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New Method to Compute the Determinant of a 3x3 Matrix

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Abstract

In this paper we will present a new method to compute the determinants of a 3x3 matrix. The advantages of this method comparing to other known methods are:

- quick computation, so it creates an easy scheme to compute the determinants of a 3x3 matrix,
- very quick appropriation, so we describe only four elements of the determinants, where it is created a new easy scheme to compute.

This new method creates opportunities to find other new methods to compute determinants of higher orders that will be our paper in the future.

Mathematics Subject Classification: 15A15, 11C20, 65F40

Keywords: methods to compute the determinant of a 3x3 matrix

1 Introduction

Let A be an nxn matrix. $A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ a_{1n} & a_{2n} & ... & a_{nn} \end{bmatrix}$

Definition 1. - Determinant of n order will be called the sum, which has $n!$ different terms $\varepsilon_{j_1,j_2,\dots j_n} a_{1j_1} a_{2j_2} \dots a_{nj_n}$ which will be formed of the matrix A elements, see [4],[7],[8],[9].

$$D = \det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{vmatrix} = \sum_{S_n} \varepsilon_{j_1,j_2,\dots j_n} a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

Where $\varepsilon_{j_1,j_2,\dots j_n} = \begin{cases} +1 \text{ , if } j_1, j_2, \dots j_n \text{ is an even permutation} \\ -1 \text{ , if } j_1, j_2, \dots j_n \text{ is an odd permutation} \end{cases}$

2 Methods to compute the determinants of third order

In base of definition 1, determinant of the third order (for n=3) can be computed in this way, see [4], [7], [8], [9]:

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{123} a_{11} a_{22} a_{33} + \varepsilon_{132} a_{11} a_{23} a_{32} + \varepsilon_{312} a_{13} a_{21} a_{32} +$$

$$+ \varepsilon_{321} a_{13} a_{22} a_{31} + \varepsilon_{231} a_{12} a_{23} a_{31} + \varepsilon_{213} a_{12} a_{21} a_{33} =$$

$$= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} - a_{12} a_{21} a_{33}$$

While the determinant of the third order expansion by the elements of whatever row we have, see [4], [5], [6], [8]:

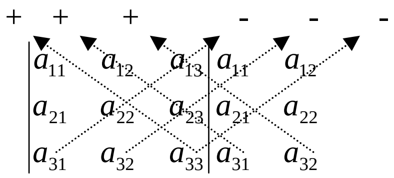
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$= -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} =$$

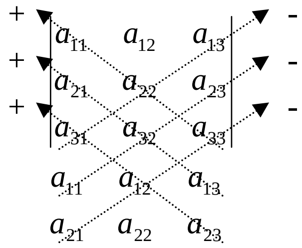
$$= a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

2.1 Sarrus's rule

Using Sarrus's rule, we have those schemes, see [4], [8], [10], [11]:



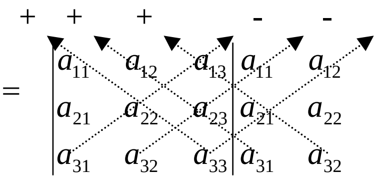
Scheme 1.



Scheme 2.

From the description of two first columns of the determinants (first and second columns) will be formed Scheme 1. respectively two rows (first and second rows) will be formed Scheme 2.The terms, which will be formed by the products of diagonal elements in the left side in both of scheme 1 and 2 become the "-"sign. In this way we get the Sarrus' rule, which are valuable just to compute the determinants of the third order. In base of the Sarrus's rule we have:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

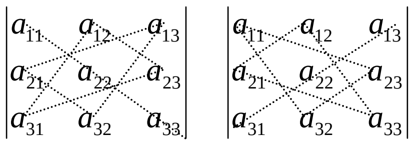


$$=$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

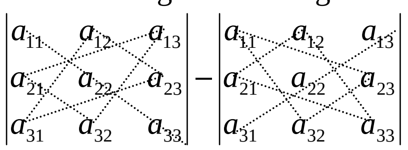
2.2 Triangle's rule

The triangle's rule will be formed with this scheme:



Scheme 3.

The product of diagonal elements and product of elements in the both vertex of two triangles of the first determinant get the "+" sign and the product of diagonal elements and product of elements in the both vertex of two triangles of the second determinant get the "-" sign. In base of triangle's rule, we have:

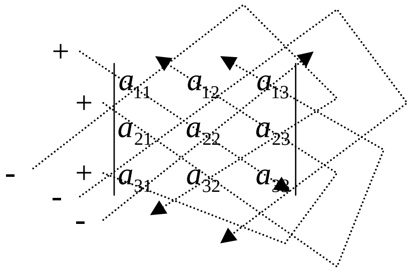


$$=$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

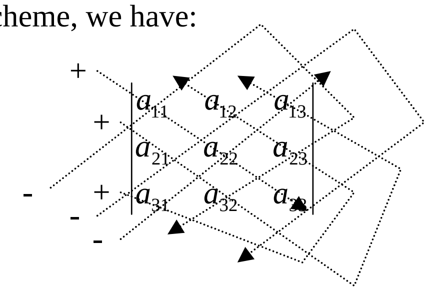
2.3 Another scheme to compute of the determinants of the third order

Another scheme to compute the determinants of the third order is Scheme 4, see [5].



Scheme 4.

In base of this scheme, we have:



$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

2.4 Chio's condensation method

Chio's condensation is a method for evaluating an nxn determinant in terms of $(n-1)$ x $(n-1)$ determinants, see [1], [3].

$$|A| = \frac{1}{a_{11}^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}$$

For $n = 3$, we obtain:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{a_{11}} \begin{Vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{Vmatrix} = \frac{1}{a_{11}} \begin{vmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{vmatrix} =$$

$$= \frac{1}{a_{11}}[(a_{11}a_{22} - a_{12}a_{21}) \cdot (a_{11}a_{33} - a_{13}a_{31}) - (a_{11}a_{32} - a_{12}a_{31}) \cdot (a_{11}a_{23} - a_{13}a_{21})] =$$

$$= \frac{1}{a_{11}}[a_{11}^2a_{22}a_{33} - a_{11}a_{22}a_{13}a_{31} - a_{12}a_{21}a_{11}a_{33} + a_{12}a_{21}a_{13}a_{31} - a_{11}^2a_{32}a_{23} + a_{11}a_{32}a_{13}a_{21} +$$

$$+ a_{12}a_{31}a_{11}a_{23} - a_{12}a_{31}a_{13}a_{21}] = a_{11}a_{22}a_{33} - a_{22}a_{13}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23} + a_{32}a_{13}a_{21} +$$

$$+ a_{12}a_{31}a_{23}$$

2.5 Dodgson's condensation method

The Dodgson's condensation method is a method, which determinants of the order nxn expansion in determinant of the $(n-1)\text{x}(n-1)$ order, than $(n-2)\text{x}(n-2)$ order and so one, see [2]. Using the Dodgson's condensation method for the determinants of the third order, we obtain:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{Vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{Vmatrix} = \begin{vmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{12}a_{23} - a_{13}a_{22} \\ a_{21}a_{32} - a_{22}a_{31} & a_{22}a_{33} - a_{23}a_{32} \end{vmatrix} =$$

$$= (a_{11}a_{22} - a_{12}a_{21}) \cdot (a_{22}a_{33} - a_{23}a_{32}) - (a_{21}a_{32} - a_{22}a_{31}) \cdot (a_{12}a_{23} - a_{13}a_{22}) =$$

$$= a_{11}a_{22}^2a_{33} - a_{11}a_{22}a_{23}a_{32} - a_{12}a_{21}a_{22}a_{33} + a_{12}a_{21}a_{23}a_{32} - a_{21}a_{32}a_{12}a_{23} + a_{21}a_{32}a_{13}a_{22} +$$

$$+ a_{22}a_{31}a_{12}a_{23} - a_{22}^2a_{31}a_{13} = a_{11}a_{22}^2a_{33} - a_{11}a_{22}a_{23}a_{32} - a_{12}a_{21}a_{22}a_{33} + a_{21}a_{32}a_{13}a_{22} +$$

$$+ a_{22}a_{31}a_{12}a_{23} - a_{22}^2a_{31}a_{13}$$

In base of Dodgson's condensation method the final result will be divided with a_{22} term, so we have:

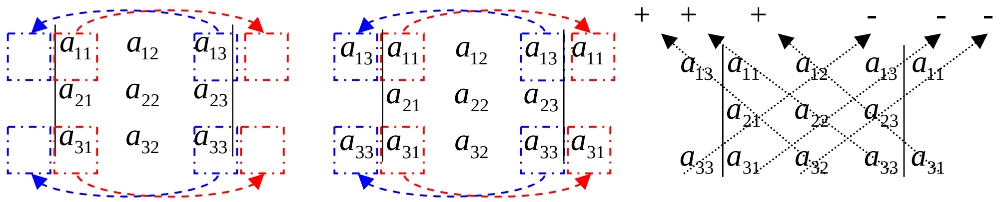
$$|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{22}a_{31}a_{13}$$

3 A new method to compute the determinant of the third order

The new method to compute the determinant of the third order might be one of the easiest methods to compute the determinants of the third order.
Assume a determinant of the third order:

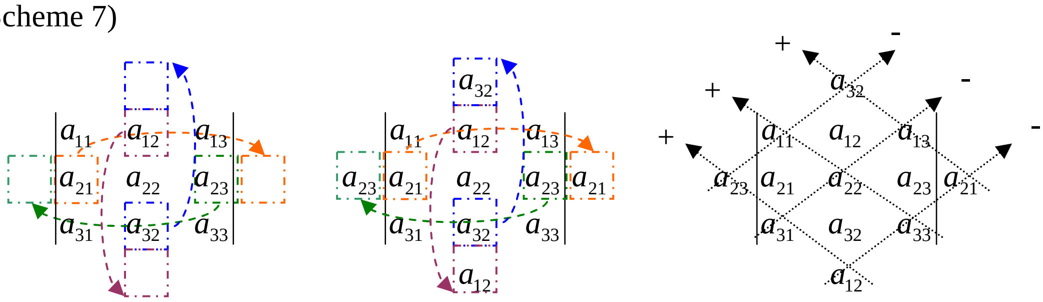
$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let's start by describing before the first row element which lie in cutting the first row with third column and before third row element which lie in cutting the third row with third column (a_{13} and a_{33}), as well in same manner let's describe after first and third row elements (a_{11} and a_{31}), respectively. Now get such a scheme (Scheme 5.):

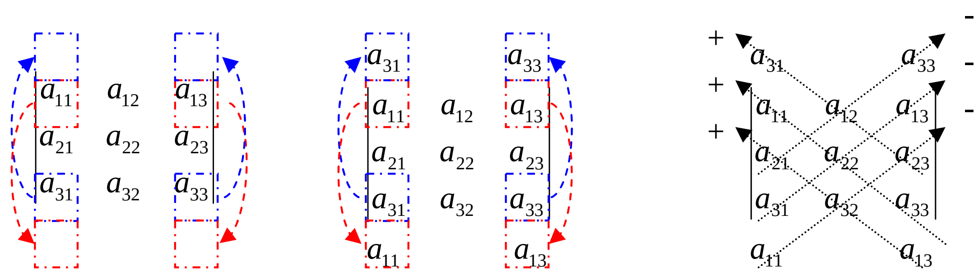


Scheme 5.

This scheme will be formed of six diagonals with three different elements of determinants. The elements products in three diagonals in left side will get the "+" sign, in the other hand the elements products in three other different diagonals in right side will get "-"sign. This will produce three terms with "+"sign and three other terms with "-"sign, which in fact presents the definition formula to compute the determinants of third order.
This new method consists of two other schemes, which will be formed in the same way like the preliminary scheme (Scheme 5) but these two other different schemes manipulate with elements in other rows and columns from the Scheme 5.
The other forms of this method are shown in the following schemes (Scheme 6, Scheme 7)



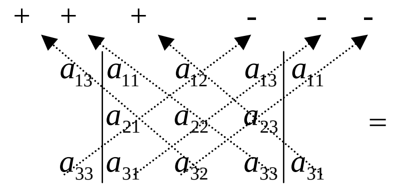
Scheme 6.



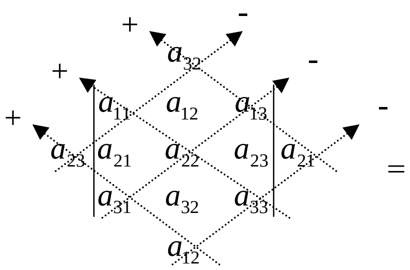
Scheme 7.

Proof

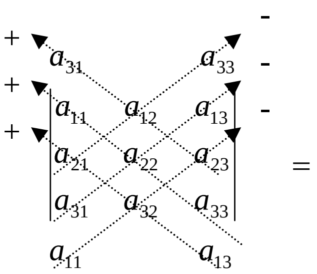
While applying the new method with the Scheme 5, Scheme 6, Scheme 7, to compute the determinants of the third order, we have:



$$= a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}$$



$$= a_{12}a_{23}a_{31} + a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}$$



$$= a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}$$

The results acquired by using the “NEW METHOD” are entirely equal with the results acquired by the other known methods until now (definition of determinants, Sarrus’s Rule, Triangle’s rule, expansion by the elements of whatever row or column, Chio’s condensation, Dodgson’s condensation). In base of this, we can conclude that this new method to compute the determinants of third order is true and can be used only for the third order determinant. Also we must pronounce, that this new method has relates and sameness with also new methods to compute determinants of higher order than that of the third order (n>3).

Conclusion

This new method, comparing with other known methods, is one of the most usable ones, based on quickness and easiness of computing the third order determinant.
Furthermore, this new method enables the further research in computing methods of higher than third order determinants. What is more, a new method, enabling the sum computation of two third order determinants will be possible by combining the schemes of this method.

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