Lattices of $\ast$-Congruences on a Quasi Regular $\ast$-Semigroup\textsuperscript{1}

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Abstract

In this paper, we define a relation $\theta$ by $\theta = \{(\rho_1, \rho_2) \in C^\ast(S) \times C^\ast(S) : \text{Ptr}\rho_1 = \text{Ptr}\rho_2\}$ on the complete lattice $C^\ast(S)$ of $\ast$-congruences under set inclusion relation on a quasi regular $\ast$-semigroup $S$. We prove that $\theta$ is a congruence on $C^\ast(S)$ and each $\theta$-class is a complete sublattice of $C^\ast(S)$. The largest $\ast$-congruence of each $\theta$-class and the largest projection separating $\ast$-congruence on $S$ are characterized. We also consider the projection kernel normal systems of the meet and union of the regular $\ast$-congruences $\rho$ and $\sigma$ on $S$, where $(\rho, \sigma) \in \theta$.

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1 Introduction

A semigroup $S$ equipped with a unary operation $\ast : x \mapsto x^\ast$ is said to be a unary semigroup which is denoted by $(S, \cdot, \ast)$ or $(S, \ast)$. In brevity, we denote the set $\{x^\ast | x \in S\}$ by $S^\ast$. Now, for any $x \in (S, \ast)$, we write $(x^\ast)^\ast$ and $((x^\ast)^\ast)^\ast$ by $x^{**}$ and $x^{***}$, respectively. An equivalence $\rho$ on $(S, \ast)$ is called a $\ast$-congruence

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by Nordahl and Scheiblich [6] if it is a congruence on the semigroup $S$ and satisfies the condition

$$(\forall x, y \in S) \quad (x, y) \in \rho \Rightarrow (x^*, y^*) \in \rho.$$  

We now call a *-congruence $\rho$ on a *-semigroup $S$ a regular *-congruence on $S$ if $S/\rho$ is a regular *-semigroup. As a generalization of regular *-semigroups, the author has recently introduced the concept of quasi regular *-semigroups [2]. In particular, projection kernel normal systems and regular *-congruence pairs on a quasi regular *-semigroup have been recently studied. It is proved that every regular *-congruence on a quasi regular *-semigroup can be uniquely determined by its projection kernel normal systems and regular *-congruence pairs, see [2] and [3], respectively. It should be noted that the congruence pairs and the quasi regular semigroups discussed in this paper are different from the congruence pairs and the quasi regular semigroups described by Shum, Guo and Ren in [9]. In this paper, we generalize some results of Y. Chae, S.Y. Lee and C.Y. Park (see [1]) from regular *-semigroup to quasi regular *-semigroup.

We first give the definition of quasi regular *-semigroups.

**Definition 1.1** [2] A unary operation * on a semigroup $S$ is said to be a quasi regular involution if it satisfies exactly the identities

$$x^{**} = x^*, \quad (xy)^* = y^*x^*, \quad x^*x**x^* = x^*.$$  

A semigroup $S$ with a quasi regular involution is called a quasi regular *-semigroup.

An idempotent $e$ of a quasi regular *-semigroup $S$ is called a projection of the quasi regular *-semigroup $S$ if $e^* = e$. Now, we denote the set of projections of $S$ by $P$, that is, $P = \{x^*x**|x \in S\} = \{x**x^*|x \in S\}$. For a *-congruence $\rho$ on $S$, the set of all the $\rho$-classes containing the elements in $P$ is called a projection kernel of $\rho$ which is denoted by $\text{Pker}\rho = \{e\rho : e \in P\}$. In addition, the restriction of $\rho$ on $P$ is written by $\text{Ptr}\rho$. Denote the set of all *-congruences on a quasi regular *-semigroup $S$ by $C^*(S)$. It is clear that the set inclusion relation “$\subseteq$” on $C^*(S)$ is a partially ordered relation. For other terminologies and notations not given in this paper, the reader is referred to [4] and [7].

### 2 The relation $\theta$

**Lemma 2.1** [5] A partially ordered set with the greatest element such that each nonempty subset has the greatest lower bound is a complete lattice. □
Lemma 2.2 Let $S$ be a quasi regular $*$-semigroup. Then the partially ordered set $(C^*(S), \subseteq)$ forms a complete lattice.

Proof. Clearly, the universe relation on $S$ is the greatest element of $(C^*(S), \subseteq)$. Let $\{\rho_i : i \in I\}$ be any nonempty subset of $C^*(S)$ and $\rho = \cap \{\rho_i : i \in I\}$. Then it can be easily seen that $\rho$ is a congruence on $S$ and $\rho \subseteq \rho_i$ for any $i \in I$. Moreover, if $(a, b) \in \rho$, then we have $(a, b) \in \rho_i$ for any $i \in I$. Hence, $(a^*, b^*) \in \rho_i$ implies that $(a^*, b^*) \in \rho$. This shows that $\rho$ is a $*$-congruence on a quasi regular $*$-semigroup $S$.

From the set inclusion relation, $\rho$ is clearly one of the lower bounds of $\{\rho_i : i \in I\}$. If $\sigma \in C^*(S)$ is another lower bound of $\{\rho_i : i \in I\}$, then $\sigma \subseteq \rho_i$ for every $i \in I$, and whence $\sigma \subseteq \cap \{\rho_i : i \in I\} = \rho$. This shows that $\rho$ is the greatest lower bound of $\{\rho_i : i \in I\}$. By Lemma 2.1, $(C^*(S), \subseteq)$ forms a complete lattice. \hfill \Box

In the following, we simply abbreviate $(C^*(S), \subseteq)$ by $C^*(S)$. Now, any equivalence relation $\rho$ on a lattice $L$ is a congruence [7] if

$$(\forall x, y, z \in L) \quad x\rho y \Rightarrow x \land z \\rho \quad y \land z, \quad x \lor z \\rho \quad y \lor z.$$ 

**Theorem 2.3** Let $S$ be a quasi regular $*$-semigroup and define a relation $\theta$ on $C^*(S)$ by:

$$\theta = \{ (\rho_1, \rho_2) \in C^*(S) \times C^*(S) : \text{Ptr}_1 = \text{Ptr}_2 \}.$$ 

Then $\theta$ is a congruence on $C^*(S)$ and each $\theta$-class is a complete sublattice of $C^*(S)$.

Proof. It is obvious that $\theta$ is an equivalence on $C^*(S)$. Let $(\rho_1, \rho_2) \in \theta, \rho_3 \in C^*(S)$. Then $\rho_1 \cap (P \times P) = \rho_2 \cap (P \times P)$ and so $\rho_1 \cap \rho_3 \cap (P \times P) = \rho_2 \cap \rho_3 \cap (P \times P)$, that is,

$$\rho_1 \cap \rho_3, \rho_2 \cap \rho_3 \in \theta.$$ 

Now let $e \in P$ and $f \in e(\rho_1 \vee \rho_3) \cap P$. Then there exist $x_1, x_2, \ldots, x_n \in S$ such that $(e, x_1) \in \rho_1, (x_1, x_2) \in \rho_3, (x_2, x_3) \in \rho_1, \cdots, (x_{n-1}, x_n) \in \rho_3$. Since $\rho_1, \rho_3$ are $*$-congruences on $S$, $(e, x_1^*x_1^*) \in \rho_1, (x_1^*x_1^*, x_2^*x_2^*) \in \rho_3, (x_2^*x_2^*, x_3^*x_3^*) \in \rho_1, \cdots, (x_n^*x_n^*, f) \in \rho_3$. But $(\rho_1, \rho_2) \in \theta$ and so we have $(e, x_1^*x_1^*) \in \rho_2, (x_1^*x_1^*, x_2^*x_2^*) \in \rho_2, (x_2^*x_2^*, x_3^*x_3^*) \in \rho_2, \cdots, (x_n^*x_n^*, f) \in \rho_3$. Thus $(e, f) \in \rho_2 \lor \rho_3$ and $f \in e(\rho_2 \lor \rho_3) \cap P$, i.e.,

$$e(\rho_1 \lor \rho_3) \cap P \subseteq e(\rho_2 \lor \rho_3) \cap P.$$ 

Similarly, we can also deduce that

$$e(\rho_1 \lor \rho_3) \cap P \geq e(\rho_2 \lor \rho_3) \cap P.$$ 

Therefore, $e(\rho_1 \lor \rho_3) \cap P \subseteq e(\rho_2 \lor \rho_3) \cap P$. \hfill \Box
Hence, \((\rho_1 \lor \rho_3, \rho_2 \lor \rho_3) \in \theta\).

Let \(A\) be a \(\theta\)-class and let \(\sigma \in A\) and \(C\) be nonempty subsets of \(A\). Then, by Lemma 2.2, \(C^*(S)\) is a complete lattice. Thus, \(\vee_{\rho \in C^*} \rho \land_{\rho \in C} \rho \in C^*(S)\) and \(\land_{\rho \in C} \rho = \bigcap_{\rho \in C} \rho \subseteq \rho\), for each \(\rho \in C \subseteq A\). Hence, \(\land_{\rho \in C} \rho \in A\). Let \((e, f) \in P\). If \((e, f) \in \bigvee_{\rho \in C^*} \rho\), then there exist \(x_1, x_2, \ldots, x_n \in S\) and \(\rho_1, \rho_2, \ldots, \rho_{n+1} \in C\) such that \((e, x_1) \in \rho_1, (x_1, x_2) \in \rho_2, \ldots, (x_n, f) \in \rho_{n+1}\). These imply that \((e, x_1^1 x_1^2) \in \rho_1, (x_1^1 x_1^2, x_2^1 x_2^2) \in \rho_2, \ldots, (x_n^1 x_n^2, f) \in \rho_{n+1}\). But \((\rho_i, \sigma) \in \theta\) for each \(i = 1, 2, \ldots, n+1\) and so \((e, x_1^1 x_1^2) \in \sigma, (x_1^1 x_1^2, x_2^1 x_2^2) \in \sigma, \ldots, (x_n^1 x_n^2, f) \in \sigma\). Thus we have \((e, f) \in \sigma\). Conversely, if \((e, f) \in \sigma\), then clearly \((e, f) \in \bigvee_{\rho \in C^*} \rho\). Consequently, we conclude that \((\forall_{\rho \in C^*} \rho, \sigma) \in \theta\). This shows that \(\bigvee_{\rho \in C^*} \rho \subseteq A\).

**Theorem 2.4** Let \(S\) be a quasi regular \(*\)-semigroup and \([\rho]\) be a \(\theta\)-class of \(C^*(S)/\theta\). Define

\[\rho_{\text{max}} = \{(a, b) \in S \times S : (\forall e \in P)(a^*ea^*, b^*eb^*) \in \rho, (a^*ea^*, b^*eb^*) \in \rho\} .\]

Then, \(\rho_{\text{max}}\) is the largest \(*\)-congruence of \([\rho]\).

**Proof.** Obviously, \(\rho_{\text{max}}\) is an equivalence relation on \(S\). Let \((a, b) \in \rho_{\text{max}}, c \in S\). Then \((a^*ea^*, b^*eb^*) \in \rho\) and \((a^*ea^*, b^*eb^*) \in \rho\) for each \(e \in P\). Now for all \(e \in P\), we have

\[((a)^*c(a)^*, (b)^*c(b)^*) = (c^*(a)^*ea^*)c^*, (b)^*(b^*eb^*)c^*) \in \rho,\]

\[\quad (a)^*e(a)^*, (b)^*e(b)^*) = (a^*(c^*ec^*)a, b^*(c^*ec^*)b) \in \rho.\]

Hence \((ac, bc) \in \rho_{\text{max}}\). Similarly, \((ca, cb) \in \rho_{\text{max}}\). Thus \(\rho_{\text{max}}\) is a congruence on \(S\). It is straightforward to verify that \(\rho_{\text{max}}\) is a \(*\)-congruence on \(S\).

Let \((e, f) \in \rho \cap (P \times P)\). Then for all \(g \in P\), we have

\[ (e^*ge^*, f^*gf^*) = (ege, fgf) \in P, \quad (e^*ge^*, f^*gf^*) = (ege, fgf) \in P.\]

Hence \((e, f) \in \rho_{\text{max}} \cap (P \times P)\). Let \((e, f) \in \rho_{\text{max}} \cap (P \times P)\). Then \((ege, fgf) \in \rho\) for all \(g \in P\). This leads to \((e, ef), (ef, f) \in \rho\). Thus

\[ e = ee \quad \rho \quad e(ef) = (ef)e \quad \rho \quad ff = f.\]

and so \((e, f) \in \rho \cap (P \times P)\), that is, \(\rho_{\text{max}} \in [\rho]\).

Finally, suppose that \(\tau\) is a \(*\)-congruence on \(S\) such that \(\tau \in [\rho]\). If \((a, b) \in \tau\), then \((a^*, b^*) \in \tau, (a^*, b^*) \in \tau\). Hence for each \(e \in P\), we have

\[ (a^*ea^*, b^*eb^*) \in \tau \cap (P \times P) = \rho \cap (P \times P) \subseteq \rho,\]
Lattices of ∗-congruences

\[(a^{**}ea^*, b^{**}eb^*) \in \tau \cap (P \times P) = \rho \cap (P \times P) \subseteq \rho,\]
and hence, \((a, b) \in \rho_{\text{max}}\). This proves that \(\tau \subseteq \rho_{\text{max}}\). \(\square\)

We now call a ∗-congruence \(\rho\) on a quasi regular ∗-semigroup \(S\) a projection-separating ∗-congruence if each \(\rho\)-class contains at most one projection. The next corollary follows immediately.

**Corollary 2.5**

\[\mu = \rho_{\text{max}} = \{(a, b) \in S \times S : (\forall e \in P) a^{*}e a^{**} = b^{**}e b^{*}, a^{**}e a^{*} = b^{**}e b^{*}\}\]
is the largest projection-separating ∗-congruence on \(S\). \(\square\)

3 The projection kernel normal system

Let \(T\) be a semigroup, \(\tau\) a congruence on \(T\) and \(B \subseteq T\). Denote \(B_{\rho} = \{x \in T : (x, b) \in \rho \text{ for some } b \in B\}\). The following lemmas play crucial roles in the rest of the paper.

**Lemma 3.1** [8] Let \(\rho_{1}, \rho_{2}\) be congruences on a semigroup \(T\) and \(B \subseteq T\). If \((B_{\rho_{1}})_{\rho_{2}} = (B_{\rho_{2}})_{\rho_{1}}\), then \((B_{\rho_{1}})_{\rho_{2}} = B(\rho_{1} \lor \rho_{2})\). \(\square\)

Let \(S\) be a quasi regular ∗-semigroup. Let \(\rho\) and \(\sigma\) be regular ∗-congruences on \(S\) such that \((\rho, \sigma) \in \theta\). Denote the projection kernel normal systems of \(\rho\) and \(\sigma\) by \(\{A_{i} : i \in I\}\) and \(\{B_{i} : i \in I\}\), respectively. For any \(i \in I\), we define

\[P_{i} = A_{i} \cap P = B_{i} \cap P,\]

\[(A \lor B)_{i} = \{x \in S : x^{*}x^{**}, x^{**}x^{*} \in P_{i} \text{ and } x^{*}a^{**}, a^{**}x^{*} \in B_{i} \text{ for some } a \in A_{i}\},\]

and

\[(A \land B)_{i} = A_{i} \cap B_{i}.\]

Then the following property holds :

**Lemma 3.2** \((A \lor B)_{i} = A_{i} \sigma = B_{i} \rho.\)

*Proof.* We first prove that \((A \lor B)_{i} \subseteq A_{i} \sigma, (A \lor B)_{i} \subseteq B_{i} \rho.\) Let \(x \in (A \lor B)_{i}.\) Then \(x^{*}x^{**}, x^{**}x^{*} \in P_{i}\) and there exists \(a \in A_{i}\) such that

\[(x^{**}a^*)^{*} = a^{**}x^{*}, x^{*}a^{**} = B_{i}, \ a^{*}a^{**}, a^{**}a^{*} \in P_{i} = A_{i} \cap P = B_{i} \cap P.\]
Hence \((x, a) \in \sigma\) and \(x \in A_i\sigma\). Now since \(x \in A_i\sigma\), there exists \(c \in A_i\) such that \((x, c) \in \sigma\). Hence, we have \(c^*c^*, c^*c^* \in A_i \cap P = P_i\). Since \(\sigma\) is a \(*\)-congruence on \(S\), \((x^*x^*, c^*c^*), (x^*x^*, c^*c^*) \in \sigma\). Thus
\[
x^*x^*, x^*x^* \in B_i \cap P = P_i \subseteq A_i.
\]
Consequently, \((x^*x^*, a^*a^*) \in \rho\), and therefore \((x^*, x^*a^*) = (x^*x^*x^*, x^*a^*) \in \rho\). By the regularity of \(\rho\), we have \((x, a^*x^*) \in \rho\). Since \(a^*x^* = (x^*a^*)^* \in B_i\), \(x \in B_i\rho\).

We now proceed to prove that \(A_i\sigma \subseteq (A \lor B)_i\). For this purpose, we let \(x \in A_i\sigma\). Then, there exists \(a \in A_i\) such that \((x, a) \in \sigma\). For \(a \in A_i\), we have \(a^*a^*, a^*a^* \in A_i \cap P = P_i\). Since \(\sigma\) is a \(*\)-congruence on \(S\), \((x^*x^*, a^*a^*), (x^*x^*, a^*a^*) \in \sigma\) and so
\[
x^*x^*, x^*x^* \in B_i \cap P = P_i, \quad (x^*a^*, a^*a^*), (a^*x^*, a^*a^*) \in \sigma.
\]
Thus, \(x^*a^*, a^*a^* \in B_i\) and whence, \(x \in (A \lor B)_i\). Similarly, we have \(B_i\rho \subseteq (A \lor B)_i\).

In closing this paper, we formulate the following theorem for the regular \(*\)-congruences on a quasi regular \(*\)-semigroup \(S\).

**Theorem 3.3** Let \(\rho\) and \(\sigma\) be regular \(*\)-congruences on a quasi regular \(*\)-semigroup \(S\) such that \((\rho, \sigma) \in \theta\). If \(\{A_i : i \in I\}\) and \(\{B_i : i \in I\}\) are the respectively projection kernel normal systems of \(\rho\) and \(\sigma\), then
\[
Pker(\rho \lor \sigma) = \{(A \lor B)_i : i \in I\}, \quad Pker(\rho \land \sigma) = \{(A \land B)_i : i \in I\}.
\]

*Proof.* It is clear that for each \(e \in P_i\), we have \(e(\rho \land \sigma) = e\rho \land e\sigma = A_i \land B_i\). Hence, \(\{(A \land B)_i : i \in I\}\) is a projection kernel normal system of \(\rho \land \sigma\).

If \(e \in P_i\), then \(e\rho = P_i\rho = A_i\) and \(e\sigma = P_i\sigma = B_i\). Since each \(\theta\)-class is a complete sublattice of \(C^*(S)\), we see immediately that \(e(\rho \lor \sigma) = P_i(\rho \lor \sigma)\), for all \(e \in P_i\). It hence follows from lemma 3.2 that
\[
(P_i\rho)\sigma = A_i\sigma = B_i\rho = (P_i\sigma)\rho.
\]

Thus by Lemma 3.1, we have
\[
P_i(\rho \lor \sigma) = (P_i\rho)\sigma = A_i\sigma = (A \lor B)_i.
\]
This shows that \(\{(A \lor B)_i : i \in I\}\) is a projection kernel normal system of \(\rho \lor \sigma\). \(\square\)
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References


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