On the Structure of Some Groups of Degree $23k$ Containing $M_{23}$

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Abstract

In this paper, we will show the structure of some groups containing the Mathieu group $M_{23}$ and generated by the four 5-cycles $(3k, 17k, 10k, 7k, 9k)$ $(4k, 13k, 14k, 19k, 5k) (8k, 18k, 11k, 12k, 23k) (15k, 20k, 22k, 21k, 16k)$ and the one $23k$-cycle $(1, 2, \cdots, 23k)$ of degree $23k$ for all $k \geq 2$. The structure of the groups constructed is obtained in terms of wreath product. Some related cases are also studied.

Mathematics Subject Classification: 20B99

Keywords: Mathieu group, group generated by $n$-cycle, wreath product of groups

1. INTRODUCTION

The structure of all groups generated by an $n$-cycle and a 2-cycle or a 3-cycle or a product of two 2-cycles or a 4-cycle are determined by Al-Amri (see [1] and [2]). Al-Amri also determined the structure of groups generated by the $n$-cycle $(1, 2, \cdots, n)$ and a 5-cycle of the form $(a_1, a_2, a_3, a_4, a_5)$ for all $1 \leq a_i \leq n, a_i \neq a_j$ for all $1 \leq i, j \leq 5$ (see [3]).

Al-Amri in [4], has shown that the group generated by the four 5-cycles $(k, 16k, 7k, 18k, 13k) (2k, 14k, 3k, 15k, 6k) (4k, 19k, 20k, 21k, 11k) (5k, 17k, 12k, 10k, 8k)$ and the two $11k$-cycles $(1, 2, \cdots, 11k) (11k + 1, 11k + 2, \cdots, 22k)$ of degree $22k$ for all $2 \leq k$ is the wreath product of $M_{22}$ by $C_k$. He also gave generating sets for

$$M_{22wrS_k}, \ M_{11wrA_k} \text{ and } M_{22wr(S_mwrC_a)}$$

The Mathieu group $M_{23}$ of order 10200960 is one of the well known simple groups. In [5], $M_{23}$ is fully described. As a matter of fact, $M_{23}$ can be generated
by using two permutations, the first is of order 23 and the second is of order 5 as follows;

\[ M_{23} = \langle (1, 2, \cdots, 23), (3, 17, 10, 7, 9)(4, 13, 14, 19, 5) \]
\[ (8, 18, 11, 12, 23)(15, 20, 22, 21, 16) \rangle. \]

In this paper, we will show the structure of the group generated by two permutations, the first is of order \(23^k\) and the second of order 5. We will show that the group obtained is the wreath product of \(M_{23}\) by \(C_k\). Some related cases are also discussed.

2. PRELIMINARY RESULTS

**DEFINITION 2.1.** Let \(A\) and \(B\) be groups of permutations on non empty sets \(\Omega_1\) and \(\Omega_2\), respectively, where \(\Omega_1 \cap \Omega_2 = \phi\). The wreath product of \(A\) and \(B\) is denote by \(AwrB\) and defined as \(AwrB = A^{\Omega_2} \times \theta B\), i.e., the direct product of \(|\Omega_2|\) copies of \(A\) and a mapping \(\theta\), where \(\theta : B \to \text{Aut}(A^{\Omega_2})\) is defined by \(\theta_{y}(x) = x^{y}\), for all \(x \in A^{\Omega_2}\). It follows that

\[ |AwrB| = (|A|)^{|\Omega_2|}|B|. \]

**THEOREM 2.2** [1] Let \(G\) be the group generated by the \(n\)-cycle \((1, 2, \cdots, n)\) and the 2-cycle \((n, a)\). If \(1 < a < n\), is an integer with \(n = am\), then

\[ G \cong S_mwrC_a. \]

3. THE RESULTS

**THEOREM 3.1** Let \(G\) be a group generated by the one 23\(k\)-cycle \((1, 2, \cdots, 23k)\) and the four 5-cycles \((3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k)(15k, 20k, 22k, 21k, 16k)\). If \(k > 1\) is an integer, then

\[ G \cong M_{23}wrC_k \]

of order \((|M_{23}|)^k \times k\), where \(M_{23}\) is the Mathieu group of order 10200960.

**Proof:** Let \(\sigma = (1, 2, \cdots, 23k)\) and

\[ \tau = (3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k) \\
(15k, 20k, 22k, 21k, 16k). \]
Let
\[ \delta = \prod_{i=0}^{k-1} \tau^{\sigma^i}. \]

Since
\[ \delta = (k, 23k, 22k, 21k, 20k, 19k, 18k, 17k, 16k, 15k, 14k, 13k, 12k, 11k, 10k, 9k, 8k, 7k, 6k, 5k, 4k, 3k, 2k), \]
then
\[ G_i = \langle \delta^{\sigma^i}, \tau^{\sigma^i} \rangle \cong M_{23} \]
for all \( 1 \leq i \leq k \). Since each \( G_i \) acts on the set
\[ \Gamma_i = \{ k^{\sigma^i}, (2k)^{\sigma^i}, (3k)^{\sigma^i}, (4k)^{\sigma^i}, (5k)^{\sigma^i}, (6k)^{\sigma^i}, (7k)^{\sigma^i}, (8k)^{\sigma^i}, (9k)^{\sigma^i}, \\
(10k)^{\sigma^i}, (11k)^{\sigma^i}, (12k)^{\sigma^i}, (13k)^{\sigma^i}, (14k)^{\sigma^i}, (15k)^{\sigma^i}, (16k)^{\sigma^i}, (17k)^{\sigma^i}, \\
(18k)^{\sigma^i}, (19k)^{\sigma^i}, (20k)^{\sigma^i}, (21k)^{\sigma^i}, (22k)^{\sigma^i}, (23k)^{\sigma^i} \}, \]
for all \( i = 1, 2, \ldots, k \), respectively, and since
\[ \bigcap_{i=1}^{k} \Gamma_i = \emptyset, \]
then we get the direct product \( G_1 \times G_2 \times \cdots \times G_k \). Let
\[ \beta = \delta \sigma = (1, 2, \cdots, k)(k+1, k+2, \cdots, 3k) \cdots (22k+1, 22k+2, \cdots, 23k), \]
and
\[ H = \langle \beta \rangle \cong C_k. \]

Then \( H \) conjugates \( G_1 \) into \( G_2 \), \( G_2 \) into \( G_3 \), \ldots, and \( G_k \) into \( G_1 \). Hence we get the wreath product
\[ M_{23} \text{wr} C_k \subseteq G. \]

On the other hand, since
\[ \delta \beta(1, 2, \ldots, 23k) = \sigma, \]
then \( \sigma \in M_{23} \text{wr} C_k \). Hence
\[ G = \langle \sigma, \tau \rangle \cong M_{23} \text{wr} C_k. \]
**REMARK:** Since

\[ \delta = (k, 23k, 22k, 21k, \ldots, 4k, 3k, 2k), \]

\[ \beta = (1, 2, \ldots, k)(k + 1, k + 2, \ldots, 3k) \cdots (22k + 1, 22k + 2, \ldots, 23k), \]

\[ \tau = (3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k) \]
\[ (15k, 20k, 22k, 21k, 16k), \]

are in the group \( G \) as described above, and since \( \langle \delta, \tau \rangle \cong M_{23} \), we have

\[ M_{23} \text{wr} C_k \cong \langle \delta, \beta, \tau \rangle. \]

Hence \( M_{23} \text{wr} C_k \) can be finitely presented as

\[ M_{23} \text{wr} C_k \cong \langle X, Y, T | \langle X, Y \rangle \cong M_{23}, T^k = 1, (XT)^{11k} = (YT)^{5k} = 1 \rangle. \]

**THEOREM 3.2** Let \( G \) be the group generated by the one 23k-cycle \((1, 2, \ldots, 23k)\) and the four 5-cycles \((3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k) \) and the involution \((1, 2)(k + 1, k + 2)(2k + 1, 2k + 2) \cdots (21k + 1, 21k + 2)\). If \( k > 2 \) is an integer, then

\[ G \cong M_{23} \text{wr} S_k \]

of order \((|M_{23}|)^k \times k!\).

**Proof:** Let \( \sigma = (1, 2, \ldots, 23k), \)

\[ \tau = (3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k) \]
\[ (15k, 20k, 22k, 21k, 16k) \]

and

\[ \mu = (1, 2)(k + 1, k + 2)(2k + 1, 2k + 2) \cdots (21k + 1, 21k + 2). \]

As in the proof of the previous theorem,

\[ \langle \sigma, \tau \rangle \cong M_{23} \text{wr} C_k \]

Since

\[ C_k = \langle (1, 2, \ldots, k)(k + 1, k + 2, \ldots, 3k) \cdots (22k + 1, 22k + 2, \ldots, 23k) \rangle, \]
then
\[ S_k \cong \langle (1, 2, \ldots, k)(k + 1, k + 2, \ldots, 3k) \cdots (22k + 1, 22k + 2, \ldots, 23k), \mu \rangle. \]

Hence
\[ G = \langle \sigma, \tau, \mu \rangle \cong M_{23} \wr S_k. \]

As a consequence of the previous theorem we have the following result:

**Theorem 3.3** Let \( k = am \geq 4 \) be any integer. Let \( G \) be the group generated by the one \( 23k \)-cycle \((1, 2, \ldots, 23k)\) and the four 5-cycles \((3k, 17k, 10k, 7k, 9k)\) \((4k, 13k, 14k, 19k, 5k)\) \((8k, 18k, 11k, 12k, 23k)\) \((15k, 20k, 22k, 21k, 16k)\) and the product of 3-cycles \((k + 1, k + 2, k + 3)\) \((2k + 1, 2k + 2, 2k + 3)\) \((22k + 1, 22k + 2, 22k + 3)\). If \( k > 3 \) is an odd integer, then \( G \cong M_{23} \wr A_k \) of order \(|M_{23}|^k \times \frac{k!}{2} \).

**Proof:** The proof is similar to the proof of the previous result. \(\Box\)

**Theorem 3.4** Let \( k = am \geq 4 \) be any integer. Let \( G \) be the group generated by the one \( 23k \)-cycle \((1, 2, \ldots, 23k)\) and the four 5-cycles \((3k, 17k, 10k, 7k, 9k)\) \((4k, 13k, 14k, 19k, 5k)\) \((8k, 18k, 11k, 12k, 23k)\) \((15k, 20k, 22k, 21k, 16k)\) and the involution \((k, a)(2k, k + a)(3k, 2k + a)\cdots(23k, 22k + a)\). Then
\[ G \cong M_{23} \wr (S_m \wr C_a) \]
of order \(|M_{23}|^k \times (m!)^a \times a\).

**Proof:** Let \( \sigma = (1, 2, \ldots, 23k) \),
\[ \tau = (3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k)(15k, 20k, 22k, 21k, 16k) \]
and
\[ \mu = (k, a)(2k, k + a)(3k, 2k + a)\cdots(23k, 22k + a). \]

As in the proof of Theorem 3.2,
\[ \langle \sigma, \tau \rangle \cong M_{23} \wr C_k, \]
and since
\[ C_k = \langle (1, 2, \ldots, k)(k + 1, k + 2, \ldots, 3k) \cdots (22k + 1, 22k + 2, \ldots, 23k) \rangle, \]
then
\[ \langle (1, 2, \cdots, k)(k + 1, k + 2, \cdots, 3k) \cdots (22k + 1, 22k + 2, \cdots, 23k), \mu \rangle \cong (S_m \wr C_a). \]

Hence
\[ G = \langle \sigma, \tau, \mu \rangle \cong M_{23} \wr (S_m \wr C_a). \]

References


Received: October, 2008