On $R$-Strong Jordan Ideals

Anita Verma

Department of Mathematics
University of Delhi, Delhi 110 007, India
verma.anitaverma.anita945@gmail.com

Abstract. R-strong Jordan Ideals have been defined. Examples have been given to show their existence. It has been proved that the sum of two $R$-strong Jordan Ideals is an $R$-strong Jordan Ideal. Also, it has been proved that the intersection of an arbitrary number of $R$-strong Jordan ideals is an $R$-strong Jordan ideal. Further, we prove that the product of two $R$-strong Jordan ideals is also an $R$-strong Jordan ideal. Finally, a set $A_V$ associated with an $R$-strong Jordan ideal $V$ has been defined and sufficient condition, under which $A_V \cap V$ is an $R$-strong Jordan ideal, has been given.

Mathematics Subject Classification: 16A66, 16A72

Keywords: Ideals, Jordan Ideal, R-strong Jordan Ideal

1. Introduction

Throughout the paper, we assume that $R$ is a non-commutative ring, the symbol $J$ denotes the Jordan ideal of $R$. A ring $R$ is said to be prime if for $a, b \in R, aRb = (0)$ implies $a = 0$ or $b = 0$. An additive subgroup $J$ of $R$ is said to be a Jordan ideal of $R$ if $ur + ru \in J$ for all $u \in J, r \in R$. For $x, y \in R$, by $[x, y]$, we mean $xy - yx$.

An additive subgroup $U$ of $R$ is said to be a Lie ideal of $R$ if $[a, r] \in U$, for all $a \in A, r \in R$.

One may observe that if char $R = 2$, then Jordan ideal and Lie ideal of $R$ are same. Also every ideal of $R$ is Jordan ideal of $R$ but converse need not be true. Further, one may verify that the intersection of an arbitrary number of $R$-strong Jordan ideals of $R$ is also a Jordan ideal of $R$.
2. Jordan Ideals

It may be observed that if \( I_1 \) and \( I_2 \) are two ideals of \( R \), then \( I_1 + I_2 \) is an ideal of \( R \). However, it is not true in case of Jordan ideals. Indeed, let \( R \) be a ring of \( 2 \times 2 \) matrices over integers and let \( a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R \). Then

\[
aR = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in \mathbb{Z} \right\}
\]

Since

\[
\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t & s \\ u & v \end{pmatrix} + \begin{pmatrix} t & s \\ u & v \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} xt + yu & xs + yv \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} tx & ty \\ ux & uv \end{pmatrix} \notin aR,
\]

\( aR \) is not a Jordan ideal of \( R \).

Similarly, \( Ra \) is not a Jordan ideal of \( R \). Also, one may verify that \( aR + Ra \) is a Jordan ideal of \( R \).

**Lemma 2.1.** Let \( R \) be a ring with unity and \( 2R = R \). If \( J \) is the Jordan ideal of \( R \) and \( 1 \in J \). Then \( J = R \).

**Proof.** Obviously \( J \subseteq R \). Let \( r \in R \) be any element. Since \( 1 \in J \) and \( J \) is Jordan ideal of \( R \), \( 1 \cdot r + r \cdot 1 \in J \). This gives \( 2r \in J \). Hence \( r \in J \).

**Lemma 2.2.** If \( J \) is a Jordan ideal of \( R \) and \( b \in J \), then \( [[x, y], b] \in J \), for all \( x, y \in R \).

**Proof.** Let \( b \in J \). Since \( J \) is Jordan, \( xb + bx \) and \( yb + by \in J \). Also, since \( J \) is an additive subgroup of \( R \),

\[
(bx + xb)y + y(bx + xb) \in J \quad \text{and} \quad (by + yb)x + x(by + yb) \in J.
\]

This gives \( b[x, y] - [x, y]b \in J \). Hence \( [[x, y], b] \in J \).

3. \( R \)-Strong Jordan Ideals

Throughout this section by ring \( R \), we mean a prime ring.

**Definition 3.1** ([1]). Let \( R \) be a prime ring. A Jordan ideal \( V \) of \( R \), is said to be \( R \)-strong Jordan ideal of \( R \), if \( avb \in V \), for all \( v \in V \) and for all \( a, b \in R \).

Towards the existence of \( R \)-strong Jordan ideals, we give the following example.
Example 3.2. (i) Let $R = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in \mathbb{Z} \right\}$ and let $V = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}, a \neq 0 \right\}$.

Then $V$ is $R$-strong Jordan ideal of $R$. Indeed, let if $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R$ and $Y = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in V$, then

$XY + YX = \begin{pmatrix} xa + ax & 0 \\ 0 & 0 \end{pmatrix} \in V$

Also, $XYX = \begin{pmatrix} xax & 0 \\ 0 & 0 \end{pmatrix} \in V$.

Hence $V$ is an $R$-strong Jordan ideal of $R$.

(ii) Let $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & d \\ 0 & f \end{pmatrix} \mid a, d, e, f \in \mathbb{Z} \right\}$ and $V = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}, a \neq 0 \right\}$.

Then $V$ is a $R$-strong Jordan ideal of $R$.

Note. $R$-strong Jordan ideal of $R$ is a Jordan ideal of $R$.

Theorem 3.3. If $V_1$ and $V_2$ are two $R$-strong Jordan ideals of $R$, then $V_1 + V_2$ is also $R$-strong Jordan ideal of $R$.

Proof. Clearly $V_1 + V_2$ is a Jordan ideal of $R$. Let $x \in V_1 + V_2$ and $a, b \in R$. Then $x = y + z, y \in V_1, z \in V_2$. Since $V_1$ is $R$-strong Jordan ideal of $R$, for $y \in V_1$ and $a, b \in R, ayb \in V_1$. Similarly, $azb \in V_2$. Also, $axb = ayb + azb \in V_1 + V_2$.

Hence $V_1 + V_2$ is an $R$-strong Jordan ideal of $R$.

Theorem 3.4. Let $\left\{ V_t : t \in T, where T is an indexed set \right\}$ be a family of $R$-strong Jordan ideals of $R$. Then $\bigcap_{t \in T} V_t$ is an $R$-strong Jordan ideal of $R$.

Proof. Let $V = \bigcap_{t \in T} V_t$. Let $x \in V$ and $a, b \in R$. Since $x \in V, x \in V_t$, for all $t \in T$. Now $x \in V_t$ and $V_t$ is $R$-strong Jordan ideal, therefore $axb \in V_t$, for all $t \in T$. Hence $axb \in \bigcap_{t \in T} V_t = V$.

Remark 3.5. Union of two $R$-strong Jordan ideals need not be an $R$-strong Jordan ideal. Indeed, if

$R = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in \mathbb{Z} \right\}$, $V_1 = \left\{ \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$
and \( V_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\} \), then both \( V_1 \) and \( V_2 \) are \( R \)-strong Jordan ideals of \( R \). But \( V_1 \cup V_2 \) is not even a Jordan ideal of \( R \). Indeed,
\[
\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} xa & bx \\ ax & by \end{pmatrix} \notin V_1 \cup V_2.
\]

Regarding product of two \( R \)-strong Jordan ideals, we give the following result.

**Theorem 3.6.** Let \( R \) be a ring with unity. If \( V_1 \) and \( V_2 \) are \( R \)-strong Jordan ideals of \( R \), then \( V_1 V_2 \) is also an \( R \)-strong Jordan ideal of \( R \).

**Proof.** Note that
\[
V_1 V_2 = \left\{ \sum_{i=1}^{n} a_i b_i \mid a_i \in V_1, b_i \in V_2, n \in \mathbb{Z} \right\}
\]
clearly \( V_1 V_2 \) is a Jordan ideal of \( R \).

Let \( x \in V_1 V_2 \) and \( r \in R \). Then \( x = \sum_{i=1}^{n} a_i b_i, a_i \in V_1, b_i \in V_2 \). Now \( a_i \in V_1, r \in R \) and \( V_1 \) is \( R \)-strong Jordan ideal, therefore, \( ra_i r \in V_1 \). Similarly, \( rb_i r \in V_2, i = 1, 2, \ldots , n \).

Now
\[
(3.3.1) \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (ra_i r \cdot rb_i r) + \sum_{i=1}^{n} (ra_i - ra_i r)(b_i r + rb_i r)
\]
Since \( r_1 a, r_2 \in V_1 \), for all \( r_1, r_2 \in R \) and \( i = 1, 2, \ldots , n \). Taking \( r_1 = r - 1, r_2 = r \), we get \( (ra_i r - a_i r) \in V_1, i = 1, 2, \ldots , n \). Similarly \( b_i r + rb_i r \in V_2 \).

Thus (3.1) gives, \( r(\sum_{i=1}^{n} a_i b_i) r \in V_1 V_2 \).

**Lemma 3.7.** Let \( V \) be an \( R \)-strong Jordan ideal of \( R \), where \( R \) is a ring with unity. If \( v \in V \) and \( a, b \in R \), then \( avb + vba \in V \).

**Proof.** Let \( avb \) and \( b av \in V \). Then
\[
(3.3.2) (v+a)(v+b) \in V
\]
Now since \( V \) is a Jordan ideal, \( a(\underbrace{avb + bav}_V + (avb + vba)a \in V \). Therefore, by (3.2), \( avba + abva \in V \). Replacing \( a \) by \( (a - 1) \), we get
\[
(\underbrace{avb - vba}_V + (abv - bv)(a - 1) \in V
\]
This gives \( -vba - abv \in V \). Hence \( vba + abv \in V \).

Let \( V \) be a \( R \)-strong Jordan ideal of \( R \). If \( a, b \in R \), we associate \( V \) with the set \( A_V = \{ b \in R : ab + ba \in V, \text{ for all } a \in R \} \)
Theorem 3.8. If $V$ is an $R$-strong Jordan ideal of $R$, then $AV$ is an $R$-strong Jordan ideal of $R$.

Proof. Let $x \in AV$ and $r \in R$. Since $x \in AV, xr + rx \in V$. Also, since $V$ is a Jordan ideal of $R$, $(xr + rx)y + y(xr + rx) \in V$.

This gives $xr + rx \in AV$. Hence $AV$ is a Jordan ideal of $R$.

Let $b \in BJ, x, y \in R$. Since $b \in BJ, x, y \in R, xb + bx, yb + by \in V$. Since $V$ is an $R$-strong Jordan ideal, $(yb + yb)y \in V$. This implies that

$y^2by \in V$. 

Similarly, $x(by + yb)y \in V$ and $y(by + yb)x \in V$. This gives $xb + yby \in V$ and $ybyx + y^2bx \in V$.

Hence, by (3.3) $x(by + yb) + (by)x \in V$. Hence $AV$ is $R$-strong.

Theorem 3.9. If $R$ is a ring with $2R = R$ and $V$ is an $R$-strong Jordan ideal of $R$, then $AV \cap V$ is a non-zero right ideal of $R$.

Proof. Note that $AV \cap V \neq (0)$. Let $b \in AV \cap V, x, y \in R$. Then $bx + xb \in V$. So, $bx + xb \in AV$. Hence $bx + xb \in AV \cap V$. Now

$x + bx = xb + bx + xb - xb$
$= bx - xb + 2xb$
$= bx - xb + xb$ (::* $2R = R$)
$= bx \in AV \cap V$

Since $x \in R$ is arbitrary, $bx \in AV \cap V$, for all $x \in R$. Hence $AV \cap V$ is a non-zero right ideal of $R$.

Theorem 3.10. If $e$ is an idempotent and $V$ is a Jordan ideal of $R$, then $eVe$ is an $eRe$-strong Jordan ideal of $R$.

Proof. Let $x \in eVe$ and $r \in eRe$. Then

$x + rx = e(vr_1 + r_1v), \ v \in V, r_1 \in R$
$\in eVe$ (::* $r_1e = r_1 = er_1$)

Again, let $u \in eVe$ and $x, y \in eRe$. Then

$xuy = e(xvy)e \in eVe$.

References


Received: March, 2009