On Central Extensions of Nulfiliform Leibniz Algebras

Rakhimov Isamiddin Sattarovich\textsuperscript{1}, Seyed Jalal Langari\textsuperscript{2} and Mouna Bibi Langari\textsuperscript{3}

\textsuperscript{1,2,3}Institute for Mathematical Research (INSPEM) & Department of Mathematics, FS, UPM, 43400, Serdang Selangor Darul Ehsan, Malaysia
\textsuperscript{1}isamiddin@science.upm.edu.my & risamiddin@mail.ru
\textsuperscript{2}jalal.langari@yahoo.com
\textsuperscript{3}moattar.amir@yahoo.com

Abstract

We establish a bijective correspondence between all central extensions of fixed Leibniz algebra $L$ by $V$ (with $k$–dimensional centers), and certain orbits in the set of all $k$–dimensional subspaces in the second cohomology group $H^2(L,V)$, under the canonical action of $\text{Aut}(L)$. As an application we describe the set of all nulfiliform Leibniz algebras.

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1 Introduction

In this article we develop a method \cite{[16]} of constructing Leibniz algebras of dimension $n$ given those algebras of dimension $< n$, and their automorphism groups. Roughly speaking, we establish a bijective correspondence between all central extensions of a fixed Leibniz algebra $L$ by $V$ with $k$–dimensional centers, and certain orbits in the set of all $k$–dimensional subspaces in the second cohomology group $H^2(L,V)$, under the canonical action of $\text{Aut}(L)$. For natural reasons we are working in the space $S$ of all bilinear forms on $L$ with values in $V$ satisfying the Leibniz identity, rather than in $H^2(L,V)$. If $\theta \in S$, the action of $\text{Aut}(L)$ on $\theta$ is simply given by

$$(\alpha, \theta) \longrightarrow \theta^\alpha, \text{ where } \theta^\alpha(x,y) = \theta(\alpha x, \alpha y); \ x, y \in L, \ \alpha \in \text{Aut}(L).$$
As an application of this procedure, we find all nilfiliform Leibniz algebras over algebraically closed field. Actually, such a method of classification of algebras is not new. This method was before used to classify nilpotent Lie algebras [16], [11] and [10]. In this paper we consider a generalization of Lie algebra called Leibniz algebra by J.-L.Loday [13], [15]. (For this reason, they have also been called “Loday algebras”). A skew-symmetric Leibniz algebra is a Lie algebra. The main motivation of J.-L.Loday to introduce this class of algebras was the search of an “obstruction” to the periodicity of algebraic $K$–theory. Beside this purely algebraic motivation some relationships with classical geometry, non-commutative geometry and physics have been recently discovered.

The (co)homology theory, representations and related problems of Leibniz algebras were studied by Loday, J.-L. and Pirashvili, T. [15], Frabetti, A. [9] and others. A good survey about all these and related problems is [14].

The problems related to the group theoretical realizations of Leibniz algebras are studied by Kinyon, M.K., Weinstein, A. [12] and others.

Deformation theory of Leibniz algebras and related physical applications of it are initiated by Fialowski, A., Mandal, A., Mukherjee, G. [8].

The papers [5], [2], [1], [4], [3] and the preprints [6], [7] concern the structural theory of Leibniz algebras.

The outline of the paper is as follows. Section 2 is a gentle reminder on relationships between Leibniz algebras extensions and automorphisms. This relations will be used in the classification method. Section 3 deals with the application of Skjelbred-Sund method to Leibniz algebras. At the end of the section we give a result for nilfiliform Leibniz algebras case.

2 Central extensions and automorphisms

**Definition 2.1.** A Leibniz algebra $L$ is a vector space over a field $K$ equipped with a bilinear map

$$[\cdot, \cdot] : L \times L \rightarrow L$$

satisfying the Leibniz identity

$$[[x, y], z] = [[x, y], z] - [[x, z], y], \text{ all } x, y, z \in L$$

Let $L$ be a Leibniz algebra over algebraically closed field $K$, $V$ be a vector space over $K$, and $\theta : L \times L \rightarrow V$ be a bilinear form satisfying the Leibniz identity

$$\theta([x, y, z]) = \theta([x, y], z) - \theta([x, z], y), \text{ all } x, y, z \in L. \quad (2.1)$$
Such forms are said to be \textbf{cocycle}. The set of all cocycles is denoted by $Z^2(L, V)$. Let $\theta \in Z^2(L, V)$, we construct a Leibniz algebra on $L \oplus V$, letting

$$\left[\begin{pmatrix} x \\ y \\ v \end{pmatrix}, \begin{pmatrix} y \\ w \end{pmatrix}\right] = \begin{pmatrix} [x, y]_L \\ \theta(x, y) \end{pmatrix}; \quad x, y \in L, \ v, w \in V.$$  

Denote this Leibniz algebra by $L_\theta$. The Leibniz algebra $L_\theta$ is called a \textbf{central extension} of $L$ by $V$.

Let $\vartheta : L \rightarrow V$ be a linear map, and define $\eta (x, y) = \vartheta ([x, y])$. Then,

$$\eta (x_1, [x_2, x_3]) - \eta ([x_1, x_2], x_3) + \eta ([x_1, x_3], x_2) =$$

$$\vartheta ([x_1, [x_2, x_3]]) - \vartheta ([[x_1, x_2], x_3]) + \vartheta ([[x_1, x_3], x_2]) =$$

$$\vartheta ([x_1, [x_2, x_3]] - [[x_1, x_2], x_3] + [[x_1, x_3], x_2]) = \vartheta (0) = 0.$$

This means that $\eta$ is a cocycle, called a \textbf{coboundary}. The set of all coboundaries is denoted by $B^2(L, V)$. Clearly, $B^2(L, V)$ is a subgroup of $Z^2(L, V)$. We call the factor space, denoted by $H^2(L, V) = Z^2(L, V) / B^2(L, V)$, the \textbf{second cohomology} group of $L$ by $V$.

Let now $\tilde{L}$ be a Leibniz algebra with $k$-dimensional center $C(\tilde{L})$, $\nu : \tilde{L} \rightarrow V$ be linear and such that $\nu (C(\tilde{L})) = V$. We put $L = \tilde{L} / C(\tilde{L})$ and get an isomorphism $\tilde{L} \cong L \oplus V$, where $\tilde{x} \leftrightarrow y + u$, $\nu (\tilde{x}) = u$, and $y = \tilde{x} + C(\tilde{L}) \in \tilde{L} / C(\tilde{L}) = L$.

We put $\theta = \nu \circ [\cdot , \cdot ]$, that is

$$\theta (x, y) = \nu [x', y']; \quad \text{where } x' + C(\tilde{L}) = x, \ y' + C(\tilde{L}) = y.$$

This shows $\tilde{L}$ and $L_\theta$ are isomorphic. Hence each Leibniz algebra with center of dimension $k$ is on the form $L_\theta$, where $\theta -$ is a cocycle.

Let $\theta : L \times L \rightarrow V$ be a cocycle. Then the center $C(L_\theta)$ of $\tilde{L} = L_\theta$ is equal to

$$C(L_\theta) = (\theta^\perp \cap C(L)) \oplus V,$$

where $\theta^\perp = \{ x \in L ; \theta (x, L) = \theta (L, x) = 0 \}$, which is called the radical of $\theta$ (Rad($\theta$)$= \theta^\perp$), and $C(L) = \{ x \in L ; [x, L] = [L, x] = 0 \}$.

We want to restrict to $\theta$ such that $\theta^\perp \cap C(L) = 0$. This way we avoid constructing the same Leibniz algebra as central extension of different Leibniz algebra.

Given two such cocycles $\theta_1, \theta_2 : L \times L \rightarrow V$, and assume the extended algebras $L_{\theta_1}$ and $L_{\theta_2}$ are isomorphic and that their centers $C(L_{\theta_1})$ and $C(L_{\theta_2})$
both are equal to $V$. Let $\alpha : L_{\theta_1} \longrightarrow L_{\theta_2}$ be an isomorphism. Dividing with the common center $V$ we obtain an automorphism $\alpha_0 : L \longrightarrow L$. Let us fix a basis $\{e_1, e_2, \ldots, e_n\}$ for $L$, and supplement it with a basis for $V$ to get a basis $E$ for $L \oplus V$. We may represent $\alpha$ as a matrix relative to $E$:

$$\alpha = \begin{pmatrix} \alpha_0 & 0 \\ \varphi & \psi \end{pmatrix}, \quad (2.2)$$

where $\alpha_0 \in \text{Aut}(L)$, $\psi = \alpha \mid_{C(L)} \in GL(k)$, and $\varphi \in \text{Hom}(L, V)$.

Now $\alpha$ preserves the brackets, and writing $[,]_i$ for the products in $L_{\theta_i}$, $i = 1, 2$, and $[,]$ for the product in $L$, we have

$$\theta_2(\alpha_0 x, \alpha_0 y) = \varphi[x, y] + \psi \theta_1(x, y), \quad \text{all } x, y \in L,$$

In case $\theta_1 = \theta_2 = \theta$, we get the following description of the automorphism group $\text{Aut}(L_{\theta})$.

**Proposition 2.1.** Let $\theta$ be a cocycle on the Leibniz algebra $L$ with values in $V$, and assume $\theta^1 \cap C(L) = (0)$. Then the automorphism group $\text{Aut}(L_{\theta})$ of the extension algebra $L_{\theta}$ consists of all linear operators of the matrix form

$$\alpha = \begin{pmatrix} \alpha_0 & 0 \\ \varphi & \psi \end{pmatrix}$$

as in (2), where

$$\theta(\alpha_0 x, \alpha_0 y) = \varphi[x, y] + \psi \theta(x, y), \quad \text{all } x, y \in L.$$
Example 2.1. The Leibniz algebra $L$ with non-zero bracket $[e_1, e_1] = e_2$ between the basis elements $e_1, e_2$ is a central extension of the Abelian algebra $K e_1 \times K e_1$ by $K e_2$, given by the bilinear form

$$\Delta_{11} : (x_1 e_1 + x_2 e_2, y_1 e_1 + y_2 e_2) \longrightarrow x_1 y_1$$

Example 2.2. The Leibniz algebra with non-zero bracket $[e_1, e_1] = e_2, [e_2, e_1] = e_3$ between the basis elements $e_1, e_2, e_3$ is a central extension of $L$ from the previous example:

$\text{Aut}(L)$ consists of all operators

$$\Phi = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{11}^3 \end{bmatrix}, \text{ where } a_{11} \neq 0$$

$H^2(L, F)$ consists of $\theta = a \Delta_{21}$. Moreover, $\theta^\perp \cap C(L) = 0$ if and only if $a \neq 0$. The automorphism group acts as follows $a \longrightarrow a^3$. (Here we write $a \longrightarrow a'$ to indicate that the coefficient of $\Delta_{21}$ in $\Phi \theta$ is $a'$.) Then we choose $a_{11} = \frac{1}{\sqrt{a}}$ such that $a$ is mapped to 1. This yields

$$[e_1, e_1] = e_2, [e_2, e_1] = e_3$$

Remark 1. There is no one-dimensional Leibniz algebra except for abelian.

Remark 2. Up to isomorphism there is only one nulfiliform Leibniz algebra in each dimension [4].

3 On Central extensions of Leibniz algebras

We continue our study of central extensions $L_\theta$ of a Leibniz algebra $L$ given by cocycles $\theta : L \times L \longrightarrow V$, and proceed to exclude forms $\theta$ such that $L_\theta \cong K \times L_{\theta'}$, where $\theta' : L \times L \longrightarrow V'$, and dim$V' = k - 1$. Let $J$ be the set of all linear maps $F : V \longrightarrow V$ such that there exists a linear map $\varphi : L \longrightarrow V$ with the property

$$F(\theta(x, y)) = \varphi[x, y], \text{ all } x, y \in L. \quad (3.1)$$

Clearly, $J$ is a left ideal of $F \circ (1 - \pi)$, i.e. $J = \text{Hom}(V, V) \circ J$, and therefore $J$ is generated by a projection $\pi : J = \text{Hom}(V, V) \circ \pi$. We have

$$\pi(\theta(x, y)) = \varphi_\pi[x, y], \varphi_\pi \in \text{Hom}(L, V).$$

But

$$\theta'(x, y) = \theta(x, y) - \varphi_\pi[x, y] = (1 - \pi)\theta(x, y), \quad (3.2)$$
in particular \( \theta' \) is cohomologous to \( \theta \).

Let \( e_1, \ldots, e_s \) be a basis of \( V \), and \( \theta \in Z^2(L, V) \). Then \( \theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y)e_i \), where \( \theta_i \in Z^2(L, F) \). Furthermore, \( \theta \) is a coboundary if and only if all \( \theta_i \) are.

The automorphism group \( \text{Aut}(L) \) acts on \( Z^2(L, V) \) by \( \phi\theta(x, y) = \theta(\phi(x), \phi(y)) \). Also, \( \eta \in B^2(L, V) \) if and only if \( \phi\eta \in B^2(L, V) \) so \( \text{Aut}(L) \) acts on \( H^2(L, V) \).

The following lemmas and theorem 3.1 are Leibniz analogies of the corresponding statements for Lie algebras case (see [11], [16]).

**Lemma 3.1.** Let \( \theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y)e_i \) and \( \eta(x, y) = \sum_{i=1}^{s} \eta_i(x, y)e_i \) be two elements of \( H^2(L, V) \). Suppose that \( \theta^\perp \cap C(L) = \eta^\perp \cap C(L) = 0 \). Then \( L_\theta \cong L_\eta \) if and only if there is a \( \varphi \in \text{Aut}(L) \) such that the \( \varphi\eta_i \) span the same subspace of \( H^2(L, V) \) as the \( \theta_i \).

**Proof.** As vector spaces \( L_\theta = L \oplus V \) and \( L_\eta = L \oplus V \). Let \( \sigma : L_\theta \rightarrow L_\eta \) be an isomorphism. Since \( V \) is the center of both Leibniz algebras, we have \( \sigma(V) = V \) so \( \sigma \) induces an isomorphism of \( L_\theta/V = L \) to \( L_\eta/V = L \), that is an automorphism of \( L \). Denote this automorphism by \( \varphi \). Let \( L \) be spanned by \( x_1, x_2, \ldots, x_n \). Then we write \( \sigma(x_i) = \varphi(x_i) + v_i \), where \( v_i \) and \( \sigma(e_i) = \sum a_{ji}e_j \). Also write \( [x_i, x_j]_L = \sum c_{ij}^kx_k \) and \( v_i = \sum \beta_{il}e_l \). Then the relation

\[
\sigma([x_i, x_j]_{L_\theta}) = [\sigma(x_i), \sigma(x_j)]_{L_\eta}
\]

amounts to

\[
\eta_l(\varphi(x_i), \varphi(x_j)) = \sum_{k=1}^{s} a_{lk}\theta_k(x_i, x_j) + \sum_{k=1}^{s} c_{ij}^k\beta_{kl}, \quad \text{for } 1 \leq l \leq s \quad (3.3)
\]

and

\[
[x_i, x_j]_{\theta} = [x_i, x_j]_L + \theta(x_i, x_j) = \sum_{k=1}^{n} c_{ij}^kx_k + \theta(x_i, x_j)
\]

\[
\sigma([x_i, x_j]_{\theta}) = \sum_{k=1}^{n} c_{ij}^k\sigma(x_k) + \sum_{k=1}^{s} \theta_k(x_i, x_j)\sigma(e_k) =
\]

\[
= \sum_{k=1}^{n} c_{ij}^k\varphi(x_k) + \sum_{k=1}^{n} c_{ij}^k\nu_k + \sum_{l=1}^{s} \left( \sum_{k=1}^{s} a_{lk}\theta_k(x_i, x_j) \right) e_l
\]

\[
= \varphi \left( \sum_{k=1}^{n} c_{ij}^kx_k \right) + \sum_{l=1}^{s} \left( \sum_{k=1}^{n} \beta_{kl}c_{ij}^k \right) e_l + \sum_{l=1}^{s} \left( \sum_{k=1}^{s} a_{lk}\theta_k(x_i, x_j) \right) e_l
\]
Now define the linear function \( \sigma \) holds. This means that, if we define \( \phi \eta \) then

\[
\sigma (x_i) \eta = [\varphi (x_i) + v_i, \varphi (x_j) + v_j] \eta 
\]

\[
= [\varphi (x_i), \varphi (x_j)]_L + \eta (\varphi (x_i), \varphi (x_j))
\]

\[
= \varphi ([x_i, x_j]_L) + \sum_{l=1}^s (\eta (\varphi (x_i), \varphi (x_j))) e_l.
\]

If \( \sigma ([x_i, x_j]_\eta) = [\sigma (x_i), \sigma (x_j)]_\eta \) then

\[
\eta (\varphi (x_i), \varphi (x_j)) = \sum_{k=1}^n a_{lk} \theta_k (x_i, x_j) + \sum_{k=1}^n c_{ij}^k \beta_{kl}, \text{ for } 1 \leq l \leq s.
\]

Now define the linear function \( f_i : L \rightarrow F \) by \( f_i(x_k) = \beta_{kl} \), then \( f_i([x_i, x_j]) = \sum_{k=1}^n c_{ij}^k \beta_{kl} \) because \([x_i, x_j]_L = \sum_{k=1}^n c_{ij}^k x_k \) and \( f_i \) is linear function thus

\[
f_i([x_i, x_j]) = f_i(\sum_{k=1}^n c_{ij}^k x_k) = \sum_{k=1}^n c_{ij}^k f_i(x_k) = \sum_{k=1}^n c_{ij}^k \beta_{kl}.
\]

We see that modulo \( B^2 (L, F) \), the \( \varphi \eta_i \) and \( \theta_i \) span the same space. Suppose that the \( \varphi \eta_i \) and \( \theta_i \) span the same space in \( Z^2 (L, K) \), modulo \( B^2 (L, F) \). Then there are linear functions \( f_i : L \rightarrow F \) and \( a_{lk} \in F \) so that \( \varphi \eta_i (x_i, x_j) = \sum_{k=1}^n a_{lk} \theta_k (x_i, x_j) + f_i([x_i, x_j]) \). If we set \( \beta_{kl} = f_i(x_k) \), then we see that (3.3) holds. This means that, if we define \( \sigma : L_\eta \rightarrow L_\eta \) by

\[
\sigma (x_i) = \varphi (x_i) + \sum_{l=1}^s \beta_i e_l, \sigma (e_i) = \sum_{j=1}^s a_{ji} e_j
\]

then \( \sigma \) is an isomorphism. \( \square \)

Let \( L = I_1 \oplus I_2 \) be the direct sum of two ideals. Suppose that \( I_2 \) is contained the centre of \( L \). Then \( I_2 \) is called a central component of \( L \). To avoid the Leibniz algebras with central components we use the following criterion.

**Lemma 3.2.** Let \( \theta (x, y) = \sum_{i=1}^n \theta_i (x, y) e_i \in H^2 (L, V) \) be such that \( \theta^+ \cap C(L) = 0 \). Then \( L_\theta \) has no central components if and only if \( e_1, \ldots, e_s \) are linearly independent in \( H^2 (L, K) \).
All our previous observations can be summarized as follows:

**Theorem 3.1.** Let $L$ be a Leibniz algebra over a field $K$. The isomorphism classes of Leibniz algebras $\widetilde{L}$ with $k$-dimensional center $C(\widetilde{L})$, $\widetilde{L}/C(\widetilde{L}) \cong L$, and without Abelian direct factors, are in bijective correspondence with those $Aut(L)$-orbits in the set of all $k$-dimensional subspace of $S/S'$ which have no kernel in $C(L)$.

**Definition 3.1.** An $n$-dimensional Leibniz algebra $L$ is said to be nulfiliform if $dim L^1 = n - i + 1$, where $2 \leq i \leq n + 1$.

Now we want to apply the theorem to find the central extensions without Abelian factors of $n$-dimensional nulfiliform Leibniz algebra

$$L_n : [e_i, e_1] = e_{i+1}, \quad i = 1, 2, ..., n - 1$$

over any algebraically closed field.

One can compute without difficulty the automorphism group of $L_n$ as follows:

$$\begin{bmatrix}
    a_{11} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
    a_{21} & a_{11}^2 & 0 & 0 & 0 & \ldots & 0 & 0 \\
    a_{31} & a_{41}a_{11} & a_{11}^3 & 0 & 0 & \ldots & 0 & 0 \\
    a_{41} & a_{31}a_{11} & a_{21}a_{11}^3 & a_{11}^4 & 0 & \ldots & 0 & 0 \\
    a_{51} & a_{41}a_{11} & a_{31}a_{11}^3 & a_{21}a_{11}^5 & a_{11}^5 & \ldots & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{(n-1)1} & a_{(n-2)1}a_{11} & a_{(n-3)1}a_{11}^2 & a_{(n-4)1}a_{11}^3 & a_{(n-5)1}a_{11}^4 & \ldots & a_{11}^{n-1} & 0 \\
    a_{n1} & a_{(n-1)1}a_{11} & a_{(n-2)1}a_{11}^2 & a_{(n-3)1}a_{11}^3 & a_{(n-4)1}a_{11}^4 & \ldots & a_{21}a_{11}^{n-2} & a_{11}^n \\
\end{bmatrix}_{n \times n}$$

Easy computations show that $c_{11} = c_{21} = \ldots = c_{(n-1)1} = 0$. Then

$$\theta \in Z^2(L, V) \iff \theta(e_i, [e_j, e_k]) = \theta(e_i, e_j, e_k) = \theta([e_i, e_k], e_j), \quad i, j, k = 1, n - 1$$

yields $c_{n(i+1)} = 0, \quad i = 1, 2, ..., n - 1$ and $c_{ik} = 0; \quad i = 1, 2, ..., n - 1$ for $k = 2, 3, ..., n$. This means that, $H^2(L_n, K)$ consists of $\theta = b\Delta_n$. Moreover, $\theta^⊥ \cap C(L_n) = 0$ if and only if $b \neq 0$. The automorphism group $Aut(L_n)$ acts on $H^2(L_n, K)$ as $b \rightarrow ba_{11}^k$. Then we choose $a_{11} = \frac{1}{\sqrt{b}}$ then $b$ is mapped to 1. This yields

$$L_{n+1} : [e_i, e_1] = e_{i+1}, \quad i = 1, 2, ..., n.$$  

The result for nulfiliform Leibniz algebras is then spelled out in the following theorem:
Theorem 3.2. A nilfiliform Leibniz algebra in dimension \((n+1)\) is the central extension of a nilfiliform Leibniz algebra in dimension \(n\).

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References


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