T_F(\hat{g})(\xi, v) \] (3.8)
for any $f \in L^1(G)$ and $g \in L^1(G)$, where $\xi = ((\eta, \lambda); \mu)$,
\[ \langle \xi, X \rangle = \eta z + \sum_{i=1}^{n} \lambda_i Y_i + \sum_{i=1}^{n} \mu_i x_i \text{ and } \] (3.9)
\[ \langle v, t \rangle = \sum_{i=1}^{n} v_i t_i. \]

Now the question is.
Can we define a new structure of algebra on $L^1(G)$ suitable for $T_F$?
References


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