Small Pseudo Projective Modules

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Abstract
In this paper the concept of small quasi projectivity has been generalized to small pseudo projectivity and some results on small pseudo projective modules and small pseudo stable submodules have been obtained. Here the basic ring R is supposed to be ring with unity and all modules are supposed to be unitary left R-modules.

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1 Definitions
Let M be an R-module, a submodule K of M is said to be small in M if K+L=M ⇒ L=M for any submodule L⊆M. An R-module M is said to be hollow if all proper submodules of M are small in M. An R-module M is S.F. if zero is the only small submodule in M. An R-module M is called quasi projective if for any given module A, any homomorphism f: M→A and epimorphism g: M→A, there exists an h in End (M) such that f=g o h. An R-module M is said to be small quasi projective, if for any small epimorphism g: M→N, any homomorphism f: M→N can be lifted to an endomorphism h of M such that f=g o h. An R-module M is called
pseudo projective if for any given module \( A \) and epimorphisms \( f: M \rightarrow A \) and \( g: M \rightarrow A \), there exists an \( h \) in \( \text{End}(M) \) such that \( f = g \circ h \). An \( R \)-module \( M \) is said to be small pseudo projective if for any module \( A \), with small epimorphism \( g: M \rightarrow A \) and epimorphism \( f: M \rightarrow A \) there exists an \( h \in \text{End}(M) \) such that the following diagram is commutative

\[
\begin{array}{ccc}
M & \xrightarrow{f} & A \\
\downarrow{g} & & \downarrow{\text{g} \circ h} \\
M & \xrightarrow{h} & A
\end{array}
\]

i.e. \( f = g \circ h \).

A submodule \( N \) of an \( R \)-module \( M \) is said to be small pseudo stable if for any epimorphism \( f:M \rightarrow A \) and any small epimorphism \( g:M \rightarrow A \) with \( N \subseteq \ker g \cap \ker f \)

\( \exists h \in \text{End}(M) \) such that \( f = g \circ h \), then \( h(N) \subseteq N \).

2 Main Results

**Proposition 1:** For a hollow module \( M \) the following conditions are equivalent:

(i) \( M \) is small pseudo projective.

(ii) \( M \) is pseudo projective.

**Proof:** (i) \( \Rightarrow \) (ii) Let \( M \) be a small pseudo projective module, \( A \) be any module and \( f, g : M \rightarrow A \) be epimorphisms. \( M \) is hollow \( \Rightarrow \) \( g \) is small epimorphism. So, by small pseudo projectivity of \( M \), \( \exists h \in \text{End}(M) \) such that \( f = g \circ h \Rightarrow M \) is pseudo projective.

(ii) \( \Rightarrow \) (i) follows from the definition.

**Proposition 2:** Every S.F. module is small pseudo projective.

**Proof:** Let \( M \) be an S.F. module. Let \( f : M \rightarrow B \) be any epimorphism and \( g : M \rightarrow B \) any small epimorphism. Since \( M \) is S.F., \( g \) is one-one and hence an isomorphism. So, \( g^{-1} \) exists and \( g^{-1} \circ f \) makes the following diagram commutative.

\[
\begin{array}{ccc}
M & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{g^{-1} \circ f} \\
M & \xrightarrow{g^{-1}} & B
\end{array}
\]

Hence \( M \) is small pseudo projective.
**Remark**: A sufficient condition for a small pseudo projective module to be a S.F. module has been provided in the following.

**Proposition 3**: Let \( M \) be a small pseudo projective module then \( M \) is S.F. if \( M/N \), where \( N \) is any small submodule of \( M \), is isomorphic to a direct summand of \( M \).

**Proof**: - Let \( Q \) be a direct summand of \( M \) and \( N \) be any small submodule of \( M \) such that \( M/N \cong Q \). Let \( \phi : Q \to M/N \) be the isomorphism, \( v : M \to M/N \) be the natural map, \( \pi_Q : M \to Q \) be the projection map and \( J_Q : Q \to M \) be the injection map. \( M \) being small pseudo projective implies that there exists an \( h \) in \( \text{End}(M) \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
M & \xrightarrow{\pi_Q} & Q \\
\downarrow{h} & & \downarrow{\beta} \\
M/N & \xrightarrow{v} & M
\end{array}
\]

i.e. \( \phi \circ \pi_Q = v \circ h \).

Define \( \beta : Q \to M \) by \( \beta = h \circ J_Q \).

and \( v' : M/N \to M \) by

\[v'(x+N) = \beta o \phi^{-1}(x+N)\]

Now, \( v = v' \circ \phi^{-1} = \phi \circ J_Q \circ \phi^{-1} \)

\[= \phi \circ \pi_Q \circ J_Q \circ \phi^{-1} = \phi \circ I_Q \circ \phi^{-1} = \phi \circ \phi^{-1} = I_{M/N} \]

Thus the sequence

\[O \to N \to M \to M/N \to O\]

splits and therefore \( N \) is a direct summand of \( M \). So, \( \exists N' \subseteq M \) such that \( M = N \oplus N' \). Now \( M = N + N' \Rightarrow N' = M \) and \( N \cap N' = 0 \Rightarrow N \cap M = 0 \Rightarrow N = 0 \). Hence \( M \) is an S.F. module.

**Proposition 4**: Let \( M \) be a small pseudo projective module and \( \phi : M \to N \) be a small epimorphism then there exists an epi-endomorphism \( h \) in \( \text{End}(M) \) such that \( \text{Ker} \phi = \text{Ker}(\phi \circ h) \) is stable under \( h \).

**Proof**: The small epimorphism \( \phi : M \to N \) induces an isomorphism.

\[\phi^* : M/\text{Ker} \phi \to N\]

Let \( f : M \to M/\text{Ker} \phi \) be the natural map.

By small pseudo projectivity of \( M \) \( \exists h \in \text{End}(M) \) such that \( \phi^* \circ f = \phi \circ h \).

Since \( \phi(M) = N = \phi^* (M) = \phi \circ h(M) \);

for any \( m \in M \), \( \phi(m) = \phi \circ h(m) \).
\[ \Rightarrow m - h (m) \in \text{Ker} \phi \Rightarrow m \in \text{Im} h + \text{Ker} \phi \]
\[ \Rightarrow M \subseteq \text{Im} h + \text{Ker} \phi \Rightarrow M = \text{Im} h + \text{Ker} \phi \Rightarrow M = \text{Im} h, \]
\[ \Rightarrow \text{h is onto}. \]

Now, \( x \in \text{Ker} \phi \Rightarrow x \in \text{Ker} f \Rightarrow f(x) = 0 \)
\[ \Rightarrow \phi * \text{o} f (x) = 0 \Rightarrow \phi \circ h (x) = 0 \]
\[ \Rightarrow x \in \text{Ker} \phi \circ h \Rightarrow \text{Ker} \phi \subseteq \text{Ker} \phi \circ h \]

Let \( y \in \text{Ker} \phi \circ h \)
\[ \Rightarrow \phi \circ h (y) = 0 \Rightarrow \phi \circ f (y) = 0 \]
\[ \Rightarrow f (y) \in \text{Ker} \phi \Rightarrow f (y) = o \text{as } \phi \text{ is one - one} \]
\[ \Rightarrow y \in \text{Ker} f = \text{Ker} \phi \Rightarrow \text{Ker} \phi \circ h \subseteq \text{Ker} \phi \]

and therefore \( \text{Ker} \phi = \text{Ker} \phi \circ h \)

Now let, \( t \in \text{Ker} \phi \) then \( \phi (t) = 0 = \phi \circ h (t) \Rightarrow \phi \{t - h (t)\} = 0 \)
\[ \Rightarrow t - h (t) \in \text{Ker} \phi \Rightarrow h (t) \in \text{Ker} \phi \Rightarrow h (\text{Ker} \phi) \subseteq \text{Ker} \phi. \]

Hence \( \text{Ker} \phi \) is invariant under epi-endomorphism of \( M \).

**Proposition 5:** Let \( M \) be a small pseudo projective module and \( K \subseteq M \) be a small submodule of \( M \) then \( M / K \) is small pseudo projective if \( K \) is stable under endomorphisms of \( M \).

**Proof:** Let \( v : M \to M / K \) be the natural map, \( f : M / K \to A \) be an epimorphism and \( g : M / K \to A \) be a small epimorphism, where \( A \) is any \( R \)-module. Then by small pseudo projectivity of \( M \) there exist \( \phi \in \text{End} (M) \) such that \( f \circ v = g \circ v \circ o \phi \).

Define
\[ v : M / K \to M / K \]
\[ \text{as } v (x + K) = \phi (x) + K \]

Then \( v \) is well defined and
\[ v \circ o v = v \circ o \phi \Rightarrow g \circ v \circ o v = g \circ o v \circ o \phi \]
\[ \Rightarrow g \circ o v \circ o v = f \circ o v \Rightarrow g \circ v = f, \text{ since } v \text{ is onto} \]
\[ \Rightarrow M / K \text{ is small pseudo projective.} \]

**Proposition 6:** Let \( M \) be a small pseudo projective module and \( g : M \to N \) be a small epimorphism then \( N \) is small pseudo projective.

**Proof:** Follows from Proposition 4 and 5.

**Proposition 7:** Let \( (M_i)_{i \in I} \) be a family of small pseudo stable submodules of an \( R \)-module \( M \) then \( \sum_{i \in I} M_i \) is also small pseudo stable.

**Proof:** Follows from Proposition 3.2 in [2].

**Lemma 8:** Let \( M \) be a small pseudo projective module and \( N \) be a small submodule of \( M \) stable under epi-endomorphisms of \( M \) then \( N \) is small pseudo stable.
**Proposition 9**: Let $M$ be a small pseudo projective module and $T$ be a small pseudo stable submodule of $M$ then $M/T$ is small pseudo projective.

**Proof**: Let $g : M/T \rightarrow A$ be a small epimorphism, $f : M/T \rightarrow A$ be any epimorphism and $v : M \rightarrow M/T$ be the natural map, where $A$ is an $R$-module. Then $v$ is a small epimorphism. As $T \subseteq Kerv \cap Ker v$ by small pseudo projectivity of $M \exists h \in \text{End}(M)$ such that $f \circ v = g \circ v \circ h$. Since $T$ is small pseudo stable, $h(T) \subseteq T$. Hence by Proposition (5) $M/T$ is small pseudo projective.

**Proposition 10**: If $T$ is a small pseudo stable submodule of a small quasi projective module $Q$ and $A$ is a submodule of $T$, then $T/A$ is a small pseudo stable submodule of $Q/A$.

**Proof**: Follows from Lemma 1.6 in [1] with minor changes.

**Lemma 11**: Let $Q$ be a small quasi projective module and $T$ be a small pseudo stable submodule of $Q$. If $C$ containing $T$, is not a small pseudo stable submodule of $Q$ then $C/T$ is not a small pseudo stable submodule of $Q/T$.

**Proof**: Follows from Lemma 1.7 in [1] with suitable changes.

**Proposition 12**: If $M$ is a small pseudo projective module and $\phi : M \rightarrow N$ is any small epimorphism then $\text{Ker} \phi = K$ is a small pseudo stable submodule of $M$.

**Proof**: $\phi : M \rightarrow N$ induces an isomorphism $\phi^* : M/K \rightarrow N$. Let $f : M \rightarrow M/K$ be the natural map, then $K \subseteq Kerf \cap Kerf^*$. As $M$ is small pseudo projective $\exists h \in \text{End}(M)$ such that $\phi^* f = \phi o h$. Now, let $h(K) \not\subseteq K$, then there exists $k \in K$ such that $h(k) \in h(K)$ and $h(k) \not\in K$.

Then $0 \neq \phi o h(k) = \phi^* f(k) = 0$, which is a contradiction, since $K \subseteq Ker \phi^*$. Thus $h(K) \subseteq K \Rightarrow K$ is small pseudo stable.

**Proposition 13**: Let $M$ be a small pseudo projective module, $A$ and $B$ be invariant submodules of $M$. Then $A \cap B$ is a small pseudo stable submodule of $M$ if either $A$ or $B$ is small in $M$.

**Proof**: Let $A$ be small in $M$. Let $g : M/A \rightarrow T$ be any small epimorphism, $f : M/A \rightarrow T$ be any epimorphism, where $T$ is any $R$-module and $v : M \rightarrow M/A$ be the natural map. Then $A \cap B \subseteq Ker g o v \cap Ker f o v$. By small pseudo projectivity of $M \exists h \in \text{End}(M)$ such that the following diagram is commutative.
i.e. \( f \circ v = g \circ v \circ h \).

To show, \( h(A \cap B) \subseteq A \cap B \).

We have, \( h(A \cap B) \subseteq h(A) \) and \( h(A \cap B) \subseteq h(B) \)

\[ \Rightarrow h(A \cap B) \subseteq A \cap h(A \cap B) \subseteq B \]

as \( A \) and \( B \) are invariant submodules.

Thus \( h(A \cap B) \subseteq A \cap B \). Hence it follows that \( A \cap B \) is small pseudo stable.

References


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