On $P$-Hereditary and $P$-Semihereditary Rings

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Abstract

In this paper, we introduce the notion of “$P$-semihereditary” (resp., “$P$-hereditary”) rings which is a generalization of the notion of “semihereditary” (resp., “hereditary”) rings. Then we establish the transfer of this notion to trivial ring extensions and direct products. We conclude with a brief discussion of the scope and limits of our results.

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1 Introduction

All rings considered below are commutative with unit and all modules are unital. Recall that a ring $R$ is called hereditary (resp., semihereditary) if every ideal (resp., finitely generated ideal) of $R$ is projective. Any hereditary (resp., semihereditary) ring is coherent (recall that a ring is called coherent if every finitely generated ideal of $R$ is finitely presented). We introduce a new concept of a “$P$-hereditary (resp., $P$-semihereditary)” ring. A ring $R$ is called $P$-hereditary (resp., $P$-semihereditary) if every prime ideal (resp., finitely generated prime ideal) of $R$ is projective. A hereditary (resp., semihereditary) ring is naturally a $P$-hereditary (resp., $P$-semihereditary) ring.

Let $A$ be a ring, $E$ be an $A$-module and $R := A \otimes E$ be the set of pairs $(a, e)$ with pairwise addition and multiplication given by: $(a, e)(b, f) = (ab, af + be)$. $R$ is called the trivial ring extension of $A$ by $E$. Recall that a prime ideal of $R$ has always the form $Q \otimes E$, where $Q$ is a prime ideal of $A$ [3, Theorem 25.1]. Considerable work, part of it summarized in Glaz’s book [2] and Huckaba’s book [3], has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts.
in (commutative and non-commutative) ring theory. See for instance \[2, 3, 5\].

Our aim in this paper is to prove that $P$-hereditary (resp., $P$-semihereditary) rings are not hereditary (resp., semihereditary)” in general. Further, we investigate the possible transfer of the $P$-hereditary (resp., $P$-semihereditary) property to various trivial extension constructions and to direct products.

2 Main Results

In this section, we study the possible transfer of the $P$-hereditary (resp., $P$-semihereditary) property to various trivial extension contexts and to direct products. First, we examine the context of trivial ring extensions of a domain by its quotient field.

The next theorem not only serves as a prelude to the construction of examples, but also contributes to the study of the homological algebra of trivial ring extensions.

**Theorem 2.1** Let $A$ be a domain which is not a field, $K = qf(A)$, and $R := A \times K$ be the trivial ring extension of $A$ by $K$. Then:

1) $R$ is a $P$-semihereditary ring if and only if $A$ is a $P$-semihereditary ring.
2) $R$ is not a $P$-hereditary ring.
3) $R$ is not coherent. In particular, $R$ is neither hereditary nor semihereditary.

**Proof.** 1) Assume that $R$ is $P$-semihereditary and let $Q := \sum_{i=1}^{n} Ab_i$ be a nonzero finitely generated prime ideal of $A$, where $b_i \in Q$ for each $i = 1, \ldots, n$. Set $P := Q \times K$. Then, $P \in \text{Spec}(R)$ by [3, Theorem 25.1] and $P = \sum_{i=1}^{n} R(b_i, 0)$ since $bK = K$ for each $b \in A - \{0\}$; so $P(= Q \otimes_A R)$ is a projectif ideal of $R$. Therefore, $Q$ is a projectif ideal of $A$ since $R$ is a faithfully flat $A$-module.

Conversely, assume that $A$ is $P$-semihereditary and let $P$ be a finitely generated prime ideal of $R$. Then, $P = Q \times K$, where $Q$ is a finitely generated prime ideal of $A$. Hence, $Q$ is a projectif ideal of $A$ since $A$ is $P$-semihereditary. Therefore, $P(= Q \otimes_A R)$ is a projectif ideal of $R$ since $R$ is a faithfully flat $A$-module.

2) We claim that $0 \times K$ which is a prime ideal of $R$ is not projectif. Deny. Then $0 \times K$ is a projectif ideal of $R(= A \times K)$. Let $p$ be a nonzero prime ideal of $A$ and let $T = A_p \times K$. Then $R \subseteq T = S^{-1}R$, where $S = A - p$ i a multiplicative set of $A$ and of $R$. Therefore, $0 \times K = (0 \times K)T = (0 \times
$K \otimes_R T$ is a projective ideal of $T$ (since $0 \otimes K$ is a projective ideal of $R$ and since $T$ is a flat $A$-module). Hence, $0 \otimes K$ is a free ideal of $T$ since $T$ is a local ring, a contradiction since $(0 \otimes K)(0, e) = 0$ for each $0 \neq e \in K$.

Hence, $0 \otimes K$ is not projective ideal of $R$ and this shows that $R$ is not $P$-hereditary.

3) $R$ is not coherent since $R(0, 1)$ is a finitely generated ideal of $R$ which is not finitely presented by the exact sequence of $R$-modules:

$$0 \to 0 \otimes K \to R \overset{u}{\to} R(0, 1) \to 0$$

where $u(a, e) = (a, e)(0, 1) = (0, a)$ (since $0 \otimes K$ is not a finitely generated ideal of $R$).

If $A$ is a Prüfer domain, we obtain by Theorem 2.1:

**Corollary 2.2** Let $A$ be a Prüfer domain which is not a field, $K = qf(D)$, and $R := A \otimes K$ be the trivial ring extension of $A$ by $K$. Then $R$ is $P$-semihereditary which is neither $P$-hereditary nor coherent.

Now, we prove that the condition "$A$ is not a field" is necessary in Theorem 2.1.

**Example 2.3** Let $K$ be a field and $n$ be a positive integer. The trivial ring extension $R := K \otimes K^n$ of $K$ by $K^n$ is not $P$-semihereditary.

**Proof.** Let $P := 0 \otimes K^n$ which is a finitely generated prime ideal of $R$. We claim that $P$ is not projective. Deny. Then $P$ is free since $R$ is a local ring, a contradiction since $(0 \otimes K^n)(0, e) = 0$ for each $0 \neq e \in K^n$. Hence, $R$ is not a $P$-semihereditary ring.

Nevertheless, if $A$ is a field and $E$ is a $K$-vector space with infinite rank, we have:

**Theorem 2.4** Let $K$ be a field, $E$ be a $K$-vector space with infinite rank and let $R := K \otimes E$ be the trivial ring extension of $K$ by $E$. Then:

1) $R$ is a $P$-semihereditary ring.
2) $R$ is not a $P$-hereditary ring.
3) $R$ is not coherent. In particular, $R$ is neither hereditary nor semihereditary.

**Proof.** 1) $R$ is a $P$-semihereditary ring since the only proper prime ideal of $R$ is $0 \otimes E$ which is not a finitely generated ideal of $R$ (since $E$ is a $K$-vector space with infinite rank).
2) $R$ is not $P$-hereditary since $0 \propto E$ is a prime ideal of $R$ which is not projectif since it is not free (since $R$ is local and $(0 \propto E)(0, e) = 0$ for each $0 \neq e \in E$).

3) $R$ is not coherent since $R$ is a 2-Von Neumann regular ring which is not a Von Neumann regular ring by [4, Theorem 3.4].

Next, we explore a different context, namely, the trivial ring extension of a local domain $(A, M)$ by an $A$-module $E$ such that $ME = 0$.

The next theorem not only serves as a prelude to the construction of examples, but also contributes to the study of the homological algebra of trivial ring extensions.

**Theorem 2.5** Let $(A, M)$ be a local domain, $E$ an $A$-module with $ME = 0$, and let $R := A \propto E$ be the trivial ring extension of $A$ by $E$. Then:

1) $R$ is a $P$-semihereditary ring if and only if $E$ is an $(A/M)$-vector space of infinite rank.

2) $R$ is not a $P$-hereditary ring.

Before proving Theorem 2.5, we establish the following Lemma.

**Lemma 2.6** Let $(A, M)$ be a local ring, $E$ an $A$-module with $ME = 0$, and let $R := A \propto E$ be the trivial ring extension of $A$ by $E$. Then $R$ does not contain any proper projectif ideal.

**Proof.** Let $J$ be a proper ideal of $R$. We claim that $J$ is not projectif. 
Deny. Then $J$ is free since $R$ is local. But $J \subseteq (M \propto E)$ (since $R$ is a local ring and $M \propto E$ is its maximal ideal) and $(M \propto E)(0, e) = 0$ for each $0 \neq e \in E$, so $J(0, e) = 0$ for each $0 \neq e \in E$, a contradiction since $J$ is free. Hence $J$ is not projectif as desired.

**Proof of Theorem 2.5.**

1) It is clear by Lemma 2.6 that $R$ is $P$-semihereditary if and only if there is no proper finitely generated prime ideal of $R$.

Assume that $R$ is $P$-semihereditary. We claim that $E$ is an $(A/M)$-vector space of infinite rank. Deny. Then $E$ is an $(A/M)$-vector space of finite rank and let $(x_i)_{i=1,...,m}$ be its basis. Then $P := 0 \propto E = \sum_{i=1}^m R(0, x_i)$ is a proper finitely generated prime ideal of $R$, a contradiction by Lemma 2.6 and since $R$ is $P$-semihereditary. Hence, $E$ is an $(A/M)$-vector space of infinite rank.

Conversely, assume that $E$ is an $(A/M)$-vector space of infinite rank. We claim that there is no proper finitely generated prime ideal of $R$. Deny. Let $P := Q \propto E = \sum_{i=1}^n R(b_i, x_i)$ be a proper finitely generated prime ideal of $R$, where $Q$ is a prime ideal of $A$ and $b_i \in Q$, $x_i \in E$ for each $i = 1, \ldots, n$. 

Then \( E \subseteq \sum_{i=1}^{n}(A/M)x_i \) (since \( b_iE = 0 \) for each \( i = 1, \ldots, n \)), and hence \( E \) is an \((A/M)\)-vector space of finite rank, a contradiction. Therefore, there is no proper finitely generated prime ideal of \( R \) and so \( R \) is \( P \)-semihereditary.

2) \( R \) is not \( P \)-hereditary since \( M \propto E \) is a proper prime ideal which is not projectif by Lemma 2.6 and this completes the proof of Theorem 2.5.

**Remark 2.7** In Theorem 2.5, the surprise is that the \( P \)-semihereditary and \( P \)-hereditary properties hold for a trivial ring extension of a local ring \((A, M)\) by an \((A/M)\)-vector space without any hypothesis on the basic ring \( A \).

Next, we study the transfer of the \( P \)-semihereditary (resp., \( P \)-hereditary) property to direct products.

**Proposition 2.8** Let \((R_i)_{i=1}^{n}\) be a family of rings. Then, \( \prod_{i=1}^{n} R_i \) is \( P \)-semihereditary (resp., \( P \)-hereditary) if and only if \( R_i \) is \( P \)-semihereditary (resp., \( P \)-hereditary) for each \( i = 1, \ldots, n \).

We need the following Lemma before proving Proposition 2.8.

**Lemma 2.9** ([4, Lemma 2.5]) Let \((R_i)_{i=1,2}\) be a family of rings and \( E_i \) an \( R_i \)-module for \( i = 1, 2 \). Then:

1) \( E_1 \prod E_2 \) is a finitely generated \( R_1 \prod R_2 \)-module if and only if \( E_i \) is a finitely generated \( R_i \)-module for \( i = 1, 2 \).

2) \( E_1 \prod E_2 \) is a projectif \( R_1 \prod R_2 \)-module if and only if \( E_i \) is a projectif \( R_i \)-module for \( i = 1, 2 \).

**Proof of Proposition 2.8.** By induction on \( n \), it suffices to prove the assertion for \( n = 2 \). Since a prime ideal of \( R_1 \prod R_2 \) is of the form \( P_1 \prod R_2 \) or \( R_1 \prod P_2 \), where \( P_i \) is a prime ideal of \( R_i \) for \( i = 1, 2 \), the conclusion follows easily from Lemma 2.9.

**References**


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