

# On a Variant of Bertini's Theorem and Generators of Ideals of a Polynomial Ring with Monic Polynomials<sup>1</sup>

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**Abstract.** In this paper we discuss different versions of Bertini's theorem and prove a variant of a theorem of Bertini. We also prove the following theorem. Suppose  $A$  is a semilocal regular reduced affine algebra over an algebraically closed field of characteristic zero such that every maximal ideal in  $A[T]$  is complete intersection, with  $\dim(A) = n > 0$ . Suppose  $I$  is an ideal of  $A[T]$ , with  $\text{height}(I) = n + 1$ . Assume co-normal module  $I/I^2$  is generated by  $n + 1$  elements over  $A[T]/I$ . Then we can find a minimal set of generators of  $I$  with a monic polynomial.

## 1. INTRODUCTION

For an ideal  $I$  of a commutative ring  $R$ , the comparison of the minimal number of generators of  $I$  and of its conormal module  $I/I^2$ , denoted by  $\mu(I)$  and  $\mu(I/I^2)$ , has been an enlivened area of interest (See [2], [3], [4], [7], [9], [10]). In ([2]) it has been shown that if  $I$  be an ideal in  $R = K[X_1, \dots, X_n]$ , where  $K$  is a field or principal ideal domain and  $\mu(I/I^2) \geq \dim(R/I) + 2$ , then  $\mu(I) = \mu(I/I^2)$ . A sharp and dimension-dependent case was proved by S. Mandal [3] for the case of a polynomial ring  $A[T]$ , with  $A$  Noetherian and  $I$  containing a monic polynomial, proving that the minimal number of generators of  $I$  and of its conormal module  $I/I^2$  are same. In [7] equality is shown to take place under the situations, especially if  $A$  is a local ring. The occurrence of monic polynomials in minimal generating sets is also discussed. In [8] it is shown that if  $R$  is an arbitrary ring,  $J$  an ideal contained in the Jacobson radical of  $R$ , and  $I$  a left ideal of  $R[T]$  containing a monic polynomial, then every monic in  $(I + JR[T])/JR[T]$  is of the form  $g + JR[T]$  for some monic

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polynomial  $g$  in  $I$ . From this it follows that if  $R$  is a commutative quasilocal ring, then each invertible ideal  $I$  of  $R[T]$  that contains a monic polynomial is actually principal.

Let  $R$  be a commutative unitary ring. In [9] it has been shown that if  $I$  is an invertible ideal of the polynomial ring  $R[X]$  which contains a monic polynomial, then there exists a decomposition  $R = R_1 \oplus \cdots \oplus R_k$  of  $R$  as a direct sum of ideals such that, for each  $i$ ,  $IR_i[X]$  is generated by a monic polynomial of  $R_i[X]$ . Using this result, it is proved that if  $R$  is indecomposable (e.g. if  $R$  is quasilocal) and if  $I$  and  $J$  are comaximal ideals of  $R[X]$  such that  $I \cap J$  is generated by a monic polynomial  $f$ , then  $I$  and  $J$  are generated by monic polynomials  $g$  and  $h$  such that  $f = gh$ . In general it is known that if  $I$  is an invertible ideal of  $R[X]$  such that  $I$  contains a monic polynomial, then  $I$  is a principal ideal ( see [10] see p. 116, Theorem 3.14). Necessary and sufficient conditions in order that  $I$  as above be generated by a monic polynomial is discussed. It is shown that if  $I$  is an invertible ideal of the polynomial ring  $R[X_1, \cdots, X_n]$  which contains a monic polynomial, then  $I$  is principal. If  $R$  is reduced and indecomposable and if  $I$  and  $J$  are comaximal ideals of  $R[X_1, \cdots, X_n]$ , an extension is given of the above-mentioned result on generators for  $I \cap J$ ,  $I$  and  $J$ . It is shown, however, that this no longer holds if  $R$  is indecomposable but not reduced. For  $R$  a reduced ring, a result is obtained on the "monic lifting problem".

Consider the following example: Let  $A = \frac{R[X, Y, Z, W]}{(X^2 + Y^2 + Z^2 + W^2 - 1)}$  be the coordinate ring of the real sphere, where  $R$  is the field of real numbers. Let  $x, y, z$  denote the image of  $X, Y, Z$  respectively in  $A$ . Then  $m = (x - 1, y, z, w)$  is a maximal ideal in  $A$ . Assume  $I = mA[T]$ . Then  $\mu(I/I^2) = 3 = \dim(A[T]/I) + 2$ , where  $\mu(I/I^2)$  denotes the minimal number of generators of  $I/I^2$  over  $A[T]/I$ , but  $I$  can not be generated by three elements. However if  $I$  contains a monic polynomial, then the Theorem 2.1 (see [3]) says that  $I$  can be generated by three elements. Further if  $A$  is an integral domain, then a monic polynomial is irreducible and it generates a prime ideal in the polynomial ring  $A[T]$ . These examples show the importance of monic polynomials. Further consider the Quillen-Suslin Theorem (Serre's conjecture): Every finitely generated projective module over a polynomial ring over a field, is free. In the proof of this theorem a monic polynomial played a crucial role. It is a natural question whether a monic polynomial is a part of some minimal set of generators if  $I$  contains a monic. The following theorem is known (see the Lemma 4.5 in ([12])).

**Theorem 1.1.** *Let  $A$  be a commutative Noetherian ring and  $I$  be an ideal of the polynomial ring  $A[T]$ , which contains a monic polynomial and let  $\dim(A/J(A)) =$*

$r$ , where  $J(A)$  denotes the Jacobson radical of  $A$ . Suppose that  $I/I^2$  is generated by  $m$  elements,  $m \geq r + 2$  and  $\bar{\phi} : A[T]^{r+2} \rightarrow I/I^2$  is a surjective homomorphism. Then  $\bar{\phi}$  can be lifted to a surjective homomorphism  $\phi : A[T]^{r+2} \rightarrow I$  such that  $\phi(e_1)$  is monic, where  $e_i$  is a usual basis of  $A[T]^{r+2}$ .

The above result is also true if  $J(A) = 0$ . In this paper we have shown that if the co-normal module  $I/I^2$  is generated by  $r + 2$  elements over  $A[T]/I$  and  $I$  contains a monic polynomial, then a minimal generating set of  $I$  also contains a monic polynomial.

In general, in [4] it has been shown that even if  $I$  contains a monic polynomial, the generating set of  $I$  may not contain a monic polynomial.

*Remark 1.1.* There is an interesting relationship between the factorization of monic polynomials and the behaviour of prime ideals in an integral extension. In [23] uniqueness of factorization of a monic polynomial into comaximal factors have been discussed. A Kronecker polynomial is a monic polynomial with integer coefficients having roots in the unit disc (see [29]). Then

1. number of Kronecker polynomial of degree  $n$  are finite.
2. non-zero roots of Kronecker polynomial are on the boundary of the unit disc.
3. such polynomial are essentially product of cyclotomic polynomials.

In section 3 we discuss different versions of Bertini's theorems and prove a variant of theorem of Bertini. Main results in this paper are Theorem 3.3 and Theorem 4.5 and rest of the paper is devoted to a review of the literature.

## 2. PRELIMINARIES

For a module  $M$  over a ring  $R$ ,  $\mu(M)$  denotes the minimal number of generators of  $M$  over  $R$ . We say that an ideal  $I$  of a ring  $R$  is said to be efficiently generated, if  $\mu(I) = \mu(I/I^2)$ . Recall that if  $R$  is a Noetherian ring, then  $I$  is called a complete intersection ideal if  $ht(I) = \mu(I)$ . Note that in general  $ht(I) \leq \mu(I)$ .

Let  $K$  be a field. Then an affine  $K$ -algebra is a finitely generated commutative  $K$ -algebra. Let  $A$  be a reduced affine  $K$ -algebra of dimension  $n$ . Then  $F^n(K_0(A))$  denotes the subgroup of the Grothendieck group  $K_0(A)$  generated by all the residue fields of all smooth maximal ideals of height  $n$  i.e.  $F^n(K_0(A)) = \langle \{[A/m] \in K_0(A) | m \in Max(A) \text{ such that } A_m \text{ is a regular local ring of dim } n\} \rangle$ .

Let  $A$  be a geometrically reduced affine algebra over an algebraically closed field  $k$ , Let  $dim(A) = n$ . Let  $P$  be a projective  $A$ -module of rank  $n$  and  $P^* = Hom_A(P, A)$ . Then the  $n$ th Chern class is defined by  $C_n(P) = \sum_{i=0}^n (-1)^i (\Lambda^i P^*) \in K_0(A)$ .

Let  $B = K[Y_1, \dots, Y_n]$  be an affine algebra over an infinite field  $K$  and let  $Sing(Spec(B)) = \{p \in Spec(B) | B_p \text{ is not regular}\}$  denote the singular locus of  $Spec(B)$ .

**Example :** Consider the ideal  $(X^2) \subset K[X, Y]$ , where  $K$  is an infinite field. Let  $B = \frac{K[X, Y]}{(X^2)} = K[x, Y]$ ,  $x^2 = 0$ . Then the ideal  $(x)$  is prime in  $B$  and the localization  $B_{(x)}$  is not regular. Note that  $Sing(Spec(B))$  and  $Spec(B)$  have a common irreducible component. Every point in  $spec(B)$  is singular, for  $B$  is not an integral domain and any maximal ideal in  $B$  is of the form  $(x, y - \alpha)$ , for some  $\alpha \in K$ . A variety is irreducible if and only if affine ring of that variety is a domain. For example of reducible variety. Let  $f = X$  and  $g = Y$  be two polynomials in  $K[X, Y]$ , where  $K$  is a field, then consider the variety  $V(fg) = V(XY)$ . Since  $XY = 0 \Rightarrow X = 0$  or  $Y = 0$ , we have  $V(XY) = V(X) \cup V(Y)$ . This shows that  $V(XY)$  is reducible and  $A = \frac{K[X, Y]}{(XY)}$  is not an integral domain.

We say that an ideal  $I$  of a ring  $R$  is efficiently generated if  $\mu(I) = \mu(I/I^2)$ . We now state results, which we have used in the proof of our main theorem.

**Theorem 2.1.** ([3]) *Let  $I$  be an ideal of  $A[T]$  over a commutative Noetherian ring  $A$ . Suppose that  $I$  contains a monic polynomial and  $\mu(I/I^2) \geq \dim(A[T]/I) + 2$ . Then ideal  $I$  is efficiently generated .*

### 3. DIFFERENT VERSIONS OF BERTINI THEOREM

In this section we discuss different versions of Bertini's theorems.

Let  $V$  be an algebraic variety over an algebraically closed field  $K$  of characteristic 0, let  $L$  be a linear system without fixed components on  $V$  and let  $W$  be the image of the variety  $V$  under the mapping given by  $L$ . The following two theorems are known as the first and the second Bertini theorem, respectively (See [17]).

1. If  $Dim(W) > 1$  , then almost all the divisors of the linear system  $L$  (i.e. all except a closed subset in the parameter space  $P(L)$  not equal to  $P(L)$ ) are irreducible reduced algebraic varieties.
2. Almost all divisors of  $L$  have no singular points outside the basis points of the linear system  $L$  and the singular points of the variety  $V$ .

Both Bertini theorems are invalid if the characteristic of the field is non-zero.

Conditions under which Bertini's theorems are valid for the case of a finite characteristic of the field have been studied (see [15], [19]). If  $Dim(W) = 1$ , Bertini's theorem is replaced by the following theorem: Almost all fibres of the mapping  $\phi_L : V \rightarrow W$  are irreducible and reduced if the function field  $K(W)$  is algebraically closed in the field  $K(V)$  under the imbedding  $\phi_L^* : K(W) \rightarrow K(V)$ . If the characteristic of  $K$  is finite, the corresponding theorem is true if the extension  $K(V)/K(W)$  is separable. The Bertini theorems apply to linear systems of hyperplane sections, without restrictions on the characteristic of the field ( See [21]).

Bertini's theorem on variable singular points may fail in characteristic  $p$  (see [23]); there are algebraic fibrations that are pathological in the sense that all

fibres are non-smooth though the total space admits only a finite number of singular points. If an algebraic fibration by plane projective quadratic curves in characteristic 7 is pathological then, after an eventual cyclic base extension of degree 3, it is, up to birational equivalence, obtained by a base extension from the pencil of quartic curves cut out by the equations  $zx^3 + xy^3 + tyz^3 = 0$ .

**Theorem 3.1. (Bertini's Theorem)** (*MathSciNet review of [24]*) *Let  $K$  be an infinite field. If a projective  $K$ -scheme  $X$  is smooth and/ or geometrically irreducible with  $\dim(X) \geq 2$ , then a generic hypersurface section  $H_\alpha$  of  $X$  is also smooth and/ or geometrically irreducible.*

Considering a linear systems of codimension one subvarieties of an algebraic variety, and calling it irreducible if its generic member is irreducible, Bertini's Theorems in [13] are stated as follows:

Bertini I Or Singular Bertini: Outside the singularities of the variety and outside its base points, members of an irreducible linear system do not have variable singularities.

Bertini II Or Reducible Bertini: If a linear system, without fixed components, is not composite with a pencil, then it is irreducible.

These were obtained by Bertini in his 1882 paper [14]. They were revisited by Zariski in [15] and [16].

In [25] Bertini's Theorem is stated as follows :

The general curve of a system which is linearly dependent on a certain number of given irreducible curves will not have a singular point which is not fixed for all the curves of the system.

In 1957 S. S. Abhayankar raised the question whether the theorem of Bertini also holds for local domain. In [28] Chow answered the question of Abhayankar affirmatively by formulating and proving such a theorem for arbitrary complete local domains.

**Theorem 3.2.** (*Theorem of Bertini for local domains, see [28]*) *Let  $(R, p)$  be a complete local domain of  $\dim(n) > 2$ . Let  $(A, q)$  be a complete local domain of dimension  $n$  contained in  $R$  such that  $R$  is a finite module over  $A$  and is separable over  $A$ , and let  $x_1, x_2, x_3$  be three elements in a minimal basis of  $q$ . We set  $u(c) = (x_1 + cx_2)/x_2$ , where  $c$  is an element in  $A$ . Then, for sufficiently general  $c$ , the ideal  $R[u(c)]$  with respect to  $R[u(c)]p$  is prime in  $R[u(c)]$  and the quotient ring  $R_{u(c)}$  of  $R[u(c)]$  with respect to  $R[u(c)]p$  is analytically irreducible.*

The following theorem is a variant of Bertini's Theorem and this is used in our main theorem. We give a proof of this theorem due to lack of reference in this form.

**Theorem 3.3.** *Let  $A$  be a finitely generated  $K$  algebra with  $K$  characteristic 0, algebraically closed field. Assume  $A$  reduced and  $\dim(A) \geq 2$ . Let  $f, g \in A$  with that  $ht(f, h) \geq 2$ . Then for all but a finite number of  $\lambda \in K$ ,  $A/(f + \lambda h)$  is reduced.*

*Proof.* Let  $A = \frac{K[X_1, \dots, X_n]}{(g_1, \dots, g_r)}$  be reduced affine algebra of dimension  $\geq 2$ , where  $K$  is a field of characteristic zero and let  $V$  be an affine variety corresponding to  $A$ . Consider  $f + Th \in A[T]$  and let  $W$  be the affine variety corresponding to  $(g_1, \dots, g_r, f + Th) \subset K[X_1, \dots, X_n, T]$ . Suppose  $ht(g_1, \dots, g_r) = d$ . To find singular locus of  $W$ , consider the following Jacobian matrix

$$M = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial g_r}{\partial x_1} & \dots & \frac{\partial g_r}{\partial x_n} & \dots & 0 \\ f_{X_1} + Th_{X_1} & \dots & f_{X_n} + Th_{X_n} & \dots & h \end{pmatrix}.$$

In the common locus of  $(d+1) \times (r+1)$  minors of the matrix, we certainly have  $d \times d$  minors  $(\frac{\partial g_i}{\partial x_j})h, i = 1, 2, \dots, r, j = 1, \dots, n$ . Consider the singular locus of  $(g_1, \dots, g_r, f + Th), (h \cdot \{ \text{all the } d \times d \text{ minors of } (\frac{\partial g_i}{\partial x_j}) \})$ . Let  $I_1 =$  the ideal generated by  $(g_1, \dots, g_r, \text{all the minors of } (\frac{\partial g_i}{\partial x_j}))$  and  $I_2 =$  ideal generated by  $(g_1, \dots, g_r, f + Th, h)$ . Hence singular locus of  $W$  is contained in the variety  $V(I_1) \cup V(I_2)$ . Consider the map  $t : W \rightarrow A^1$ , where  $t =$  residue of  $T$  in  $\frac{K[X_1, \dots, X_n, T]}{(g_1, \dots, g_r, f + Th)} = \frac{A[T]}{(f + Th)}$  defined by taking the projection on the  $T$ -th component and then take the fibre of a generic point. Note that every field of characteristic 0 is also infinite, we apply Bertini Theorem 3.1 on variable singular points, which says that for the general  $t = \lambda$ , the inverse image is smooth outside the singular locus of  $W$ . Since  $\dim W > \dim V - 2$ , we can assume that singular locus of  $W$  and singular locus of  $V$  have no common irreducible component. We conclude that the singular locus of  $W$  is a proper subset of  $W$ . Since if  $B$  is a finitely generated affine algebra and singular locus of  $\text{Spec}(B)$  has no irreducible component common with  $\text{Spec}(B)$ , therefore  $\text{Spec}(B)$  is reduced. Take  $W = \text{Spec}(B)$ . This implies that  $B = \frac{A}{(f + \lambda h)}$  is reduced ring over  $K$ .

□

**Remark :** It seems that the Theorem 3.3 should have a proof without interesting geometry. For simple case consider  $A = K[X, Y]$ , a polynomial ring in

two variables over a field,  $f = X^2$  and  $g = Y^2$  and consider  $h_\lambda = f + \lambda g = X^2 + \lambda Y^2$ . If  $\lambda = 0$ , then  $h_\lambda = X^2$  and  $A/h_\lambda$  is not reduced. If  $\lambda \neq 0$ , then  $h_\lambda = (X + \sqrt{\lambda}Y)(X - \sqrt{\lambda}Y)$ , coprime factors, so  $A/h_\lambda$  is reduced (but of course not a domain). May be such arguments can be generalized, at least for polynomial case. It is interesting to note that if  $\text{char}K = p = 2$ , then  $(X + \sqrt{\lambda}Y) = (X - \sqrt{\lambda}Y)$ . So  $h_\lambda = (X + \sqrt{\lambda}Y)^2$ , for every  $\lambda$ , then  $A/h_\lambda$  is not reduced for any  $\lambda$ .

#### 4. GENERATORS OF IDEALS OF A POLYNOMIAL RING WITH MONIC POLYNOMIALS

Before proving our main theorem on generators of ideals of a polynomial ring with monic polynomial, we state some useful results, which are relevant to our main theorem.

**Theorem 4.1.** [6] *Let  $A$  be a reduced affine algebra of  $\dim(n) \geq 2$ . Let  $I \subset A$  be a local complete intersection ideal of height  $n$ . Suppose  $F^n K_0(A)$  has no  $(n-1)!$  torsion. Then  $[A/I] = 0$  in  $K_0(A)$  if and only if  $I$  is complete intersection ideal.*

*Remark 4.1.* It is known that  $F^n K_0(A)$  is torsion free in the following cases.

1.  $\text{Char}(K) = 0$  (See [27])
2.  $A$  is regular in codimension one (See [1]).
3.  $A$  be reduced affine algebra of  $\dim n \geq 2$  and closure of  $\mathcal{R}$ -rational points in  $\text{Spec}(A)$  has  $\dim \leq n - 1$ .

The relevant known facts (see [6]) needed in the sequel about torsion in  $F^n K_0(A)$  are given in the following theorem.

**Theorem 4.2.** ([1], [27]) *Let  $A$  be an integral  $K$  affine algebra of dimension  $n \geq 2$ . Suppose  $K$  is algebraically closed or  $K = \mathbb{R}$  (real) or and the closure of  $\mathbb{R}$ -rational points of  $\text{Spec}(A)$  is not dense in  $\text{Spec}(A)$ .*

1. *Suppose one of the following condition holds:*
  - (a)  $n = 2$ .
  - (b)  $\text{Char}K = p \geq n$
  - (c)  $n \geq 3$  and  $A$  is regular in codimension one.  
Then  $F^n K_0(A)$  has no  $(n-1)!$  torsion.
2. *Suppose  $A$  is regular. Then the natural map  $\psi : CH^n(\text{Spec}A) \longrightarrow F^n K_0(A)$  given by the  $n$ th Chern class, is an isomorphism.*

**Corollary 4.3.** *Let  $A$  be a reduced affine algebra of  $\dim n$  over a field  $K$  and  $I \subset A$  be a local complete intersection ideal of height  $n$ . Suppose that the following conditions hold:*

1.  $F^n K_0(A)$  has no  $(n-1)!$  torsion (i.e.  $(n-1)!x = 0, x \in F^n K_0(A) \Rightarrow x = 0$ ).
2. Let  $K$  be algebraically closed or  $K = \text{mathcal}R$  and closure of  $\text{mathcal}R$ -rational points in  $\text{Spec}(A)$  has  $\dim \leq n - 1$ .

Then  $I$  is complete intersection iff  $[A/I] = 0$  in  $K_0(A)$ .

**Theorem 4.4.** ( see [6]) Let  $A$  be a reduced algebra of  $\dim n$  over an algebraically closed field  $K$ . Then the following are equivalent:

1. Every finite  $A$ -module  $M$  is generated by  $\delta(M) = \sup \{ \mu_p(M) + \dim(A/p) \mid p \in \text{Supp}(M), \dim(A/p) < n \}$  elements.
2. Every locally complete intersection ideal is generated by  $n$  elements.
3. Every locally complete intersection ideal of height  $n$  is a complete intersection.
4. Every smooth maximal ideal is a complete intersection.
5.  $F^n K_0(A) = 0$ .
6. Every projective  $A$ -module of rank  $n$  has a free direct summand of rank one.

**Theorem 4.5.** Let  $A$  be a semi local regular reduced affine algebra of  $\dim n > 0$  over an algebraically closed field of characteristic zero. Suppose that every maximal ideal of  $A$  is a complete intersection. Suppose  $I$  is a zero dimensional locally complete intersection ideal in  $A[T]$  containing a monic polynomial. Assume co-normal module  $I/I^2$  is generated by  $n + 1$  elements over  $A[T]/I$ . Then there exist a generating set  $f_1, \dots, f_{n+1}$  of  $I$  with  $f_1$  monic.

*Proof.* Since  $I$  is a zero dimensional,  $\dim(A[T]/I) = 0$ , By Theorem 2.1,  $\mu(I) = \mu(I/I^2) = n + 1$ . Let  $g_1, \dots, g_{n+1}$  in  $I$  be such that their images in  $I/I^2$  generate  $I/I^2$  over  $A[T]/I$ . Let  $h$  be a monic polynomial in  $I$  and let  $f_1 = g_1 + h^N$ , for sufficiently large  $N$ , such that  $f_1$  is a monic polynomial. Let  $R = A[T]/(f_1)$  and  $I' = I/(f_1)$ . We choose  $g$  in  $I$  such that the height of the ideal generated by  $\{f_1, g\} \geq 2$ . Then by the Theorem 3.3, for some  $\lambda \in K$ ,  $f_1 + \lambda g$  is reduced i.e.  $R/(f_1 + \lambda g)$  is reduced ring. Hence we can assume  $R$  is a reduced algebra over  $K$ . We have  $I'/I'^2 = I/(f_1) + I^2 = I/(g_1) + I^2$ . Since  $g_1$  is a member of minimal generating set of  $I/I^2$ ,  $\mu(I'/I'^2) = n$ . Since  $R$  is an integral extension of  $A$ , it is a semi local ring. We show that  $\mu(I') = n$ . Let  $I' = \langle g_2, \dots, g_{n+1} \rangle + I'^2$ . If  $I' \subset \text{Rad}(R)$ , then by Nakayama lemma,  $I' = \langle g_2, \dots, g_{n+1} \rangle$ . If  $I'$  is not contained in  $\text{Rad}(R)$ , then suppose that  $m_1, \dots, m_k$  are the maximal ideals that do not contain  $I'$ . We can choose  $g_2$  in  $m_1, \dots, m_l$  and not in  $m_{l+1}, \dots, m_k$ . Choose  $g' \in I'^2 \cap (m_{l+1} \cap \dots \cap m_k) - (m_1 \cap \dots \cap m_l)$ . Write  $F_1 = g_2 + g'$ . Then  $F_1$  does not belong to  $m_1, \dots, m_k$ . Then  $I' = (F_1, g_3, \dots, g_{n+1})$ . This proves that  $\mu(I') = n$ .

Since  $R$  is a reduced algebra over  $K$  and  $\text{char}(K) = 0$ , by remark 4.1,  $F^n K_0(R)$  has no  $(n - 1)!$  torsion. Since every maximal ideal in  $R$  is complete intersection, by the Theorem 4.1,  $[A/m] = 0$  in  $K_0(A)$ . Since by definition  $F^n(K_0(A)) = \langle \{ [A/m] \in K_0(A) \mid m \in \text{Max}(A) \text{ such that } A_m \text{ is a regular local ring of dim } n \} \rangle$ , we have  $F^n(K_0(A)) = 0$ . Then by the Theorem 4.4, every locally complete intersection ideal is generated by  $n$  elements. Hence  $I'/I'^2$  has a unimodular element. Therefore image of  $f_1$  in  $I'/I'^2$  generates a direct

summand isomorphic to  $A[T]/I$ . This implies that  $I'$  is a locally complete intersection ideal in  $R/I'$ . Hence the generators of  $I'/I'^2$  can be lifted to a generating set of  $I'$ , say  $g_2, \dots, g_{n+1}$ . Therefore  $I = \langle f_1, \dots, f_{n+1} \rangle$ . This proves the result.  $\square$

*Remark 4.2.* For example of  $A$  in the above theorem consider the localization of any regular affine algebra with the properties as stated.

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