Galois Extensions with a Galois Commutator Subring

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Abstract

Let $B$ be a Galois extension of $B^G$ with Galois group $G$, $\Delta$ the commutator subring of $B^G$ in $B$, and $G|\Delta$ the restriction of $G$ to $\Delta$. Equivalent conditions are given for a Galois extension $\Delta$ of $\Delta^G$ with Galois group $G|\Delta$. It is shown that the following statements are equivalent: (1) $\Delta$ is a Galois extension of $\Delta^G$ with Galois group induced by and isomorphic with $G/N$ where $N = \{g \in G \mid g(x) = x \text{ for all } x \in \Delta\}$. (2) $B^G\Delta$ is a Galois extension of $B^G$ with Galois group induced by and isomorphic with $G/N$ and $\Delta$ is a finitely generated and projective module over $\Delta^G$. (3) $B$ is a composition of two Galois extensions: $B \supseteq B^G\Delta$ with Galois group $N$ and $B^G\Delta \supseteq B^G$ with Galois group induced by and isomorphic with $G/N$ such that $\Delta$ is a finitely generated and projective module over $\Delta^G$. Consequently, more results can be derived for several well known classes of Galois extensions such as DeMeyer-Kanzaki Galois extensions, Azumaya Galois extensions, and Hirata separable Galois extensions.

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1 Introduction

Let $T$ be a ring extension of $S$ and $V_T(S)$ the commutator subring of $S$ in $T$. Properties of $V_T(S)$ play an important role for central simple algebras, Azumaya algebras, Hirata separable extensions, Galois extensions for rings, and Hopf Galois extensions ([1]–[2], [4]–[9]). Let $B$ be a Galois extension of $B^G$ with Galois group $G$. Then $V_B(B^G) = \oplus \sum_{g \in G} J_g$ where $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ ([4], Proposition 1). In [1], it was shown that if $B$ is a Galois extension of an Azumaya algebra $B^G$ over $C^G$ where $C$ is the center of $B$, then $V_B(B^G)$ is a Galois algebra over $C^G$ with Galois group induced by and isomorphic with $G$ ([1], Theorem 2). Also, $V_B(B^G)$ is investigated for a Hirata separable Galois extension $B$ with Galois group $G$ ([8]). The purpose of the present paper is to characterize a Galois extension $B$ of $B^G$ with Galois group $G$ such that $V_B(B^G)$ is a Galois extension with Galois group induced by $G$. We shall show the following equivalent conditions: Let $B$ be a Galois extension of $B^G$ with Galois group $G$ and $\Delta = V_B(B^G)$. (1) $\Delta$ is a Galois extension of $G^G$ with Galois group isomorphic with $G/N$ where $N = \{ g \in G \mid g(x) = x \text{ for all } x \in \Delta \}$. (2) $B^G \Delta$ is a Galois extension of $B^G$ with Galois group isomorphic with $G/N$ and $\Delta$ is a finitely generated and projective module over $G^G$. (3) $B$ is a composition of two Galois extensions: (i) $B \supset B^G \Delta$ with Galois group $N$ and (ii) $B^G \Delta \supset B^G$ with Galois group isomorphic with $G/N$ such that $J_{\Delta}^g(\Delta)$ is a finitely generated projective module over $G^G$ for each $g \in G/N$ where $J_{\Delta}^g(\Delta) = \{ b \in \Delta \mid bx = g(x)b \text{ for all } x \in \Delta \}$. Consequently, more results can be derived for several well known classes of Galois extensions such as DeMeyer-Kanzaki Galois extensions, Azumaya Galois extensions, and Hirata separable Galois extensions.

2 Definitions and Notations

Throughout this paper, $B$ will represent a ring with 1, $C$ the center of $B$, $G$ a finite automorphism group of $B$, $B^G$ the set of elements in $B$ fixed under each element in $G$, and $B \ast G$ the skew group ring of $G$ over $B$, that is, $B \ast G$ is a free left $B$-module in which the multiplication is given by $gb = g(b)g$ for $b \in B$ and $g \in G$.

Let $A$ be a subring of a ring $B$ with the same identity 1. Following the definitions and notations as given in [9], we denote $V_B(A)$ the commutator (also called centralizer) subring of $A$ in $B$. We call $B$ a separable extension of $A$ if there exist \{ $a_i, b_i$ in $B$, $i = 1, 2, ..., k$ for some integer $k$ \} such that $\sum a_i b_i = 1$, and $\sum b_i \otimes a_i b_i = \sum a_i \otimes b_i b$ for all $b$ in $B$ where $\otimes$ is over $A$. An Azumaya algebra is a separable extension of its center. We call $B$ a Galois extension of $B^G$ with Galois group $G$ if there exist elements $\{ a_i, b_i$ in $B$, $i = 1, 2, ..., m \$
for some integer \( m \) such that \( \sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g} \) for each \( g \in G \). Such a set \( \{a_i, b_i\} \) is called a \( G \)-Galois system for \( B \). A Galois extension \( B \) of \( B^G \) is called a Galois algebra over \( B^G \) if \( B^G \) is contained in \( C \), and a central Galois algebra if \( B \) is a Galois extension of \( C \). We called \( B \) a center Galois extension with Galois group \( G \) if \( C \) is a Galois algebra over \( C^G \) with Galois group \( G|_C \cong G \), and a commutator Galois extension of \( B^G \) with Galois group \( G \) if \( V_B(B^G) \) is a Galois extension of \( (V_B(B^G))^G \) with Galois group \( G|_{V_B(B^G)} \cong G \). A Galois extension \( B \) of \( B^G \) with Galois group \( G \) is called an Azumaya Galois extension if \( B \) is an Azumaya \( C^G \)-algebra. A Galois extension \( B \) of \( B^G \) with Galois group \( G \) is called a DeMeyer-Kanzaki Galois extension if \( B \) is an Azumaya algebra over \( C \) which is a Galois algebra over \( C^G \) with Galois group \( G|_C \cong G \).

A ring \( B \) is called a Hirata separable extension of \( A \) if \( B \otimes_A B \) is isomorphic to a direct summand of a finite direct sum of \( B \) as a \( B \)-bimodule, and \( B \) is called a Hirata separable Galois extension of \( B^G \) if it is a Galois and a Hirata separable extension of \( B^G \).

3 Characterizations

In this section, let \( B \) be a Galois extension of \( B^G \) with Galois group \( G \) and \( \Delta = V_B(B^G) \). We shall characterize \( B \) with a Galois commutator \( \Delta \) with Galois group induced by \( G \). We begin with some basic facts.

**Lemma 3.1** Let \( T \) be a ring and \( G \) an automorphism group of \( T \). Then (1) \( V_T(T^G) \) is a \( G \)-invariant subring of \( T \) and (2) \( (V_T(T^G))^G \) is contained in the center of \( V_T(T^G) \).

**Proof.** (1) For any \( g \in G, a \in V_T(T^G), \) and \( x \in T^G \), we have that \( g(a)x = g(ax) = g(xa) = xg(a) \), so \( g(a) \in V_T(T^G) \).

(2) holds because \( (V_T(T^G))^G = T^G \cap (V_T(T^G)) \) which is contained in the center of \( V_T(T^G) \).

Let \( N = \{g \in G | g(a) = a \text{ for all } a \in V_T(T^G)\} \). Then part (1) in Lemma 3.1 implies that \( N \) is a normal subgroup of \( G \), and \( V_T(T^G) \) is an algebra over \( (V_T(T^G))^G \) by part (2). We shall employ a well known fact for a Galois extension.

**Lemma 3.2** Let \( B \) be a Galois extension of \( B^G \) with Galois group \( G \) and \( A \) a \( G \)-invariant subring of \( B \) under the action of \( G \). If \( A \) is a Galois extension of \( B^G \) with Galois group induced by \( G \) and isomorphic with \( G \), then \( A = B \).

Now we show the main theorem in this section.
Theorem 3.3 Let $B$ be a Galois extension of $B^G$ with Galois group $G$, $\Delta = V_B(B^G)$, and $D = \Delta^G$. Then the following statements are equivalent:

(1) $\Delta$ is a Galois algebra over $D$ with Galois group induced by and isomorphic with $G/N$ where $N = \{g \in G | g(x) = x \text{ for all } x \in \Delta\}$. (2) $B^G \Delta$ is a Galois extension of $B^G$ with Galois group induced by and isomorphic with $G/N$ and $\Delta$ is a finitely generated and projective module over $D$. (3) $B$ is a composition of two Galois extensions: $B \supset B^G \Delta$ with Galois group $N$ and $B^G \Delta \supset B^G$ with Galois group induced by and isomorphic with $G/N$ such that $J^G_{\Delta}(\Delta)$ is a finitely generated projective module over $D$ for each $g \in G/N$ where $J^G_{\Delta}(\Delta) = \{b \in D | bx = g(x)b \text{ for all } x \in \Delta\}$.

Proof. (1) $\implies$ (2) Since $\Delta$ is a Galois algebra over $D$ where $D = \Delta^G$ which in contained in the center of $\Delta$ by Lemma 3.1, $\Delta$ is a finitely generated and projective module over $D$. Let $\{a_i, b_i \in \Delta | i = 1, 2, \ldots, m\}$ be a Galois system for $\Delta$. Then $B^G \Delta$ is a Galois extension of $(B^G \Delta)^G (= B^G)$ with Galois group induced by and isomorphic with $G/N$ for $\Delta$ because $B^G \Delta$ can take $\{a_i, b_i \in B^G \Delta | i = 1, 2, \ldots, m\}$ as a Galois system.

(2) $\implies$ (1) By hypothesis, $B^G \Delta$ is a Galois extension of $B^G$ with Galois group induced by and isomorphic with $G/N$, so, by Theorem 1 in [3], the skew group ring

$$(B^G \Delta) * (G/N) \cong \text{Hom}_{B^G}(B^G \Delta, B^G \Delta).$$

Denoting $G/N$ by $\overline{G}$, we have that

$$\alpha : (B^G \Delta) * \overline{G} \cong \text{Hom}_{B^G}(B^G \Delta, B^G \Delta)$$

by $(\alpha(\sum_{\overline{g} \in \overline{G}} a_{\overline{g}}))(x) = \sum_{\overline{g} \in \overline{G}} a_{\overline{g}}(x)$ for each $x \in B^G \Delta$. Then

$$\Delta * \overline{G} = V_{B^G \Delta \overline{G}}(B^G) \cong V_{\text{Hom}_{B^G}(B^G \Delta, B^G \Delta)}(\alpha(B^G)).$$

Next we claim that $V_{\text{Hom}_{B^G}(B^G \Delta, B^G \Delta)}(\alpha(B^G)) = \text{Hom}_D(\Delta, \Delta)$ where $D = \Delta^G = \Delta^G$. In fact, let $f \in \text{Hom}_{B^G}(B^G \Delta, B^G \Delta)$ such that $f \cdot \alpha(r) = \alpha(r) \cdot f$ for each $r \in B^G$. Then for each $t \in \Delta$, $f(t) = f(tr) = f(rt) = f(\alpha(r)(t)) = (f \cdot \alpha(r))(t) = (\alpha(r) \cdot f)(t)$. This implies that $f(t) \in \Delta$. Thus $f : \Delta \rightarrow \Delta$; and so $V_{\text{Hom}_{B^G}(B^G \Delta, B^G \Delta)}(\alpha(B^G)) \subset \text{Hom}_D(\Delta, \Delta)$ (for $D = B^G \cap \Delta$ by Lemma 3.1). Conversely, let $f \in \text{Hom}_{B^G}(B^G \Delta, B^G \Delta)$ such that $f \in \text{Hom}_D(\Delta, \Delta)$. We claim that $f \cdot \alpha(r) = \alpha(r) \cdot f$ for each $r \in B^G$. In fact, for each $s \in B^G$ and each $a \in \Delta$, $(f \cdot \alpha(r))(sa) = f(ras) = f(ars) = f(a)(rs) = r f(a)s = r f(as) = r f(sa) = (\alpha(r) \cdot f)(sa)$ (for $f(a) \in \Delta$). Thus $f \cdot \alpha(r) = \alpha(r) \cdot f$ for each $r \in B^G$. But then $f \in V_{\text{Hom}_{B^G}(B^G \Delta, B^G \Delta)}(\alpha(B^G))$. This proves that

$$V_{\text{Hom}_{B^G}(B^G \Delta, B^G \Delta)}(\alpha(B^G)) = \text{Hom}_D(\Delta, \Delta).$$
Therefore, \( \alpha : \Delta * \overline{G} \cong \text{Hom}_D(\Delta, \Delta) \). Moreover, by hypothesis, \( \Delta \) is a finitely generated and projective module over \( D \), so \( \Delta \) is a Galois algebra over \( D \) with Galois group isomorphic with \( \overline{G} \) ([3], Theorem 1).

(2) \( \Rightarrow \) (3) Since \( B^G \Delta \) is a Galois extension of \( B^G \) with Galois group induced by and isomorphic with \( \overline{G} (= G/N) \), \( B^N \) containing \( B^G \Delta \) is also a Galois extension of \( B^G (= B^G) \) with Galois group isomorphic with \( \overline{G} \); and so \( B^N = B^G \Delta \) by Lemma 3.2. But then \( B \supset B^G \Delta \) is a Galois extension with Galois group \( N \) and \( B^G \Delta \supset B^G \) is a Galois extension with Galois group induced by and isomorphic with \( \overline{G} (= G/N) \) such that \( \Delta \) is a finitely generated and projective module over \( D \). Noting that \( V_B(B^G) = \Delta = \bigoplus_{\overline{\eta} \in \overline{G}} J^{(\Delta)}_{\overline{\eta}} \) ([4], Proposition 1 and Theorem 1), we have that \( J^{(\Delta)}_{\overline{\eta}} \) is a finitely generated projective module over \( D \) for each \( \overline{\eta} \in G/N \).

(3) \( \Rightarrow \) (2) is clear.

By Theorem 3.3, we shall derive some consequences for several well known classes of Galois extensions. We recall that \( B \) is a center Galois extension with Galois group \( G \) if \( B \) is a Galois extension with Galois group \( G \) such that its center \( C \) is a Galois algebra over \( C^G \) with Galois group \( G|_C \cong G \), and \( B \) is a commutator Galois extension of \( B^G \) with Galois group \( G \) if \( V_B(B^G) \) is a Galois extension of \( (V_B(B^G))^G \) with Galois group \( G|_{V_B(B^G)} \cong G \).

**Corollary 3.4** Let \( B \) be a Galois extension of \( B^G \) with Galois group \( G \). If \( B = B^G C \) such that \( C \) is finitely generated and projective over \( C^G \), then \( B \) a center Galois extension with Galois group \( G \).

**Corollary 3.5** Let \( B \) be a Galois extension of \( B^G \) with Galois group \( G \). If \( B = B^G \Delta \) such that \( \Delta \) is finitely generated and projective over \( \Delta^G \), then \( B \) a commutator Galois extension with Galois group \( G \).

**Remark 3.6** Since a DeMeyer-Kanzaki Galois extension is also a center Galois extension and an Azumaya Galois extension is a commutator Galois extension ([1], Theorem 2), Corollary 3.4 and Corollary 3.5 hold for the classes of DeMeyer-Kanzaki Galois extensions and Azumaya Galois extensions.

**Corollary 3.7** Let \( B \) be a Hirata separable Galois extension of \( B^G \) with Galois group \( G \). If \( B = B^G \Delta \), then \( \Delta \) is a Galois algebra with Galois group induced by and isomorphic with \( G/N \).
Proof. Since $B$ is a Hirata separable Galois extension of $B^G$ with Galois group $G$, $J_g$ is a finitely generated and projective rank one module over $C^G$ for each $g \in G$ ([8], Theorem 2). Hence $\Delta = \oplus g \in G J_g$ is a finitely generated and projective module over $C^G$. Thus $\Delta$ is a Galois algebra over $D$ with Galois group induced by and isomorphic with $G/N$ by Theorem 3.3.

4 The Galois commutator

In section 3, we characterize a Galois extension $B$ with a Galois commutator subring $\Delta$. In this section, we shall give an equivalent condition for $\Delta$ as a composition of a central Galois algebra and a commutative Galois algebra. Thus we derive an expression for $B$ as a composition of three Galois extensions. We keep the notations of $N$, $G$, and $J^{(\Delta)}$ as given in section 3.

Lemma 4.1 Let $B$ be a Galois extension of $B^G$ with Galois group $G$. Then $\oplus h \in N J_{gh} \subset J^{(\Delta)}$ for each $g \in G$ and $\Delta = \sum_{g \in G} J^{(\Delta)}$.

Proof. Since $B$ is a Galois extension of $B^G$ with Galois group $G$, $\Delta = V_B(B^G) = \oplus g \in G J_g = \oplus g \in G \sum h \in N J_{gh}$ ([4], Proposition 1). For any $a \in J_{gh}$ and $x \in \Delta$, we have that $ax = (gh)(x) = g(x)a = \overline{g}(x)a$, so $a \in J^{(\Delta)}$; and so $J_{gh} \subset J^{(\Delta)}$. Thus $\Delta = \oplus g \in G \sum h \in N J_{gh} = \sum_{g \in G} J^{(\Delta)}$.

Theorem 4.2 Let $B$ be a Galois extension of $B^G$ with Galois group $G$ such that $\Delta$ is a Galois algebra over $D$ with Galois group induced by and isomorphic with $G/N$, $Z$ the center of $\Delta$, and $K = \{g \in G | g(a) = a \text{ for all } a \in Z\}$. Then $\Delta$ is a central Galois algebra over $Z$ with Galois group induced by and isomorphic with $K/N$ and $Z$ is a commutative Galois algebra over $Z^G$ with Galois group induced by and isomorphic with $(G/N)/(K/N)$ if and only if $J_{gh} = \{0\}$ for each $g \notin K$ and $h \in N$.

Proof. Since $\Delta$ is a Galois algebra over $D$ with Galois group induced by and isomorphic with $G/N$, $\Delta = \oplus g \in G J^{(\Delta)}$. Moreover, $\Delta$ is a central Galois algebra over $Z$ with Galois group induced by and isomorphic with $K/N$ if and only if $J^{(\Delta)} = \{0\}$ for each $\overline{g} \notin K/N$, and in this case, $Z$ is a commutative Galois algebra over $Z^G$ with Galois group induced by and isomorphic with $(G/N)/(K/N)$ ([4], Proposition 3). But then $J^{(\Delta)} = \oplus h \in N J_{gh}$ for each $\overline{g} \in G$ by Lemma 4.1. Thus $J^{(\Delta)} = \{0\}$ for each $\overline{g} \notin K/N$ if and only if $J_{gh} = \{0\}$ for each $g \notin K$ and $h \in N$. 

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Corollary 4.3 Let $B$ be a Galois extension of $B^G$ with Galois group $G$ such that $\Delta$ is a Galois algebra over $D$ with Galois group induced by and isomorphic with $G/N$, $Z$ the center of $\Delta$, and $K = \{ g \in G | g(a) = a \text{ for all } a \in Z \}$. If $J_{gh} = \{0\}$ for each $g \notin K$ and $h \in N$, then $B$ is a composition of three Galois extensions: (1) $B \supset B^G \Delta$ with Galois group $N$, (2) $B^G \Delta \supset B^G Z$ with Galois group induced by and isomorphic with $K/N$, and (3) $B^G Z \supset B^G$ with Galois group induced by and isomorphic with $\overline{G}/\overline{K}$ where $\overline{G} = G/N$ and $\overline{K} = K/N$.

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