Compact Primitive Semigroups Having (CEP)

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Abstract

Compact completely simple semigroups having congruence extension property (in brevity (CEP)) were first studied by Dumesnil in 1997. In this paper, we study the compact primitive semigroups having (CEP) and characterize such semigroups, so that the result of Dumesnil on compact completely simple semigroups having (CEP) is extended to compact primitive semigroups.

Keywords and Phrases: Compact primitive semigroup; Congruence extension property; Regular element; Voltatic ideal

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1 Introduction

A topological semigroup is a Hausdorff space $S$ endowed with with a jointly continuous function acting on $S$ as its semigroup multiplication. We call a topological semigroup $S$ having the congruence extension property, in brevity $(CEP)$, if for each closed subsemigroup $T$ of $S$ and each closed congruence $\sigma$ on $T$, $\sigma$ has a closed extension to $S$. Obviously, if a compact semigroup $S$ has $(CEP)$ then each closed subsemigroup of $S$ also has $(CEP)$.

The congruence extension property of a compact completely simple semigroup was first studied by Dumesnil [4]. She proved that $S$ is a compact completely simple semigroup having $(CEP)$ if and only if $S$ is isomorphic to a direct product semigroup $I \times G \times \Lambda$, where $I$ is a left trivial semigroup, $\Lambda$ is a right trivial semigroup and $G$ is a compact group having $(CEP)$. In this paper, we study the compact primitive semigroup having $(CEP)$ and extend the above result of Dumesnil from compact completely simple semigroups to compact primitive semigroup having $(CEP)$ with no non-trivial nilpotent elements. For notations and terminologies of semigroups and topological semigroups not given in this paper, the reader is referred to the texts of Howie, as well as Carruth, Hilderbrandt and Koch (see [9], [2] and [3]).

2 Preliminaries

We first cite some definitions and results.

**Definition 2.1** (i) A semigroup $S$ is said to be eventually regular [10] if for all $x \in S$, $x^n$ is regular for some positive integer $n$, where $n$ depends on $x$. The smallest integer $n(x)$ for $x^{n(x)}$ to be regular is called the regular index of $x$, denoted by $r(x)$.

(ii) A semigroup $S$ is said to be a $D$-semigroup if $S$ contains exactly one regular $D$-class.

(iii) A semigroup $S$ is called a GV-semigroup [1] if $S$ is an eventually regular semigroup and every regular element of $S$ is $H$-related to an idempotent of $S$.

(iv) An idempotent $e$ of a semigroup $S$ is said to be primitive if for all $f \in E(S)$, $f \leq e$ implies $f = e$ or $f = 0$ when $S$ has a zero. Naturally, we call a semigroup $S$ primitive if every idempotent of $S$ is primitive.

**Notation 2.2** If $x^{r(x)}$ is contained in a subgroup $H$ of a semigroup $S$, then we denote the identity of such largest subgroup in $S$ by $x^\circ$. 
**Definition 2.3** If $I$ is an ideal of a semigroup $S$, then we call $I$ a voltaic ideal of $S$ provided the following conditions are satisfied:

(a) for all $a, b \in S$, either $ab \in I$ or $a^2 = b^2 = ab$;

(b) if for $a, b \in S$, $ab$ and $ba$ are both not contained in $I$, then for all $z \in I$,

\[
az \in I \iff bz \in I \quad \text{and} \quad za \in I \iff zb \in I;
\]

(c) for all $a, b \in S$, if $ab \in I$, then

\[
a \circ ab = ab = abb \circ.
\]

**Lemma 2.4** [7] If $S$ is a compact semigroup having (CEP), then $S$ is a semilattice of compact D-semigroups having (CEP).

**Lemma 2.5** [8] Let $S$ be a compact semigroup whose set of all regular elements $\text{Reg}S$ forms an ideal. Then $S$ has (CEP) if and only if $\text{Reg}S$ is a compact orthogroup having (CEP) such that $S$ is a nil extension of $\text{Reg}S$ which is a voltaic ideal of $S$.

We now consider the congruence on the 0-direct union of completely 0-simple semigroups. For the 0-direct union of semigroups, the reader can be referred to [9, p.71].

**Lemma 2.6** Let $S$ be the 0-direct union of completely 0-simple semigroups $S_\alpha$ for $\alpha \in Y$. Then an equivalent relation $\rho$ on $S$ becomes a congruence on $S$ if and only if there exist a subset $A$ of $Y\setminus A$ and a proper congruence $\rho_\beta$ on the completely 0-simple semigroup $S_\beta$ for all $\beta \in A$, such that

\[
\rho = (\bigcup_{\alpha \in A} \rho_\alpha) \cup \nabla_A.
\]

**Proof:** By routine checking, we can easily show that the converse part of the lemma holds. On the other hand, if $\rho$ is a congruence on $S$, then the restriction $\rho_\alpha = \rho|_{S_\alpha}$ of $\rho$ on $S_\alpha$ is obvious a congruence on $S_\alpha$ for all $\alpha \in Y$. Observe that the congruence class containing the zero element of $S_\alpha$ is an ideal of $S_\alpha$, so either the $\rho_\alpha$-class of $S_\alpha$ containing 0 is the $\{0\}$ or $\rho_\alpha = \nabla_{S_\alpha}$ is the universal relation on $S_\alpha$ (see, [9, p.73]). In the former case, $\rho_\alpha$ is proper. And, we denote the set $\{\alpha \in Y : \rho_\alpha \text{ is proper}\}$ by $A$. Because the congruence on $S$ is compatible with multiplication, we can deduce that for every $x \in S_\alpha \setminus \{0\}$ and $y \in S_\beta \setminus \{0\}$ with $\alpha \neq \beta$, $xypy$ implies $xpye = 0$, where $e \in L_x \cup E(S_\alpha)$ and hence $y \rho 0$. This shows that the $\rho_\alpha$-class of $S_\alpha$ containing 0 and the $\rho_\beta$-class of $S_\beta$ containing 0 are the nonzero ideals of $S_\alpha$ and $S_\beta$, respectively. Note that on a completely 0-simple semigroup, the congruence class containing 0 is $\{0\}$. Accordingly, $\rho_\alpha = \nabla_{S_\alpha}$ and $\rho_\beta = \nabla_{S_\beta}$ if $\alpha \neq \beta$ and there exist $x \in S_\alpha$ and $y \in S_\beta$ such that $xpy$. Thus $\rho = (\bigcup_{\alpha \in A} \rho_\alpha) \cup \nabla_A$. \(\Box\)
3 Main result

Let us first extend [4, Theorem 1] to compact completely 0-simple semigroups having (CEP).

**Lemma 3.1** Let $S$ be a compact completely 0-simple semigroup. Then $S$ has (CEP) if and only if $S$ is a rectangular group $I \times G \times \Lambda$ adjoining a zero element, where $I$ is a left trivial semigroup, $\Lambda$ is a right trivial semigroup and $G$ is a compact group having (CEP).

**Proof:** $\Rightarrow$ Suppose that $S$ has (CEP). Then, by Lemma 2.4, $S$ is a semilattice $Y$ of compact $D$-semigroups $S_\beta$ having (CEP) with $\beta \in Y$. Notice that $S$ has exactly two $D$-classes, namely, $D$ and $\{0\}$. Hence, $S = D^0$ and $S_{a\beta} = D$, for some $\beta \in Y$. By Lemma 2.4, this shows that $D$ is a completely simple semigroup. Now, by [4, Theorem 1], we can immediately see that $D$ is isomorphic to a compact rectangular group $I \times G \times \Lambda$ in which the group $G$ is of (CEP). This proves the necessary part of the lemma.

$\Leftarrow$ Assume that $S$ satisfies the condition in the lemma. Then $S \setminus \{0\}$ is a completely simple semigroup having (CEP). Let $T$ be a closed completely simple subsemigroup of $S$ and $\rho$ be a closed congruence on $T$. We now consider the following two cases:

1. $T$ contains no zero element $0$. In this case, $T \subseteq S \setminus \{0\}$. Hence there exists a closed congruence $\sigma$ on $S \setminus \{0\}$ such that $\sigma \cap (T \times T) = \rho$. Now, by Lemma 2.6, $\sigma \cup \Delta_S$ extends $\rho$ from $T$ to $S$.

2. $T$ contains a zero element $0$. Since $S \setminus \{0\}$ is a rectangular group, $T \setminus \{0\} = T \cap (S \setminus \{0\})$ is a closed completely simple subsemigroup of $S \setminus \{0\}$. By Lemma 2.5, either $\rho = \nabla_T$ or there exists a closed congruence $\rho'$ on $S \setminus \{0\}$ such that $\rho = \rho' \cup \Delta_T$.
   - If $\rho = \nabla_T$, then $\nabla_S$ extends $\rho$ from $T$ to $S$.
   - If $\rho = \rho' \cup \Delta_T$, then, because $S \setminus \{0\}$ has (CEP), there exists a closed congruence $\sigma'$ on $T \setminus \{0\}$ which extends $\rho'$ on $S \setminus \{0\}$. By applying Lemma 2.6 again, $\sigma = \sigma' \cup \Delta_S$ is a closed congruence on $S$ and we can easily see that $\sigma$ indeed extends $\rho$ from $T$ to $S$.

Thus, $\rho$ can be extended from $T$ to $S$, and consequently $S$ has (CEP). $\square$

We now consider the compact primitive semigroups having (CEP).
Lemma 3.2 Let $S$ be a compact semigroup which is a primitive regular semigroup. If $S$ is the 0-direct union of completely 0-simple semigroups $S_\alpha$ with $\alpha \in Y$, then $S$ has (CEP) if and only if every $S_\alpha$ is isomorphic to a rectangular group $I_\alpha \times G_\alpha \times \Lambda_\alpha$ adjoined with 0 in which each $G_\alpha$ is a compact group having (CEP).

Proof: $\Rightarrow$ Suppose that $S$ has (CEP). Because $S_\alpha$ is a principal ideal of $S$, $S_\alpha$ is a closed ideal of $S$ and each $S_\alpha$ clearly has (CEP). By Lemma 3.1, $S_\alpha$ is isomorphic to a rectangular group $I_\alpha \times G_\alpha \times \Lambda_\alpha$ adjoined with 0, where $G_\alpha$ is a compact group having (CEP).

$\Leftarrow$ Assume that $S$ satisfies the conditions of the theorem. Then, by [4, Theorem 1], every $S_\alpha$ has (CEP). Now let $T$ be a closed subsemigroup of $S$ and $\rho$ a closed congruence on $T$. Then, $T$ is obviously the 0-direct union of compact semigroups $T_\alpha = T \cap S_\alpha$, with $\alpha \in Y$. Note that each $\mathcal{H}$-class of a compact semigroup is closed, we observe that each $S_\alpha$ is a union of some closed subgroups and $T_\alpha$ is hence a union of closed cancellative semigroups. Recall that any compact cancellative semigroup is a compact group (see [2]), we observe that each $T_\alpha$ is either a completely 0-simple semigroup or a completely simple semigroup, and hence $T$ is the 0-direct union of completely 0-simple semigroups $T_\alpha$, with $\alpha \in Y$. Denote by $X$ the set $\{\alpha \in Y : T_\alpha \neq \emptyset\}$. Then, by Lemma 2.6, there exist a subset $A$ of $X$, the universal relation $\nabla_A$ on the 0-direct union of $T_\alpha$ for $\alpha \in X \setminus A$ and a proper congruence $\rho_\beta$ on $T_\beta$, for all $\beta \in A$ such that $\rho = (\bigcup_{\alpha \in A} \rho_\alpha) \cup \nabla_A$. Because each $S_\alpha$ has (CEP), there exists a closed congruence $\sigma_\alpha$ on $S_\alpha$ such that $\sigma_\alpha \cap (T_\alpha \times T_\alpha) = \rho_\alpha$, for all $\alpha \in A$. But since $\rho_\alpha$ is proper for $\alpha \in A$, $\sigma_\alpha$ is clearly proper for $\alpha \in A$. Again by Lemma 2.6, $\sigma = (\bigcup_{\alpha \in A} \sigma_\alpha) \cup (\bigcup_{\alpha, \beta \in X \setminus A} S_\alpha \times S_\beta) \cup (\bigcup_{\alpha \in Y \setminus X} \Delta_{S_\alpha})$, where $\Delta_{S_\alpha}$ is the identity relation on $S_\alpha$. Thus, $\sigma$ is a congruence on $S$ with $\sigma \cap (T \times T) = \rho$.

It remains to verify that $\sigma$ is a closed congruence on $S$. By [9, Theorem 3.5], the usual Green’s relation $\mathcal{J}$ is a congruence on $S$ such that each $S_\alpha$ is one of the $\mathcal{J}$-class. Since $S$ is itself a compact semigroup, $\mathcal{J}$ is of course a closed relation on $S$ (see [2]). This implies that the mapping $\phi : S \to S/\mathcal{J}$ induced by $\mathcal{J}$ is continuous. Accordingly, $\phi$ is a closed mapping because $S$ is a compact space and $S/\mathcal{J}$ is a Hausdorff space. Hence, the mapping

$$
\psi : S \times S \to S/\mathcal{J} \times S/\mathcal{J}; (x, y) \mapsto (x\phi, y\phi)
$$

is a closed and continuous mapping. Thus $(\rho)\psi$ is a closed relation on $S/\mathcal{J}$ and hence, $\tau = (\bigcup_{\alpha \in A} S_\alpha \times S_\alpha) \cup (\bigcup_{\alpha, \beta \in X \setminus A} S_\alpha \times S_\beta) = (\rho)\psi^{-1}$ is a closed subset of $S \times S$. Clearly, every $(S_\alpha \times S_\alpha) \setminus \sigma_\alpha$ is open, and we immediately see that $\bigcup_{\alpha \in A} [(S_\alpha \times S_\alpha) \setminus \sigma_\alpha]$ is also open. Thus $(S \times S) \setminus \sigma = ((\bigcup_{\alpha \in A} S_\alpha \times S_\alpha) \setminus \sigma_\alpha) \cup ((S \times S) \setminus \tau) \cap ((S \times S) \setminus \Delta_S)$ is open and consequently, $\sigma$ is a closed
congruence on $S$. This shows that the congruence $\sigma$ is an extension of $\rho$ to $S$. In other words, $S$ has (CEP). \hfill \Box

**Notation 3.3** Denote the set $\{ \beta \in Y : \beta \leq \alpha \}$ by $\omega(\alpha)$, where $Y$ is a semi-lattice.

We now formulate the following lemma for nil extension of rectangular groups.

**Lemma 3.4** Let $N_\alpha$ be a nil extension of rectangular group $S_\alpha = I_\alpha \times G_\alpha \times \Lambda_\alpha$. If $S = \bigcup_{\alpha \in Y} N_\alpha$ is a primitive semigroup, then $|\omega(\alpha)| \leq 2$, for all $\alpha \in Y$.

**Proof:** Suppose, on the contrary, that $|\omega(\alpha)| \geq 3$ for some $\alpha \in Y$. Let $\beta, \gamma \in \omega(\alpha)$. Then $\gamma \beta < \beta < \alpha$ and so we can let $\gamma \beta < \beta < \alpha$. We first claim that $\text{Reg}_{S_\alpha} \bigcup \text{Reg}_{S_\beta} \bigcup \text{Reg}_{S_\gamma}$ is a regular subsemigroup of $S$. To prove our claim, we let $e \in E(S_\alpha)$ and $f \in E(S_\beta)$. Then $ef \in S_\beta$. Since $\text{Reg}_{S_\beta}$ is an ideal of $S_\beta$ and by $ef = ef \circ f$ and $ef \in \text{Reg}_{S_\beta}$, we consequently obtain that $\text{Reg}_{S_\alpha} \text{Reg}_{S_\beta} \subseteq \text{Reg}_{S_\beta}$. Similarly, we can also obtain that $\text{Reg}_{S_\beta} \text{Reg}_{S_\gamma} \subseteq \text{Reg}_{S_\beta}$, $\text{Reg}_{S_\alpha} \text{Reg}_{S_\gamma} \subseteq \text{Reg}_{S_\gamma}$, $\text{Reg}_{S_\beta} \text{Reg}_{S_\alpha} \subseteq \text{Reg}_{S_\gamma}$, $\text{Reg}_{S_\beta} \text{Reg}_{S_\gamma} \subseteq \text{Reg}_{S_\beta}$, and $\text{Reg}_{S_\gamma} \text{Reg}_{S_\beta} \subseteq \text{Reg}_{S_\gamma}$. Hence, we have verified that $\text{Reg}_{S_\alpha} \bigcup \text{Reg}_{S_\beta} \bigcup \text{Reg}_{S_\gamma}$ is a regular subsemigroup of $S$ and our claim is thus established. It is now clear that the principal factors of the regular semigroup $\text{Reg}_{S_\alpha} \bigcup \text{Reg}_{S_\beta} \bigcup \text{Reg}_{S_\gamma}$ are $\text{Reg}_{S_\alpha}^0$, $\text{Reg}_{S_\beta}^0$ and $\text{Reg}_{S_\gamma}^0$. It has been known that a semigroup $S$ is an orthodox semigroup if and only if its principal factors are orthodox (see, [9, Ex.2, p.109]). Thus the semigroup $\text{Reg}_{S_\alpha} \bigcup \text{Reg}_{S_\beta} \bigcup \text{Reg}_{S_\gamma}$ is an orthodox semigroup. Now, it is easy to deduce that $(gf)g)(gf) < gf < g$, for $e \in E(S_\alpha), f \in E(S_\beta)$ and $g \in E(S_\gamma)$. However, this contradicts our hypothesis that $g$ is a primitive idempotent of $S$. Hence, $|\omega(\alpha)| \leq 2$ and our proof is now completed. \hfill \Box

Finally, we proceed to characterize the compact primitive semigroups with no non-trivial nilpotent elements. The following is the main theorem of this paper.

**Theorem 3.5 (Main Theorem)** Let $S$ be a compact primitive semigroup with zero. If $S$ contains no non-trivial nilpotent elements, then $S$ has (CEP) if and only if $\text{Reg}S$ is a voltaic ideal of $S$ such that $S$ is a nil extension of $\text{Reg}S$ and $\text{Reg}S$ can be expressible as the 0-direct union of rectangular groups $\text{RG}_\alpha = I_\alpha \times G_\alpha \times \Lambda_\alpha$ adjoined with a zero element, where $G_\alpha$ is a compact group having (CEP), for every $\alpha$. 


**Proof:** Let $S$ be a compact primitive semigroup having (CEP). Then, by the above lemma and [5, Theorem 2.1], $S$ is a semilattice $Y$ of compact $D$-semigroups $S_\alpha$, for $\alpha \in Y$. Obviously, each $S_\alpha$ has (CEP). By using a result of X. J. Guo in [7], $S_\alpha$ is a nil-extension of a compact rectangular group $I_\alpha \times G_\alpha \times \Lambda_\alpha$ and $G_\alpha$ has (CEP). Now let $0 \in S_\tau$. Then $S_\tau = \{0\}$ because $S$ contains no non-trivial nilpotent elements. On the other hand, by Lemma 3.4, $|\omega(x)| \leq 2$ for all $x \in Y$ and hence, $S_\alpha S_\beta = \{0\}$ if $\alpha \neq \beta$. This shows that $RegS$ is the 0-direct union of $RegS_\alpha^0$ for $\alpha \in (Y \setminus \{\tau\})$ and $RegS$ is an ideal of $S$. The rest of the proof follows from Lemma 2.5.

Conversely, suppose that $S$ satisfies the conditions of the theorem. Then, by Lemmas 2.5 and 3.4, it suffices to verify that $RegS$ is an ideal of $S$. In order to prove that $S$ is an eventually-regular semigroup, we suppose, on the contrary, that $x^n$ is not a regular element for all positive integer $n$. Then, since the ideal $RegS$ is a voltaic ideal of $S$, $x^3 = x \circ x^2 = x^2$ (by Definition 2.4 (a)) and so we can deduce that
\[ x^2 \bullet x^2 \bullet x^2 = x^4 = x^3 = x^2, \]
that is, $x^2$ is regular. This is clearly a contradiction. Thus, $S$ is an eventually-regular semigroup. Now, by our hypothesis, $S$ is the so called GV-semigroup and furthermore, by the results of Bogdanović [1,p 118-130], $S$ is a semilattice $Y$ of $D$-semigroups $S_\alpha$ which is a nil-extension of $RegS_\alpha$, for $\alpha \in Y$. Since $RegS$ is the 0-direct union of rectangular groups $RG_\beta$ adjoining zero, $RegS_\alpha$ can be expressed by some rectangular groups $RG_\beta$. By using the same arguments, we can also show that $RG_\alpha RG_\beta = \{0\}$ if $\alpha \neq \beta$, and thereby $RegS_\alpha RegS_\beta = \{0\}$ if $\alpha \neq \beta$. This shows that $S_\alpha S_\beta \subseteq S_0$ if $\alpha \neq \beta$, where $S_0$ denotes the semigroup $S_\alpha$ containing 0. Since $S$ contains no non-trivial nilpotent elements, $S_0 = \{0\}$ and $S_\alpha S_\beta = \{0\}$ if $\alpha \neq \beta$. Thus $RegS$ is indeed an ideal of $S$, as required. This completes the proof. \[ \square \]

From Theorem 3.5 and the fact that a semigroup without zero always contains no non-trivial nilpotent elements, we deduce the following corollary.

**Corollary 3.6** A compact primitive semigroup $S$ without zero has (CEP) if and only if $RegS$ is a compact rectangular group having (CEP) such that $S$ is a nil extension of $RegS$ and $RegS$ is a voltaic ideal of $S$.

**References**


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