The $c$–Supplemented Subgroups and $p$-Nilpotency of Finite Groups

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Abstract

The key of this paper is to determine the $p$-nilpotence of finite groups under assumption that some subgroups of Sylow subgroups of $G$ are $c$–supplemented.

Mathematics Subject Classification: 20D10, 20D20

Keywords: Sylow subgroups, $c$-supplemented subgroups, $p$-nilpotent

1 Introduction

In this paper, all groups are finite. A well-known theorem of Burnside (10.1.8 of [1]) asserts that if some prime $p$ a Sylow $p$–subgroup $P$ of a finite group $G$ lies in the center of its normalizer, then $G$ is $p$-nilpotent. Another well-known result due to Frobenius(10.3.2 of [1]) shows that a finite group $G$ is $p$-nilpotent iff every $p$-subgroup if centralized by the $p'$-elements in its normalizer. Thompson [2] improved the result of Frobenius.Zhang [3] furthered the results of Thompson. Some notions and notations, we refer to [1].

2 Some Lemmas

Lemma 2.1 (Lemma 2.1 of [4]). Let $G$ be a group. Then
(1) If $H$ is $c$-supplemented in $G$, $H \leq M \leq G$, then $H$ is $c$-supplemented in $M$.

(2) Let $N \triangleleft G$ and $N \leq H$. Then $H$ is $c$-supplemented in $G$ iff $H/N$ is $c$-supplemented in $G/N$.

(3) Let $\pi$ be a set of primes. Let $N$ be a normal $\pi'$-subgroup and let $H$ be a $\pi$-subgroup of $G$. If $H$ is $c$-supplemented in $G$, then $HN/N$ is $c$-supplemented in $G/N$. If further $N$ normalizes $H$, then converse also holds.

(4) Let $H \leq G$ and $L \leq \Phi(H)$. If $L$ is $c$-supplemented in $G$, then $L \leq \Phi(G)$ and $L \leq \Phi(G)$.

Lemma 2.2. Let $G$ be a group and $p$ a prime number.

(1) If $P$ is a minimal normal $p$-subgroup of $G$, and $x \in P$ is $c-$supplemented in $G$, then $P = \langle x \rangle$.

(2) Let $P$ be a normal $p$-subgroup of $G$ and $x$ be an element of $P - \Phi(P)$. If $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ and $x$ is $c$-supplemented in $G$, then $P = \langle x \rangle$.

Proof. (1) Since $\langle x \rangle$ is $c-$supplemented in $G$, then there exists a subgroup $K$ such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K \leq \langle x \rangle_G$. Let $P_1 = P \cap K$. Since $P$ is a minimal normal subgroup of $G$, then $P$ is either trivial or $P$. If $P_1 = 1$, then $P = P \cap G = \langle x \rangle (P \cap K) = \langle x \rangle$. Other $P_1 = P$ and hence $P \leq K, \langle x \rangle = \langle x \rangle \cap K \leq \langle x \rangle_G$. So $\langle x \rangle$ is a normal subgroup of $G$ and hence $P = \langle x \rangle$.

(2) Since $\langle x \rangle$ is $c-$supplemented in $G$, then there exists a subgroup $K$ such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K \leq \langle x \rangle_G$. Let $P_1 = P \cap K$. Hence $P_1 \triangleleft G$ and $P_1 \Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, $P_1 \Phi(P)/\Phi(P)$ is either trivial or $P/\Phi(P)$. If the former, then $P_1 \leq \Phi(P), P = P \cap G = \langle x \rangle (P \cap K) = \langle x \rangle \Phi(P) = \langle x \rangle$. Other $P_1 = P$ and hence $P \leq K, \langle x \rangle = \langle x \rangle \cap K \leq \langle x \rangle_G$. So $\langle x \rangle$ is a normal subgroup of $G$ and since $x$ is an non-identity of $P - \Phi(P)$, then $P = \langle x \rangle$. □

3 Main Results

Theorem 3.1. Let $P$ be a Sylow $p$-subgroup of $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. Then $G$ has a normal $p$-complement iff every cyclic subgroup of $P \cap O^p(G)$ with order $p$ or 4 (if $p = 2$) is $c$-supplemented in $G$. 
\textbf{Proof.} (i). First every cyclic subgroup of \( P \cap O^p(G) \) with order \( p \) or 4 (if \( p = 2 \)) is \( c \)-supplemented in \( G \). Suppose that the theorem is false and let \( G \) ba a counterexample with minimal order. Then we have the following situation:

(1) Let \( M \) be a proper subgroups of \( G \), then \( M \) is \( p \)-nilpotent.

Since \( P \) is a Sylow \( p \)-subgroup of \( G \), then \( P \cap M \) is a Sylow \( p \)-subgroup of \( M = G \cap M \). And since \( P \cap O^p(G) \cap M \leq (P \cap M) \cap (O^p(G) \cap M) \leq (P \cap M) \cap (O^p(M)) \leq P \cap O^p(G) \), then by hypotheses and lemma 2.1(1), we have that every cyclic subgroup of \((P \cap M) \cap O^p(M)\) with order \( p \) or 4 (if \( p = 2 \)) is \( c \)-supplemented in \( M \). Then \( M, (P \cap M) \cap O^p(M) \) satisfies the hypotheses. So by the minimal choice of \( G \), we have \( M \) is \( p \)-nilpotent. So \( G \) is not \( p \)-nilpotent but every proper subgroups of \( G \) is \( p \)-nilpotent. Then, by IV, 5.4 and III, 5.2(1) of [6], we have that \( G = PQ \), where \( P \) is a normal Sylow \( p \)-subgroup of \( G \), \( Q \) is a cyclic Sylow \( q \)-subgroup of \( G \) and \( Q \) is not normal in \( G \).

(2) Let \( L \) be a minimal normal subgroups of \( G \), then \( G/L \) is \( p \)-nilpotent.

Since \( P \) is a Sylow \( p \)-subgroup of \( G \), then \( PL/L \) is a Sylow \( p \)-subgroup of \( G/L \). And \((PL/L) \cap O^p(G)L/L = (P \cap O^p(G))L/L \), then by hypothesis and lemma 2.1(2) and lemma 2.1(3), we have every cyclic subgroup of \((P \cap O^p(G))L/L\) of order \( p \) or 4 is \( c \)-supplemented in \( G/L \). Thus by the minimal choice of \( G \), we have \( G/L \) is \( p \)-nilpotent.

(3) \( G/L \cap P \) is \( p \)-nilpotent.

By (1) and (2) and 2.6 of [7], we have \( G/P \cap L \leq G/P \times G/L \) is \( p \)-nilpotent.

(4) If \( \Phi(P) \neq 1 \) and \( L = P \).

If \( \Phi(P) = 1 \), then \( P \) is an abelian normal Sylow \( p \)-subgroup of \( G \), then \( P \leq Z(G) \), \( G \) is \( p \)-nilpotent, a contradiction. So we have \( \Phi(P) \neq 1 \).

If \( L \leq \Phi(P) \). Since \( P \triangleleft G \), \( \Phi(P) \leq \Phi(G) \). By (3), \( G/\Phi(P) \) is \( p \)-nilpotent, then \( G/\Phi(G) \) is \( p \)-nilpotent and so is \( G \), a contradiction. So \( L/\Phi(P) \) is a minimal normal \( p \)-subgroup of \( G/\Phi(P) \), this contradicts III, 5.2(2) of [6], then \( L = P \).

(5) To complete the proof.

By (4) and since \( P \) is a minimal normal Sylow \( p \)-subgroup of \( G \), then \( P =< x > \) by lemma 2.2 and III, 5.2(3)(4) of [6]. By lemma 2.8 of [5], we have \( P \leq Z(G) \). So \( G \) is \( p \)-nilpotent.

(ii). Second \( G \) is \( p \)-nilpotent. Then \( O^p(G) \) is a normal Hall \( p' \)-subgroup of \( G \), then \( P \cap O^p(G) = 1 \), so the necessary condition holds. □

\textbf{Theorem 3.2}. Let \( P \) be a Sylow \( p \)-subgroup of \( G \), where \( p \) is a prime divisor
of $|G|$ with that minimal subgroups of $P$ is in $Z_\infty(G)$. Then $G$ has a normal $p$-complement if and only if every cyclic subgroup of $P \cap O^p(G)$ with order 4 $(p = 2)$ is $c$-supplemented in $G$.

**Proof.** If $p$ is odd. If $G$ is $p$-nilpotent, then the minimal subgroup of $P$ is obvious in $Z_\infty(G)$. If every minimal subgroup of $P$ is in $Z_\infty(G)$, so $G$ is $p-$nilpotent by lemma 1.1 of [8]. Then we only think $p = 2$.

If $G$ is 2-nilpotent, then by theorem 1.1, we have every cyclic subgroup of $P \cap O^2(G)$ with order 4 is $c$-supplemented in $G$.

Then we suppose that every cyclic subgroup of $P \cap O^2(G)$ with order 4 $(p = 2)$ is $c$-supplemented in $G$. And suppose that the result is false, so we chose $G$ as a minimal counterexample. We will prove it in the following steps:

(1) Every proper subgroup of $G$ is 2-nilpotent.

Let $M$ be a proper subgroup of $G$. Let $P$ be a Sylow 2-subgroup of $G$, then $P \cap M$ is a Sylow 2-subgroup of $M = G \cap M$. Since $Z_\infty(G) \cap M \leq Z_\infty(M)$, then minimal subgroup of $P \cap M$ is in $Z_\infty(M)$. And since $P \cap O^2(G) \cap M \leq (P \cap M) \cap (O^2(G) \cap M) \leq (P \cap M) \cap (O^2(M)) \leq P \cap O^2(G)$, then the hypotheses and lemma 2.1(1) implies that every cyclic subgroup of $(P \cap M) \cap O^2(M)$ with order 4 is $c$-supplemented in $M$. Then $M, (P \cap M) \cap O^2(M)$ satisfies the hypotheses. The minimal choice of $G$ implies that $M$ is 2-nilpotent. So $G$ is not 2-nilpotent but every proper subgroups of $G$ is 2-nilpotent. Then by IV, 5.4 and III, 5.2(1) of [6], we have that $G = PQ$, where $P$ is a normal Sylow 2-subgroup of $G$, $Q$ is a cyclic Sylow $q$-subgroup of $G$, $q \neq 2$ and $Q$ is not normal in $G$.

(2) $\Phi(P) \neq 1$

If $\Phi(P) = 1$, then $P$ is an abelian normal Sylow 2-subgroup of $G$, then $P \leq Z(G)$, then $G$ is 2-nilpotent, a contradiction. So we have $\Phi(P) \neq 1$.

(3) Let $L$ be a minimal normal subgroup of $G$, then $G/L$ is $p$-nilpotent, $L$ is 2-subgroup, and so $L = P$.

Since $P$ is a Sylow 2-subgroup of $G$, then $PL/L$ is a Sylow 2-subgroup of $G/L$. And $(PL/L) \cap O^2(G)L/L = (P \cap O^2(G))/L$, then by hypothesis and lemma 2.1(2)(3), we have every cyclic subgroup of $(P \cap O^2(G))/L$ of order 4 is $c-$supplemented in $G/L$. Thus by the minimal choice of $G$, $G/L$ is 2-nilpotent. Also $L$ is a 2-group. Then $L \leq P$. If $L < P$. Since $P$ is a Sylow 2-subgroup of $G$, then $P/L$ is a Sylow 2-subgroup of $G/L$. And $(P/L) \cap O^2(G)L/L = (P \cap O^2(G))/L$, then by hypothesis and lemma 2.1(2)(3), we have every cyclic subgroup of $(P \cap O^2(G))/L$ of order 4 is $c$-supplemented in $G/L$. Thus
by the minimal choice of $G$, we have $G/L$ is 2-nilpotent. If $L \leq \Phi(P)$. Since $P \triangleleft G$, then $\Phi(P) \leq \Phi(G)$, then $G/\Phi(G)$ is 2-nilpotent, and so $G$ is 2-nilpotent, a contradiction. So $L/\Phi(P)$ is a minimal normal 2-subgroup of $G/\Phi(P)$, this contradicts III, 5.2(2) of [6], then $L = P$.

(4) To complete the proof.

by (3) $P$ is minimal normal 2-subgroup of $G$. Then if $\Phi(P) < P$, $P/\Phi(P)$ is a minimal normal 2-subgroup of $G/\Phi(P)$. Since $(P/\Phi(P)) \cap O^2(G)\Phi(P)/\Phi(P) = (P \cap O^2(G))\Phi(P)$, so by lemma 2.1(2)(3), we have that every cyclic subgroup of $(P \cap O^2(G))/\Phi(P)$ of order 4 is $c$-supplemented in $G$, then by the minimal choice of $G$, $G/\Phi(P)$ is 2-nilpotent, and so $G/\Phi(G)$ is 2-nilpotent as $\Phi(P) \leq \Phi(G)$. Thus $G$ is 2-nilpotent, a contradiction. If $P = \Phi(P)$, then by (1) $G/\Phi(P) = G/P \cong Q$ is nilpotent, then $G/\Phi(G)$ is nilpotent and so $G$ is nilpotent, a contradiction. □

**Corollary 3.1.** Let $G$ be a group and $p$ the smallest prime divisor of $|G|$. If every minimal subgroup of Sylow $p$-subgroups of $G$ of order $p$ or 4 (if $p = 2$) is $c$-supplemented in $G$, then $G$ is $p$-nilpotent.

**Acknowledgments** The object is partially supported by the Scientific Research Fund of School of Science of SUSE. The author is very grateful for the helpful suggestions of the referee.

**References**


Received: March, 2009