From Pro-$C$ Simplicial Groups to
Pro-$C$ Crossed Complexes

Alper Odabaş

Eskişehir Osmangazi University
Art and Science Faculty
Department of Mathematics and Computer Sciences
TR-03200, Eskişehir, Turkey
aodabas@ogu.edu.tr

Abstract

In this paper, we will give a short proof of the construction of the pro-$C$ crossed complexes of pro-$C$ groups by using the higher order Peiffer elements.

Mathematics Subject Classification: 18G50, 18G30, 55P10

Keywords: Profinite Groups, Simplicial Groups, Crossed complexes

1 Introduction

The theory of profinite and in particularly pro-$p$ groups has several times provided significant results for the theory of presentations of finite groups. The most remarkable example of this is, of course, the Golod-Šafarevič Theorem (c.f. [9]).

F.J. Korkes and T. Porter (c.f. [7]) introduced profinite or pro-$C$ crossed complexes using as motivation the idea that they might encode more of the information analogous to that in a $K(G, 1)$, than did the chain complex type constructions derived from the bar resolution. Pro-$C$ crossed complexes in combinatorial and cohomological algebra theory have the advantage of being less cumbersome than the full simplicial theory, but certain structural invariants are lost when they are used as such crossed resolutions do not represent all the possible homotopy types available. It is therefore important to be able to go from the simplicial context to the crossed one and to study what is lost in the process.

Let $G$ be a simplicial group with Moore complex $NG$ and for $n \geq 0$, let $D_n$ be the subgroup generated by the degenerate elements in dimension $n$. Carrasco
and Cegarra [5], calculated the relative homotopy group \( \pi_n(\text{sk}_n G, \text{sk}_{n-1} G) \) for a simplicial group \( G \) and proved it equal to

\[
C_n(G) = \frac{\text{NG}_n}{(\text{NG}_n \cap D_{n+1})d_{n+1}(\text{NG}_{n+1} \cap D_{n+1})}.
\]

for each \( n \). This constructs a crossed complex of groups from the Moore complex \( \text{NG} \) of \( G \). Their proof requires an understanding of hypercrossed complexes. Ehlers and Porter [6] developed a direct proof for simplicial groups/groupoids independently of [5], and their methods establish the profinite case too. Here we adapt the original approach of Carrasco and Cegarra [5], and give a short proof of this result by using the higher order Peiffer elements.

**Terminology**

In this paper \( \mathcal{C} \) will denote a class of finite groups which is closed under the formation of subgroups, homomorphic images, finite products and which contains at least one non-trivial group. Pro-\( \mathcal{C} \) groups are profinite groups whose finite quotients are in \( \mathcal{C} \).

The class \( \mathcal{C} \) will be assumed to be full in the sense that \( \mathcal{C} \) must also be closed under extension of groups.

## 2 Crossed modules and pro-\( \mathcal{C} \) crossed modules

We recall the definition of a crossed module, (see for instance Brown and Huebschmann [2] for a more detailed treatment), and introduce the pro-\( \mathcal{C} \) analogue.

A **crossed module** \( (C, G, \partial) \) consists of groups \( C \) and \( G \), a left action of \( G \) on \( C \), which will be written \((g, c) \rightarrow^g c\) for \( g \in G, c \in C \), and a group homomorphism \( \partial : C \rightarrow G \) satisfying the following conditions:

(CM1) for all \( g \in G \), and \( c \in C \)

\[
\partial(g^c) = g(\partial c)g^{-1};
\]

(CM2) for all \( c_1, c_2 \in C \)

\[
c_2c_1c_2^{-1} = \partial(c_2) c_1.
\]

((CM2) is often called the **Peiffer relation** or **identity**).

**Examples.** (1) For \( H \) a normal subgroup of \( G \), the inclusion homomorphism \( i : H \rightarrow G \) makes \((H, G, i)\) into a crossed module where \( G \) acts on \( H \) by conjugation.

(2) If \( M \) is a left \( G \)-module and \( 0 : M \rightarrow G \) is the zero homomorphism, then \((M, G, 0)\) is a crossed module.
Let \((C, G, \partial)\) and \((C', G', \partial')\) be crossed modules. A morphism

\[(\mu, \eta) : (C, G, \partial) \to (C', G', \partial')\]

of crossed modules consists of group homomorphism \(\mu : C \to C'\), \(\eta : G \to G'\) such that

(i) \(\partial' \mu = \eta \partial\) and
(ii) \(\mu(\eta g c) = \eta g \mu(c)\) for all \(g \in G\), and \(c \in C\).

This notion of morphism easily gives us a category \(C\text{mod}\) of crossed modules and crossed module morphisms. There are special classes of morphisms in which \(G = G'\) and \(\mu\) is the identity morphism. For fixed \(G\), such a morphism \((\mu, Id_G) : (C, G, \partial) \to (C', G', \partial')\) will be called a morphism of crossed modules over \(G\). These gives a subcategory \(C\text{mod}/G\) of \(C\text{mod}\).

The pro-\(C\) analogues of these concepts are now easy to give.

A **pro-\(C\) crossed module** \((C, G, \partial)\) is a crossed module in which \(C\) and \(G\) are pro-\(C\) topological groups, \(\partial\) is a continuous homomorphism and the left \(G\)-action of \(C\) is a continuous \(G\)-action. Closed normal subgroups give examples of such as do zero morphisms from pseudocompact left \(G\)-modules to \(G\), (see Brumer [3] for the theory of pseudocompact compact modules).

A morphism

\[(\mu, \eta) : (C, G, \partial) \to (C', G', \partial')\]

of pro-\(C\) crossed modules is a morphism of the underlying crossed modules in which both \(\mu\) and \(\eta\) are continuous morphisms of pro-\(C\) groups. This gives us categories \(\text{Pro-}C\text{mod}\) and \(\text{Pro-}C\text{mod}/G\) for \(G\) a pro-\(C\) group and also a functor

\[U_{C\text{mod}} : \text{Pro-}C\text{mod} \to C\text{mod}\]

which forgets the topology.

If \(G\) is a group, let \(\Omega(G)\) be directed set of normal finite index subgroup \(W\) of \(G\) with \(G/W \in \mathcal{C}\), then

\[\hat{G} \cong \lim_{W \in \Omega(G)} G/W\]

We will sometimes write \(W <_{\text{fin}} G\) as indicating that \(W \in \Omega(G)\).

### 3 Higher order Peiffer elements

Let \(S(n, n - r)\) be the set of all monotone increasing surjective continuous maps from the ordered set \([n] = \{0, 1, \ldots, n\}\) to the ordered set \([n - r] = \{0, 1, \ldots, n - r\}\)
the following rule. The composition of these generating continuous maps is subject to the following rule \( \sigma, \sigma_i = \sigma_{i-1} \sigma_j, \ j < i \). This implies that every element \( \sigma \in S(n, n-r) \) has a unique expression as \( \sigma = \sigma_{i_1} \circ \sigma_{i_2} \circ \ldots \circ \sigma_{i_r} \) with \( 0 \leq i_1 < i_2 < \ldots < i_r \leq n-1 \), where the indices \( i_k \) are the elements of \([n]\) such that \( \{i_1, \ldots, i_r\} = \{i : \sigma(i) = \sigma(i+1)\} \). We thus can identify \( S(n, n-r) \) with the set \( \{(i_r, \ldots, i_1) : 0 \leq i_1 < i_2 < \ldots < i_r \leq n-1\} \). In particular, the single element of \( S(n, n) \), defined by the identity map on \([n]\), corresponds to the empty 0-tuple \((\ )\) denoted by \( \emptyset_n \). Similarly the only element of \( S(n, 0) \) is \((n-1, n-2, \ldots, 0) \). For all \( n \geq 0 \), let

\[ S(n) = \bigcup_{0 \leq r \leq n} S(n, n-r). \]

Let \( P(n) \) be a set consisting of pairs of elements \((\alpha, \beta)\) from \( S(n) \) with \( \alpha \cap \beta = \emptyset \), where \( \alpha = (i_r, \ldots, i_1), \beta = (j_s, \ldots, j_1) \in S(n) \). We write \#\(\alpha = r\), i.e. the length of the string \(\alpha\). The linear continuous morphisms that we will need,

\[ \{F_{\alpha, \beta} : NG_{n-\#\alpha} \times NG_{n-\#\beta} \rightarrow NG_n : (\alpha, \beta) \in P(n), \ n \geq 0\} \]

are given as composites \( F_{\alpha, \beta} = p\mu(s_{\alpha} \times s_{\beta}) \) where

\[ s_\alpha = s_{i_r} \ldots s_{i_1} : NG_{n-\#\alpha} \rightarrow G_n, \ s_\beta = s_{j_s} \ldots s_{j_1} : NG_{n-\#\beta} \rightarrow G_n, \]

\[ p : G_n \rightarrow NG_n \]

is defined by composite continuous projections \( p = p_{n-1} \ldots p_0 \), where \( p_j(z) = zs_jd_j(z)^{-1} \) with \( j = 0, 1, \ldots n-1 \) and \( \mu : G_n \times G_n \rightarrow G_n \) is given by the commutator. Thus

\[ F_{\alpha, \beta}(x_\alpha, y_\beta) = (1s_{n-1}d_{n-1}^{-1}) \ldots (1s_0d_0^{-1})[s_\alpha(x_\alpha), s_\beta(y_\beta)]. \]

Define the closed normal subgroup \( N_n \) to be that generated by elements of the form \( F_{\alpha, \beta}(x_\alpha, y_\beta) \) where \( x_\alpha \in NG_{n-\#\alpha} \) and \( y_\beta \in NG_{n-\#\beta} \).

The idea for the construction of \( N_n \) and the use of the structure maps came from examining the thesis of Carrasco [4], see also Carrasco and Cegarra, [5].

The following result is the pro-\( \mathcal{C} \) analogue of the group version given in [?] by Arvasi.

Let \( G \) be a pro-\( \mathcal{C} \) simplicial group with Moore complex \( NG \) and for \( n > 1 \), let \( D_n \) be the closed normal subgroup generated by the degenerate elements in dimension \( n \). If \( G_n = D_n \), then

\[ \partial_n(NG_n) = \partial_n(N_n) \quad \text{for all } n > 1, \]

where \( N_n \) is a closed normal subgroup in \( G_n \) generated by a fairly small explicitly given set of elements, see below.
If \( n = 2, 3 \) or \( 4 \), then the image of the Moore complex of the pro-\( \mathcal{C} \) simplicial group \( G \) can be given in the form

\[
\partial_n(NG_n) = \prod_{I,J} [K_I, K_J]
\]

where the square brackets denote the closed commutator subgroup and \( \emptyset \neq I, J \subset [n - 1] = \{0, 1, \ldots, n - 1\} \) with \( I \cup J = [n - 1] \), and where

\[
K_I = \bigcap_{i \in I} \ker d_i \quad \text{and} \quad K_J = \bigcap_{j \in J} \ker d_j.
\]

In general for \( n > 4 \), there is an inclusion

\[
\prod_{I,J} [K_I, K_J] \subseteq \partial_n(NG_n).
\]

4 From pro-\( \mathcal{C} \) simplicial groups to pro-\( \mathcal{C} \) crossed complexes

4.1 Construction of crossed complex

The final elements that we need are the definition of a pro-\( \mathcal{C} \) crossed complex of pro-\( \mathcal{C} \) groups, and a construction of a pro-\( \mathcal{C} \) crossed complex from a pro-\( \mathcal{C} \) simplicial group. The proof that this works uses these \( F_{\alpha,\beta} \) maps in a neat way.

We recall from [8] the definition of a pro-\( \mathcal{C} \) crossed complex as follows:

A pro-\( \mathcal{C} \) crossed complex of pro-\( \mathcal{C} \) groups which will be denoted \( \mathbf{C} \) consists of a sequence of pro-\( \mathcal{C} \) groups and continuous morphisms

\[
\mathbf{C} : \quad \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0
\]

satisfying the following

i) \((C_1, C_0, \partial_1)\) is a pro-\( \mathcal{C} \) crossed module;

ii) each \( C_n \), \( (n > 1) \), is a pseudocompact left \( \hat{\mathbb{Z}}_\mathcal{C} [C_0/\partial_1 C_1] \)-module and each for \( n > 1 \), \( \partial_n \) a continuous morphism of pseudocompact left \( \hat{\mathbb{Z}}_\mathcal{C} [C_0/\partial_1 C_1] \)-module, (for \( n = 2 \) this means that \( \partial_2 \) commutes with the action of \( C_0 \) and that \( \partial_2(C_2) \subset C_1 \) must be a pseudocompact \( \hat{\mathbb{Z}}_\mathcal{C} [C_0/\partial_1 C_1] \)-module);

iii) for \( n \geq 1 \), \( \partial_{n+1} \partial_n = 0 \).

The notion of a continuous morphism of pro-\( \mathcal{C} \) crossed complexes is clear.

We thus get a category, \textbf{Pro-\( \mathcal{C} \).Crs.}

We should first show that the quotient pro-\( \mathcal{C} \) group does exist.
Lemma 1. The closed subgroup \((NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})\) is a closed normal subgroup in \(G_n\).

Proof. It is straightforward from a direct calculation. \(\Box\)

Theorem 1. Let \(G\) be a pro-\(\mathcal{C}\) simplicial group, then defining

\[C_n(G) = \frac{NG_n}{(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})}\]

with

\[\partial_n(z) = \overline{d_nz}\]

gives a pro-\(\mathcal{C}\) crossed complex \(C(G)\) of pro-\(\mathcal{C}\) groups.

Proof. (i) follows since

\[C_1(G) = \frac{NG_1}{\partial_2(NG_2 \cap D_2)} = \frac{NG_1}{[\text{Ker} d_0, \text{Ker} d_1]},\]

and one can easily show that the closed normal subgroup \([\text{Ker} d_0, \text{Ker} d_1]\) contains the Peiffer elements so \((C_1(G), C_0(G), \partial)\) is a pro-\(\mathcal{C}\) crossed module, i.e.

\[d_1 : \frac{NG_1}{[\text{Ker} d_0, \text{Ker} d_1]} \rightarrow NG_0.\]

(ii) For \(n \geq 2\), \(C_2(G)\) is abelian, in fact

\[F_{(n-1)(n)}(x, y) = [s_{n-1}x, s_n y][s_n y, s_n x]\]
\[d_{n+1}F_{(n-1)(n)}(x, y) = [x, s_{n-1}d_n(y)][y, x]\]

is in \((NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})\), so \(d_{n+1}F_{(n-1)(n)}(x, y) \equiv 1 \text{ mod}(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})\) giving \(\overline{xy} = \overline{yx}\).

For \(x \in NG_n\) and \(y \in NG_m\), taking \(\alpha = (n, n - 1, \ldots, m)\), and \(\beta = (m - 1)\), it is easy to see that

\[F_{\alpha, \beta}(x, y) = \prod_{k=0}^{n-m-1} [s_n s_{n-1} \ldots s_{m+k}(x), s_{m+1+k}(y)](-1)^k [s_n \ldots s_{m}(x), s_n(y)].\]
and then
\[ d_{n+1}F_{\alpha,\beta}(x, y) = [s_{n-1} \ldots s_m(x), y] \prod_{k=0}^{n-m} [s_{n}s_{n-1} \ldots s_m(x), s_{m-1+k}(y)]^{(-1)^k}. \]

This implies that
\[ [s_m^{(n-m)}(x), y] \in (NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1}), \]
(where \( s_m^{(n-m)}(x) = s_{n-1} \ldots s_m(x) \)) which shows that the continuous actions of \( NG_m \) on \( NG_n \) defined by conjugation
\[ \varpi(y) = s_m^{(n-m)}(x)ys_m^{(n-m)}(x)^{-1} \]
via this degeneracies are trivial if \( m \geq 1 \). For \( m = 1 \), this gives \( \alpha = (n, n - 1, \ldots, 1) \), \( \beta = (0) \) and
\[ F_{(n,n-1,\ldots,1)(0)}(x, y) = [s_{n}s_{n-1} \ldots s_1(x), s_0(y)][s_{n}s_{n-1} \ldots s_1(x), s_1(y)]^{-1} \ldots \]
\[ [s_{n}s_{n-1} \ldots s_1(x), s_n(y)]^{(-1)^n} \]
where \( x \in NG_1, y \in NG_n \) and it is easily checked that
\[ d_{n+1}F_{\alpha,\beta}(x, y) = [s_{n-1} \ldots s_1(x), s_0d_n y] \cdots [s_{n-1} \ldots s_1(x), s_n d_n y][s_{n-1} \ldots s_1(x), y]^{(-1)^n} \]

Then
\[ [s_{n-1} \ldots s_1(x), y] \equiv 1 \mod (NG_n \cap D_n d_{n+1}(NG_{n+1} \cap D_{n+1})). \]

This gives the following; if \( \bar{x} \in C_1 \) then \( \bar{x} \) and \( \partial_1 \bar{x} \) act continuously on \( C_n \) in the same way, and so \( \partial_1 C_1 \) acts trivially on \( C_n \).

(iii) By defining
\[ \partial_n(\bar{z}) = d_n^n(\bar{z}) \quad \text{with} \quad \bar{z} \in NG_n, \]
one obtains a well-defined map \( \partial : C_n(G) \to C_{n-1}(G) \) verifying \( \partial \partial = 0 \). \( \square \)

References


[3] A. Brumer, Pseudocompact algebras, profinite groups and class forma-

[4] P. Carrasco, Complejos Hipercruzados Cohomologia y Extensiones,


[7] F.J. Korkes and T. Porter, Continuous Derivations, Profinite crossed
complexes and pseudocompact chain complexes. U.C.N.W. Pure Math
Preprint 87.-.

[8] F.J. Korkes and T. Porter, Profinite Homotopical and Homologcal
Algebra. In Preparation.


Received: March, 2009