Semi-Endosimple Modules
and Some Applications

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Abstract

An \( R \)-module is called semi-endosimple if it has no proper fully invariant essential submodules. For a quasi-projective retractable module \( M_R \) we show that \( M \) is finitely generated semi-endosimple if and only if the endomorphism ring of \( M \) is a finite direct sum of simple rings. For an arbitrary module \( M \), conditions equivalent to the semi-endosimplicity of its quasi-injective hull are found. As consequences of these results, new characterizations of V-rings, right Noetherian V-rings and strongly semiprime rings are obtained. As such, a hereditary left Noetherian ring \( R \) is a finite direct sum of simple Noetherian right V-rings if and only if all finitely generated right \( R \)-modules are semi-endosimple.

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1. Introduction

Throughout rings will have unit elements and modules will be right unitary. If \( M \) is a module over a ring \( R \), its quasi-injective (injective) hull will be denoted by \( \hat{M}_R \) (\( E(M_R) \)). Unexplained terminology and standard results may be found in [2] or [6]. As a simultaneous generalization of weakly primitive rings in the sense of Zelmanowitz [7] and strongly prime rings, Desale and Nicholson [2] defined an endoprimitive ring \( R \) for which one of several equivalent definitions is the existence of a faithful endosimple module, where a non-zero module is called endosimple if it has no non-trivial fully invariant submodules. With this terminology, an endosimple module \( M_R \) is strongly prime, that is, \( M \) is contained in every non-zero fully invariant submodule of \( E(M_R) \), or equivalently \( \hat{M}_R \) is endosimple; see [2, Proposition 1.2]. A strongly semiprime module \( M_R \)
was defined by Beidar and Wisbauer [1] by requiring that $H\hat{M}_R$ be semisimple (as a bimodule) where $H = \text{End}_R(\hat{M})$. They characterized strongly semiprime modules and established that $R_R$ is strongly semiprime if and only if the ring $R$ is right strongly semiprime in the sense of Handelman [4]. Generalizing the concept of endosimple, we call an $R$-module semi-endosimple if it has no proper fully invariant essential submodules. Clearly every quasi-injective strongly semiprime module is semi-endosimple, but in general strongly semiprime and semi-endosimple are different conditions; see Remark and Example 3.7. The study of semi-endosimple modules is useful as we are able to give new characterizations of some diver types of rings including right Noetherian $V$-rings and strongly semiprime rings in terms of semi-endosimplicity of certain modules. Our investigation is centered on quasi-projective and/or quasi-injective modules that are semi-endosimple. One notable feature of semi-endosimplicity is the left-right symmetry that it can impose on the ring, in contrast to the fact that being strongly semiprime is not a symmetric property for rings. Our first observation is that $R_R$ is semi-endosimple if and only if $R_R$ is semi-endosimple if and only if $R$ is isomorphic to a finite direct sum of simple rings. Replacing $R_R$ by a (non-zero) quasi-projective retractable module $M$, we prove in Theorem 2.2 that $M_R$ is finitely generated semi-endosimple if and only if $S_S$ is semi-endosimple where $S = \text{End}_R(M)$. Here $M_R$ is called retractable if $\text{Hom}_R(M, N) \neq 0$ for any non-zero submodule $N \leq M_R$. We then establish some structural results on the ring equivalent to the semi-endosimplicity of certain cyclic modules; see Theorem 2.5. The final result in §2 is a characterization of rings for which all finitely generated modules are semi-endosimple. In §3 we find conditions on a module $M_R$ that are equivalent to the semi-endosimplicity of $M_R$, and then deduce that the class of semi-endosimple $R$-modules is closed under taking quasi-injective hulls. Applying these results to $R_R$ we obtain some new characterizations of right strongly semiprime rings.

2. Finitely generated semi-endosimple modules

Throughout the paper for a ring $R$ we denote the class of semi-endosimple $R$-modules by $\mathcal{C} = \mathcal{C}(R)$. If the regular module $R_R$ is in $\mathcal{C}$ then clearly $R$ is a right strongly semiprime ring, that is, every ideal of $R$, essential as a right ideal, contains a finite subset with zero annihilator.

Proposition 2.1. The following statements are equivalent for a ring $R$.
(i) $R_R$ is semi-endosimple.
(ii) $I + r.\text{ann}_R(I) = R$ for every ideal $I$ of $R$.
(iii) $R$ is isomorphic to a finite direct sum of simple rings.
(iv) $R_R$ is semi-endosimple.
(v) $I + l.\text{ann}_R(I) = R$ for every ideal $I$ of $R$. 
Proof. (i)⇒(ii). Let $I$ be an ideal of $R$. Then $I + r.\ann_R(I)$ is an ideal which is essential as a right ideal. Thus $I + r.\ann_R(I) = R$ by (i).

(ii)⇒(iii). Suppose first that $I$ is an ideal of $R$ with zero square. Then $I \subseteq r.\ann_R(I)$, hence $I = 0$ by (ii). It follows that $R$ is a semiprime ring. This in turn implies that the right annihilator of any essential right ideal of $R$ is zero. Consequently, $R_R$ is semi-endosimple. Hence $R$ being right strongly semi-prime, it has only finitely many minimal prime ideals, say $P_1, \ldots, P_k$, and $\bigcap_{i=1}^k P_i = 0$, see [4]. Since in any ring a prime ideal is either minimal or essential as a right ideal, we deduce that any maximal ideal of $R$ is a minimal prime. Therefore each $P_i$ is a maximal ideal and $R$ is in fact isomorphic to the finite direct sum of simple rings $R/P_1 \oplus \cdots \oplus R/P_k$. The other implications are immediate.

In the following result we generalize the equivalence (i)⇔(iii) of Proposition 2.1.

Theorem 2.2. Let $M$ be a quasi-projective retractable $R$-module. Then the following are equivalent.

(i) $M$ is a finitely generated $R$-module in $\mathcal{C}$.
(ii) $\End_R(M)$ is isomorphic to a finite direct sum of simple rings.

Proof. (i)⇒(ii). Let $S = \End_R(M)$. In view of Proposition 2.1, we shall show that $S$ is semi-endosimple. Let $I$ be an ideal of $S$, which is essential as a right ideal. The fully invariant submodule $IM$ is then essential. For, let $N$ be a non-zero submodule of $S$ such that $IM \cap N = 0$. By condition on $M$, we can pick $0 \neq g \in S$ such that $g(M) \subseteq N$. Note that $IM \cap g(M) = 0$ and hence $\Hom_R(M, IM) \cap \Hom_R(M, g(M)) = 0$. Thus $I \cap gS = 0$, a contradiction, see [6, 18.4]. Now by semi-endosimplicity assumption of $M$ we have $M = IM$. Therefore $I = \Hom_R(M, IM) = S$.

(ii)⇒(i). Suppose that $S = S_1 \oplus \cdots \oplus S_n$, for some ideals $S_1, \ldots, S_n$ such that each $S_i$ is a simple ring with unity $e_i$ where $1_S = e_1 + \cdots + e_n$. The idempotent endomorphisms $e_1, \ldots, e_n$ are central and mutually orthogonal, giving $M = e_1 M \oplus \cdots \oplus e_n M$. Since $\Hom_R(e_i M, e_j M) = 0$ whenever $i \neq j$, the endomorphism ring of $R$–submodule $e_i M$ is isomorphic to $S_i$ for each $i = 1, \ldots, n$. Fix $j$, and $0 \neq x \in L := e_j M$ and set $T = \End_R(L)$. By our assumption on $M$, it is easily seen that $\Hom_R(L, xR) \neq 0$. If $0 \neq f : L \to xR$, then by simplicity of $T$ we have $TfT = T$, hence $\sum_{i=1}^n t_i ft_i' = 1_T$ for some $t_i$ and $t_i'$ in $T$ for $1 \leq i \leq n$. It follows that for any $y \in L$, $\sum_{i=1}^n t_i ft_i'(y) = y$ and so $y = t_1 f(m_1) + \cdots + t_n f(m_n)$ where $m_i' = t_i(y)$. Because $\text{Im } f \subseteq xR$ we have $y = t_1(xr_1) + \cdots + t_n(xr_n) = t_1(x)r_1 + \cdots + t_n(x)r_n$ for some $r_i \in R$. 

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Consequently, \( t_1(x)R + \cdots + t_n(x)R = L \). Therefore each \( e_iM \) is a finitely generated \( R \)-module, hence \( M_R \) is finitely generated. Let \( N \) be a fully invariant essential submodule of \( M_R \), and set \( I = \text{Hom}_R(M,N) \). Then \( I \) is a two sided ideal of \( S \), and we show below that it is essential as a right ideal. Suppose \( I \cap fS = 0 \) for some \( f \in S \). Since \( M_R \) is finitely generated quasi-projective, \( fS = \text{Hom}_R(M,f(M)) \), hence \( 0 = \text{Hom}_R(M,N) \cap fS = \text{Hom}_R(M,N \cap f(M)) \). Thus our assumption on \( M \) yields \( N \cap f(M) = 0 \). But \( N \) is assumed essential, thus \( f(M) = 0 \), and so \( f = 0 \). Therefore by Proposition 2.1, \( I = S \), and consequently \( N = M \).

**Corollary 2.3.** Over a simple ring \( R \), finitely generated projective modules are in \( \mathcal{C} \).

**Proof.** Let \( M_R \) be finitely generated projective. Then \( \text{Tr}_R(M,R) \) is a non-zero ideal of \( R \) and so is equal to \( R \) by hypothesis. It follows that \( M_R \) is a progenerator for \( \text{Mod-}R \). Consequently \( S = \text{End}_R(M) \), being Morita equivalent to \( R \), is a simple ring (see e.g., [6, 46.4]). The result is now clear by Theorem 2.2.

The next result collects some properties of \( \mathcal{C} \). We shall later show that \( \mathcal{C} \) is also closed under quasi-injective hull extensions.

**Proposition 2.4.** (i) The class \( \mathcal{C} \) is close under direct sums.
(ii) \( M \in \mathcal{C} \) if and only if \( M^{(\Lambda)} \in \mathcal{C} \) for any index set \( \Lambda \) if and only if \( M^{(\Lambda)} \in \mathcal{C} \) for some index set \( \Lambda \).
(iii) If \( M \in \mathcal{C} \) then \( (M \oplus M/N) \in \mathcal{C} \) for all \( N \leq M \).

**Proof.** (i) Consider an arbitrary family \( \{M_\lambda\}_{\lambda \in \Lambda} \) of semi-endosimple \( R \)-modules and let \( N \) be a fully invariant essential submodule of \( M = \bigoplus_{\lambda \in \Lambda} M_\lambda \). Let \( N_\lambda = \overline{M_\lambda} \cap N \) where \( \overline{M_\lambda} \) is the image of the canonical injection \( i_\lambda : M_\lambda \to M \). Clearly, each \( N_\lambda \) is an essential \( R \)-submodule of \( \overline{M_\lambda} \). Since \( \overline{M_\lambda} \) is a direct summand of \( M_R \), every element in \( \text{End}_R(\overline{M_\lambda}) \) can be extended to an element in \( \text{End}_R(M) \) and so \( N_\lambda \) is also a fully invariant \( R \)-submodule of \( \overline{M_\lambda} \). Therefore \( N_\lambda = \overline{M_\lambda} \) by semi-endosimplicity of \( \overline{M_\lambda} \) \( \simeq M_\lambda \). Thus \( \overline{M_\lambda} \subseteq N \) for all \( \lambda \in \Lambda \), giving \( M = N \), proving that \( M \in \mathcal{C} \).
(ii) This is proved by (i) and the fact that if \( N \) is a fully invariant essential submodule of \( M_R \), then \( N^{(\Lambda)} \) is an fully invariant essential submodule of \( M_R^{(\Lambda)} \) for any index set \( \Lambda \).
(iii) Let \( 0 \neq M \in \mathcal{C} \), \( N \leq M \) and \( K \) be any fully invariant submodule of \( M \oplus M/N \). As it is seen in the proof of (i), \( K = W \oplus L \) for some fully invariant essential submodules of \( M \) and \( M/N \) respectively. Because \( M \) is semi-endosimple, \( W = M \) or \( W = 0 \). If \( W = 0 \) then \( K = 0 \oplus L \) cannot be essential. Therefore \( W = M \) and hence \( K = M \oplus L \). On the other hand, the canonical
projection $\pi : M \to M/N$ can be extended to $\alpha : M \oplus M/N \to M \oplus M/N$ by $\alpha(m, \pi(m')) = (0, \pi(m))$ for all $m, m' \in M$. Thus $\alpha(K) = (0, M/N) \subseteq K$. It follows that $L = M/N$. Consequently, $K = M \oplus M/N$, proving that $(M \oplus M/N) \in C$.

We recall that a ring $R$ is said to be a right $V$-ring if any simple right $R$-module is injective or equivalently any right $R$-module has zero Jacobson radical see e.g., [6, 23.1].

**Theorem 2.5.** (i) The ring $R$ is a right $V$-ring if and only if every cyclic cocyclic $R$-module is in $C$.

(ii) $R$ is right Noetherian right $V$-ring if and only if every cyclic $R$-module which is an essential extension of a semisimple module is in $C$.

(iii) A hereditary left Noetherian ring $R$ is a finite direct sum of simple Noetherian right $V$-rings if and only if all finitely generated $R$-modules are in $C$.

**Proof.** We only prove (ii) and (iii).

(ii) The necessity follows from the fact that a ring $R$ is right Noetherian if and only if any direct sum of cocyclic $R$-modules is injective; see [6, 27.3]. Conversely, let $M$ be a direct sum of cocyclic $R$-modules and $x \in E = E(M_R)$. Then $\text{Soc}(E_R) \subseteq M$ and $xR$ is a cyclic $R$-module which is an essential extension of a semisimple module. Thus by our assumption $xR$ is semi-endosimple. Because $\text{Soc}(xR)$ is a fully invariant essential submodule, $xR$ must be semisimple. It follows that $\text{Soc}(E_R) = M = E(M_R)$.

(iii) The sufficiency is by Proposition 2.4(i) and part (ii) above. Conversely, let $R = R_1 \oplus \cdots R_n$ be a finite direct sum of hereditary simple Noetherian right $V$-rings and let $M$ be a finitely generated $R$-module. Then $M = M_1 \oplus \cdots \oplus M_n$, where each $M_i$ is a finitely generated $R_i$-module. If each $M_i$ is a semi-endosimple $R_i$-module, then all $M_i(1 \leq i \leq n)$ are semi-endosimple $R$-modules and hence $M \in C$ by Proposition 2.4(i). Therefore, we shall show that each $M_i$ is a semi-endosimple $R_i$-module. Now fix $i$ and let $L = M_i$. By [5, Lemma 5.7.4] the torsion submodule $\tau(L)$ has finite length, $L/\tau(L)$ is projective and $L \cong \tau(L) \oplus L/\tau(L)$. By Corollary 2.3, $L/\tau(L)$ is semi-endosimple. Also $\tau(L)$ being Artinian it has an essential socle, and since $R_i$ is a $V$-ring, the socle is injective. Thus $\tau(L)$ is semisimple, hence semi-endosimple. Applying Proposition 2.4(i) yields $L_{R_i}$ semi-endosimple, as wanted.

If $R$ is the Weyl algebra over the complex field, then $R$ is a simple Noetherian domain but not a $V$-ring, hence there exists a cyclic $R$-module which is not semi-endosimple. This shows that the class of rings with all finitely generated modules semi-endosimple lies properly between the class of semisimple rings and the class of rings that are finite direct sum of simple rings.
We are now going to characterize rings over which all finitely generated modules are in $\mathcal{C}$. First recall that for a right ideal $I$ in a ring $R$, $\text{Idealizer}(I) = \{ r \in R \mid rI \subseteq I \}$ and that $\text{End}_R(R/I) \cong \text{Idealizer}(I)/I$ as rings.

**Lemma 2.6.** All cyclic $R$–modules are in $\mathcal{C}$ if and only if for any pair of non-zero proper right ideals $I \subseteq J$, the condition $\text{Idealizer}(I) \subseteq \text{Idealizer}(J)$ implies that there exists an element $x \in R \setminus I$ with $xR \cap J \subseteq I$. In this case, all quasi-injective $R$-modules are in $\mathcal{C}$.

**Proof.** Suppose $0 \neq I \subseteq J$ are proper right ideals with $\text{Idealizer}(I) \subseteq \text{Idealizer}(J)$. Then $J/I$ is a proper fully invariant submodule of $R/I$. If the cyclic module $R/I$ is semi-endosimple, then $J/I$ cannot be essential in $R/I$, which is equivalent to the existence of an element $x \in R \setminus I$ with $xR \cap J \subseteq I$. The proof of the converse statement is similarly done. For the last statement, let $N$ be a fully invariant essential submodule of a quasi-injective $R$-module $M$ and $0 \neq m \in M$. It easy to verify that $L := N \cap (mR)$ is an essential submodule of $mR$. Because $M_R$ is quasi-injective, $L$ is also a fully invariant $R$-submodule of $mR$. Hence by semi-endosimplicity of $mR$, we have $L = mR$ and so $m \in N$, proving that $N = M$.

**Lemma 2.7.** Let $R$ be any ring and $S = \text{Mat}_{n \times n}(R)$ for some $n \geq 1$. Then $M_R$ is semi-endosimple if and only if $M^{(n)}$ is semi-endosimple as an $S$–module (with the natural $S$–module structure).

**Proof.** ($\Rightarrow$). Let $K$ be a fully invariant essential $S$-submodule of $M^{(n)}$. Then $K = Ke_1 \oplus \ldots \oplus Ke_n$ where $e_i$ is the $i$-th unit matrix. It is easy to check that each $Ke_i$ is an essential $R$–submodule of $M^{(n)}e_i \cong M$. Also each $Ke_i$ is a fully invariant $R$–submodule of $M^{(n)}e_i$. To see this, let $f : (M,0,\ldots,0) \to (M,0,\ldots,0) = M^{(n)}e_1$ be an $R$–map. Define $g : M^{(n)} \to M^{(n)}$ by $g[(m_1,\ldots,m_n)] = (m'_1,\ldots,m'_n)$ where $f[(m_i,0,\ldots,0)] = (m'_i,0,\ldots,0)$. We claim that $g$ is an $S$-map. If $A = (a_{ij})_{n \times n} \in S$, then $(m_1,\ldots,m_n)A = (\sum_{i=1}^n m_ia_{i1},\ldots,\sum_{i=1}^n m_ia_{in}) =: (t_1,\ldots,t_n)$. Let $g[(m_1,\ldots,m_n)]A = (t'_1,\ldots,t'_n)$ where $f[(t_j,0,\ldots,0)] = (t'_j,0,\ldots,0)$ for $j = 1,\ldots,n$. This shows that $t'_j = \sum_{i=1}^n m'_ia_{ij}$ for $j = 1,\ldots,n$. Thus $(t'_1,\ldots,t'_n) = (m'_1,\ldots,m'_n)A$, as desired. Now $f(Ke_1) = f(K \cap M^{(n)}e_1) = g(K \cap M^{(n)}e_1) \subseteq (K \cap M^{(n)}e_1) = Ke_1$, proving that $Ke_1$ is a fully invariant $R$–submodule of $M^{(n)}e_1$. Similarly, each $Ke_i$ is a fully invariant $R$–submodule of $M^{(n)}e_i$. Now by our assumption, $Ke_i = M^{(n)}e_i$ for all $i$, hence $K = M^{(n)}$.

($\Leftarrow$). If $N$ is a fully invariant essential $R$-submodule of $M$ then $N^{(n)}$ is fully invariant essential in $M_S^{(n)}$, hence $N = M$. 

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Theorem 2.8. The following are equivalent statements on a ring $R$.
(i) All finitely generated right $R$-modules are in $\mathcal{C}$.
(ii) For any $n \geq 1$, and any non-zero proper right ideals $I \subseteq J$ of $S = \text{Mat}_{n \times n}(R)$ the condition $\text{Idealizer}(I) \subseteq \text{Idealizer}(J)$ implies that $xS \cap J \subseteq I$ for some $x \in S \setminus I$.

Proof. Let $n \geq 1$ and $S = \text{Mat}_{n \times n}(R)$. Using the standard Morita equivalence of $\text{Mod-}R$ with $\text{Mod-}S$, we know that a right $R$-module generated by $n$ elements correspond to a cyclic right $S$-module, and conversely a cyclic right $S$-module corresponds to a finitely generated right $R$-module. Thus the result follows from Lemmas 2.6 and 2.7.

A concrete example of a non-semisimple ring whose finitely generated modules are semi-endosimple is provided by Cozzens’s example $R = K[x, D]$ where $K$ is a universal field with derivation $D$. It is well known that $R$ is a simple principal right (and left) ideal domain with a unique (up to isomorphism) simple injective $R$-module. Hence it follows either directly or by Theorem 2.5(iii), that any finitely generated $R$-module is in $\mathcal{C}(R)$.

3. Quasi-injective modules in $\mathcal{C}$

Proposition 3.1. The following are equivalent for a quasi-injective $R$-module $M$.
(i) $M \in \mathcal{C}$.
(ii) $M_R$ is generated by any essential submodule.
(iii) $M = \bigcap N$ where the intersection runs through the set of fully invariant essential $R$-submodules of $E(M_R)$.

Proof. (i)⇒(ii). Let $M_R$ be semi-endosimple and let $N$ be an essential submodule of $M$. Then $\text{Tr}_R(N, M)$ is a fully invariant essential submodule of $M_R$. Thus by our assumption, $\text{Tr}_R(N, M) = M$, so $M$ is generated by $N$.
(ii)⇒(iii). Let $E = E(M_R)$ and $L = \bigcap N$ where the intersection runs through the set of fully invariant essential $R$-submodules of $E_R$. Because $M$ is quasi-injective, $M$ is a fully invariant (and essential) submodule of $E_R$, hence $L \subseteq M$. Now let $N$ be any fully invariant essential submodule of $E_R$. Then it is easy to verify that $W := N \cap M$ is a fully invariant essential submodule of $M_R$. Thus by (ii), $\text{Tr}_R(W, M) = M$. On the other hand, the quasi-injectivity of $M$ implies that $\text{Tr}_R(W, M) = W$. Consequently $W = M$ that is $M \subseteq N$. Therefore $M \subseteq L$ and so $M = L$.
(iii)⇒(i). By the fact that being fully invariant and essential are transitive properties.
Theorem 3.2. The following statements are equivalent for an $R$-module $M$.

(i) $\hat{M}_R \in \mathcal{C}$.

(ii) For any essential submodule $N$ of $M$, there exists an index set $\Lambda$ such that $M \cong W/L$ for some $L \leq W \leq N^{(\Lambda)}_R$.

(iii) For any fully invariant essential submodule $N$ of $M$, there exists an index set $\Lambda$ such that $M \cong W/L$ for some $L \leq W \leq N^{(\Lambda)}_R$.

Proof. (i)$\Rightarrow$(ii). Let $N$ be an essential submodule of $M_R$. By Proposition 3.1, $\operatorname{Tr}_R(N, M_R) = M_R$. Fix $m \in M$, then $m \in f_1(N) + \cdots + f_t(N)$ for some $f_1, \ldots, f_t \in \text{Hom}_R(N, M)$. Let $W_m = \{(x_1, \ldots, x_t) \in N^{(t)} | \sum_{i=1}^t f_i(x_i) \in mR\}$. Clearly, $W_m$ is an $R$-submodule of $N^{(t)}$. Now if $\theta : W_m \to mR$ is defined by $\theta_m(n_1, \ldots, n_t) = \sum_{i=1}^t f_i(n_i)$, then $\theta_m$ is a surjective $R$-homomorphism. Consequently, there exists an $R$-epimorphism $\alpha = \sum_{m \in M} \theta_m$ from $W := \bigoplus_{m \in M} W_m$ to $M$, as desired.

(ii)$\Rightarrow$(iii). This is clear.

(iii)$\Rightarrow$(i). Let $N$ be a fully invariant essential submodule of $M_R$. Then $N \cap M$ is a fully invariant essential submodule of $M_R$. Thus by (iii), there exists a surjective $R$-homomorphism $\theta : W \to M$, where $W$ is an $R$-module of $N^{(\Lambda)}$ for some index set $\Lambda$. Now because $M_R$ is $N^{(\Lambda)}$-injective, $\theta$ can be extended to a map $\bar{\theta} \in \text{Hom}_R(N^{(\Lambda)}, M_R)$. Note that $\alpha(N) \subseteq N$ for any $\alpha \in \text{Hom}_R(N, \hat{M}_R)$. Thus if $x = \sum_{i=1}^t \iota_i(n_i) \in N^{(\Lambda)}$ where each $\iota_i : N \to N^{(\Lambda)}$ is the canonical injection, then $\bar{\theta}(x) = \sum_{i=1}^t \bar{\theta}\iota_i(n_i) \in N$. Hence $\bar{\theta}(N^{(\Lambda)}) \subseteq N$, yielding $M = \theta(W) = \bar{\theta}(W) \subseteq N$. It follows that $M_R = \text{End}_R(M_R)M \subseteq \text{End}_R(M_R)N = N$, proving that $M_R \in \mathcal{C}$.

We now give a number of applications of Theorem 3.2.

Corollary 3.3. The class $\mathcal{C}$ is closed under quasi-injective hull extensions.

Corollary 3.4. The following are equivalent for a quasi-projective $R$-module $M$.

(i) $\hat{M}_R \in \mathcal{C}$.

(ii) For any essential submodule $N$ of $M$, there exists an index set $\Lambda$ such that $M_R$ embeds in $N^{(\Lambda)}$.

(iii) For any fully invariant essential submodule $N$ of $M$, there exists an index set $\Lambda$ such that $M_R$ embeds in $N^{(\Lambda)}$.

Proof. Note that if $N \leq M_R$ and $V$ is a direct sum of copies of $N$, then $M$ is $W$-projective for any submodule $W$ of $V$. Hence, $M \cong W/L$ for some $L \leq W \leq N^{(\Lambda)}_R$ if and only if $M_R$ embeds in $N^{(\Lambda)}$. The result is now clear by Theorem 3.2.
Recall that an $R$-module $M$ is co-faithful if $R_R$ embeds in a (finite) direct sum of copies of $M_R$.

**Corollary 3.5.** The following are equivalent statements on a ring $R$.
(i) $R$ is a right strongly semiprime ring.
(ii) $E(R_R) \in \mathcal{C}$.
(iii) $E(F_R) \in \mathcal{C}$, for any free $R$-module $F$.
(iv) $	ext{Mod-}R$ contains a co-faithful semi-endosimple object.
(v) There exists $M \in \mathcal{C}$ such that $R$ embeds into $M_R$.

**Proof.** (i)$\Rightarrow$(ii). By Corollary 3.4.
(ii)$\Rightarrow$(iii). Let $E = E(R_R)$ and $F \simeq \bigoplus A R := \hat{L}$ be a free $R$-module. Then $\hat{L}$ is an essential submodule of $\bigoplus A E := W$ and we have $\hat{W} = \hat{L} \simeq \hat{F} = \text{Tr}_R(F, E(F_R)) = E(F_R)$. By (ii) and Proposition 2.4(i), $\hat{W}$ is in $\mathcal{C}$, hence $\hat{W} \in \mathcal{C}$ by Corollary 3.3. The proof is completed.
(iii)$\Rightarrow$(iv). This is clear.
(iv)$\Rightarrow$(v). By Proposition 2.4(i).
(v)$\Rightarrow$(i). Suppose that (v) holds and $I$ is a two sided ideal of $R$ which is essential as a right ideal, and let $N = \text{ann}_M(I)$. It is easily shown that $MI + N$ is a fully invariant essential submodule of $M$ (for essentiality, note that if $0 \neq K \leq M_R$, then either $KI = 0$ or $KI \neq 0$). Hence $MI + N = M$ by semi-endosimplicity of $M_R$. Because $R$ embeds into $M_R$, there exists $m \in M$ such that $r.\text{ann}_R(m) = 0$. Thus there are finitely many elements $m_j \in M$, $a_j \in I$, $j = 1, \ldots, k$, and an element $n \in N$ such that $\sum_{j=1}^k m_j a_j + n = m$. Now $(\bigcap_j r.\text{ann}_R(a_j)) \cap I \subseteq r.\text{ann}_R(m) = 0$, and by essentiality of $I$, we deduce that $\bigcap_j r.\text{ann}_R(a_j) = 0$, showing that $R$ is a right strongly semiprime ring.

**Corollary 3.6.** Over a prime right Goldie ring $R$, any injective torsion free $R$-module is in $\mathcal{C}$.

**Proof.** Let $R$ be a prime right Goldie ring and $M$ be an injective torsion free $R$-module. Then there exists a positive integer $n$ such that $M^{(n)}$ is an essential extension of a free $R$-module, see [3, Corollary 7.26]. The result is now obtained by Corollary 3.5 and Proposition 2.4(ii).

**Remark and Example 3.7.** In [1], it is shown that $R$ is a right strongly semiprime ring if and only if $R_R$ is strongly semiprime. Hence by Corollary 3.5, $E(R_R) \in \mathcal{C}$ if and only if $E(R_R)$ is strongly semiprime. However, in general an injective module in $\mathcal{C}$ need not be strongly semiprime. To see this, consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Q}$, $(p$ a prime) with $S = \text{End}_\mathbb{Z}(M)$. While this (injective) $\mathbb{Z}$-module is semi-endosimple by Proposition 2.4(iii) and Corollary 3.6, it is not strongly semiprime, since the lattice of $(S, \mathbb{Z})$-bisubmodules of $M$
has a proper essential element, namely $Z_p \oplus 0$.

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**References**


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