Lie Ideals in Prime Rings with 
$(\sigma, \tau)$–Derivation

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Abstract. Let $R$ be a prime ring, char$R \neq 2$ and $\sigma, \tau, \alpha, \beta, \lambda, \mu, \varphi, \theta$ are automorphisms of $R$. In this note the following basic results are given. (i) Let $d$ be a nonzero $(\sigma, \tau)$–derivation of $R$ such that $d\sigma = \sigma d, d\tau = \tau d$ and $[a, d(I)]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then $a \in C_{\alpha, \beta}$ or $R$ is commutative, where $I \neq 0$ is an ideal of $R$ and $a \in R$. (ii) Let $U$ be a nonzero Lie ideal of $R$ and $d_1$, a nonzero $(\sigma, \tau)$–derivation of $R$. If $d_2$ is a nonzero $(\varphi, \theta)$–derivation of $R$ such that $d_2\varphi = \varphi d_2, d_2\theta = \theta d_2$ and $I \neq 0$ an ideal of $R$ such that $[d_1(U), d_2(I)]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then $U \subset Z$. The above results are generalizations of Theorem 5 and Theorem 6 in [7] respectively.

Keywords: Prime ring, Lie ideal, $(\sigma, \tau)$–derivation

1. Introduction

Let $R$ be a ring and $\sigma, \tau$ two mappings of $R$. We set $[x, y] = xy - yx$, and $[x, y]_{\sigma, \tau} = x\sigma(y) - \tau(y)x$ for $x, y \in R$. For subsets $A, B$ of $R$, let $[A, B]_{\sigma, \tau} = \{[x, y]_{\sigma, \tau} \mid x \in A \text{ and } y \in B\}$. We recall that a Lie ideal $U$ is an additive subgroup of $R$ such that $[R, U] \subset U$. An additive mapping $d : R \rightarrow R$ is called a $(\sigma, \tau)$–derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$. Let $C_{\sigma, \tau} = \{c \in R \mid c\sigma(r) = \tau(r)c, \forall r \in R\}$. We shall use the following relations frequently.

(A) $[xy, z]_{\sigma, \tau} = x[y, z]_{\sigma, \tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma, \tau}y$
(B) $[x, yz]_{\sigma, \tau} = \tau(y)[x, z]_{\sigma, \tau} + [x, y]_{\sigma, \tau}\sigma(z)$
(C) $[[x, y]_{\sigma, \tau}, z]_{\sigma, \tau} = [[x, z]_{\sigma, \tau}, y]_{\sigma, \tau} + [x, [y, z]]_{\sigma, \tau}$

Throughout the present paper, $R$ will represent a prime ring of char$R \neq 2$ and $\varphi, \theta, \sigma, \tau, \alpha, \beta, \lambda, \mu$ will be automorphisms of $R$.

Suppose that $a$ is an element of $R$ such that $ad(x) = d(x)a$ for all $x \in R$. Then by a theorem of Herstein [5] $a$ must be central. In [8] J.C.Chang extended this result by assuming that $[a, \delta(x)] \subset Z$ for all $x \in R$, where $\delta$
is an \((\alpha, \beta)\)-derivation of \(R\) such that \(\delta \alpha = \alpha \delta\), \(\delta \beta = \beta \delta\). In this paper we extended this result as follows. Let \(d\) be a nonzero \((\sigma, \tau)\)-derivation of \(R\) such that \(d\sigma = \sigma d, d\tau = \tau d\) and \([a, d(I)]_{\alpha, \beta} \subset C_{\lambda, \mu}\) then \(a \in C_{\alpha, \beta}\) or \(R\) is commutative, where \(I \neq 0\) is an ideal of \(R\) and \(a \in R\).

If \([d(x), d(y)] = 0\) for all \(x \in R\) then \(R\) is commutative. This result is proved in [4, Theorem 2] by Herstein. J.C. Chang extended this result in [8, Theorem 2(i)] by assuming that \(\delta(R), \delta(R) \subset Z\), where \(\delta\) is an \((\alpha, \beta)\)-derivation of \(R\) such that \(\delta \alpha = \alpha \delta, \delta \beta = \beta \delta\). In this note we extended this result as follows. Let \(U\) be a nonzero Lie ideal of \(R\) and \(d_1, a\) a nonzero \((\sigma, \tau)\)-derivation of \(R\). If \(d_2\) is a nonzero \((\varphi, \theta)\)-derivation of \(R\) such that \(d_2\varphi = \varphi d_2, d_2\theta = \theta d_2\) and \(I \neq 0\) an ideal of \(R\) such that \([d_1(U), d_2(I)]_{\alpha, \beta} \subset C_{\lambda, \mu}\) then \(U \subset Z\). In addition, some results are given about Lie ideal and another generalization of the [6, Lemma 1.5].

2. Results

**Lemma 1.** [2, Lemma 1] Let \(I\) be a nonzero ideal of \(R\) and \(a, b \in R\). If \([I, a]_{\sigma, \tau}, b \alpha, \beta = 0, \text{ then } [\tau(a), \beta(b)] = 0\).

**Lemma 2.** [9, Lemma 3] Let \(a, b \in R\). If \(b, ab \in C_{\sigma, \tau}\) then \(b = 0\) or \(a \in Z\).

**Lemma 3.** Let \(d\) be a nonzero \((\sigma, \tau)\)-derivation of \(R\) and \(I \neq 0\) an ideal of \(R\). If \(a \in R\) such that \([d(I), a]_{\sigma, \tau} \subset C_{\alpha, \beta}\) then \(a \in Z\) or \(d(a) \neq 0\).

**Proof.** Suppose that \(d(a) = 0\) and \(a \notin Z\). If we use hypothesis we have \(C_{\alpha, \beta} \ni [d(xa), a]_{\sigma, \tau} = \[d(x)\sigma(a) + \tau(x)d(a), a]_{\sigma, \tau} = [d(x)\sigma(a), a]_{\sigma, \tau} = d(x)[\sigma(a), \sigma(a)] + [d(x), a]_{\sigma, \tau}\sigma(a)\) and so

\[(2.1) \quad [d(x), a]_{\sigma, \tau}\sigma(a) \in C_{\alpha, \beta}, \forall x \in I.\]

Since \([d(x), a]_{\sigma, \tau} \in C_{\alpha, \beta}\) then considering Lemma 2 and (2.1) we have \([d(x), a]_{\sigma, \tau} = 0, \forall x \in I\) or \(\sigma(a) \in Z\). That is

\[(2.2) \quad [d(I), a]_{\sigma, \tau} = 0 \text{ or } a \in Z.\]

If \([d(I), a]_{\sigma, \tau} = 0\) then \(a \in Z\) by [10, Theorem 1]. Hence we obtain contradictions for two cases in (2.2). Thus, it must be \(a \in Z\) or \(d(a) \neq 0\). \(\square\)

**Theorem 1.** Let \(U\) be a nonzero Lie ideal of \(R\) and \(d\) a nonzero \((\sigma, \tau)\) - derivation of \(R\). If \(d(U) \subset C_{\alpha, \beta}\) then \(U \subset Z\).

**Proof.** Suppose that \(U \notin Z\). Then there exist an element \(v \in U\) such that \(v \notin Z\). On the other hand \(t = vr - rv\) and \(vt = vvr - vrv, \forall r \in R\) are elements of \(U\). Thus we have, \(0 = [d(vt), s]_{\alpha, \beta} = [d(v)[\sigma(t) + \tau(v)d(t)], s]_{\alpha, \beta} = d(v)[\sigma(t), \alpha(s)] + [d(v), s]_{\alpha, \beta}[\tau(v)d(t)] + [\tau(v), \beta(s)]d(t)\) for all \(s \in R\). That is

\[(2.3) \quad d(v)[\sigma(t), \alpha(s)] + [\tau(v), \beta(s)]d(t) = 0, \forall s \in R.\]
Replacing $s$ by $\beta^{-1}\tau(v)$ in (2.3) we have $d(v)[\sigma(t), \alpha\beta^{-1}\tau(v)] = 0$. Since $d(v) \in C_{\alpha, \beta}$ and $R$ is a prime ring we get
\[
d(v) = 0 \text{ or } [[v, r], \sigma^{-1}\alpha\beta^{-1}\tau(v)] = 0, \forall r \in R.
\]
Since $v \notin Z$ then the mapping defined by $d_1(r) = [v, r], \forall r \in R$ is a nonzero derivation. Hence
\[
d(v) = 0 \text{ or } [d_1(R), \sigma^{-1}\alpha\beta^{-1}\tau(v)] = 0
\]
is obtained. If $[d_1(R), \sigma^{-1}\alpha\beta^{-1}\tau(v)] = 0$ then $v \in Z$ by [4, Theorem]. This is a contradiction.

If $d(v) = 0$ then $C_{\alpha, \beta} \ni (d(v), r) = [d(v), r]_{\sigma, r} - [d(r), v]_{\sigma, r}$ and so
\[
[d(r), v]_{\sigma, r} \in C_{\alpha, \beta}, \forall r \in R.
\]
Using Lemma 3 we get $v \in Z$ or $d(v) \neq 0$. These are contradicting with $d(v) = 0$ and $v \notin Z$. Hence we obtain that $U \subset Z$. \hfill \Box

**Corollary 1.** [7, Lemma 6(i)] Let $U$ be a Lie ideal of $R$ and $d : R \to R$ a nonzero derivation of $R$. If $d(U) \subset Z(R)$ then $U \subset Z(R)$.

**Lemma 4.** Let $I \neq 0$ be an ideal of $R$ and $d$ a nonzero $(\sigma, \tau)$-derivation of $R$ such that $d\sigma = \sigma d, d\tau = \tau d$. If $a \in R$ and $[a, d(I)]_{\alpha, \beta} = 0$ then $a \in C_{\alpha, \beta}$.

**Proof.** Let us consider the mapping defined by $d_1(r) = [a, r]_{\alpha, \beta}, \forall r \in R$. Then $d_1$ is a $(\alpha, \beta)$-derivation of $R$. Hence we have $d_1d(I) = 0$ by hypothesis. This implies that $d_1 = 0$ or $d = 0$ by [1, Theorem 2]. Since $d \neq 0$ then we obtain that $d_1 = 0$ and so $a \in C_{\alpha, \beta}$. \hfill \Box

**Theorem 2.** Let $I \neq 0$ be an ideal of $R$ and $d$ a nonzero $(\sigma, \tau)$-derivation of $R$ such that $d\sigma = \sigma d, d\tau = \tau d$. If $a \in R$ such that $[a, d(I)]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then $a \in C_{\alpha, \beta}$ or $R$ is commutative.

**Proof.** Considering hypotheses we have $C_{\lambda, \mu} \ni [a, d(x)]_{\alpha, \beta} = a\alpha d(x) - \beta d(x)a = [a, x]_{\alpha d, \beta d}, \forall x \in I$. Let $\gamma = \alpha d$ and $\delta = \beta d$. Then
\[
(a, x)_{\gamma, \delta} \in C_{\lambda, \mu}, \forall x \in I,
\]
where $\gamma = \alpha d$ and $\delta = \beta d$.

is obtained. Using (2.4) we get
\[
0 = [a, xy]_{\gamma, \delta}, t = [\delta(x)]_{\lambda, \mu}[a, x]_{\gamma, \delta} + [a, y]_{\gamma, \delta}, \forall x, y \in I, t \in R.
\]
for all $x, y \in I, t \in R$. If we take $\lambda^{-1}\gamma(y)$ instead of $t$ and use (2.4) we have
\[
[\delta(x), \mu\lambda^{-1}\gamma(y)]_{\alpha, \beta} = 0, \forall x, y \in I
\]
(2.5)

If we multiply the relation (2.5) with $\lambda(r), r \in R$ by right and use that $[a, y]_{\gamma, \delta} \in C_{\lambda, \mu}$ we obtain that $[\delta(x), \mu\lambda^{-1}\gamma(y)]\mu(r)[a, y]_{\gamma, \delta} = 0, \forall x, y \in I, r \in R$. Since $R$ is prime, for all $y \in I$ we have $[\delta(I), \mu\lambda^{-1}\gamma(y)] = 0$ or $[a, y]_{\gamma, \delta} = 0$. That is, for any $y \in I$ we get
\[
[\beta d(I), \mu\lambda^{-1}ad(y)] = 0 \text{ or } [a, y]_{\alpha d, \beta d} = 0.
\]
Let $K = \{y \in I \mid [a^{-1}\lambda d^2, y] = 0\}$ and $L = \{y \in I \mid [a, y]_{\alpha, \beta} = 0\}$. Then $K$ and $L$ are subgroups of $R$. A group can not be a union of two of its proper subgroups. Then $I = K \cup L$ implies that $I = K$ or $I = L$. That is $[\alpha^{-1}\lambda d^2, y] = 0$ or $[a, y]_{\alpha, \beta} = 0$. If $[\mu^{-1}\alpha d^2, y] = 0$ then $d(I) \subset Z$ by Lemma 4. $d(I) \subset Z$ implies that $I \subset Z$ by Theorem 1 and so $R$ is commutative. If $[a, y]_{\alpha, \beta} = 0$ then $a\alpha d^2 - \beta d^2 a = 0$ for all $x \in I$. That is $[a, d(I)]_{\alpha, \beta} = 0$. Using Lemma 4, we obtain that $a \in C_{\alpha, \beta}$.

The following Corollary is a generalization [5, Theorem 6].

**Corollary 2.** Let $U$ be a nonzero Lie ideal of $R$ and $d_1$, a nonzero $(\sigma, \tau)$-derivation of $R$. Let $d_2$ be a nonzero $(\varphi, \theta)$-derivation of $R$ such that $d_2 \varphi = \varphi d_2, d_2 \theta = \theta d_2$ and $I \neq 0$ an ideal of $R$. If $[d_1(U), d_2(I)]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then $U \subset Z$.

**Proof.** Using Theorem 2, we have $d_1(U) \subset C_{\alpha, \beta}$ or $R$ is commutative. If $d_1(U) \subset C_{\alpha, \beta}$ then $U \subset Z$ by Theorem 1. On the other hand if $R$ is commutative then $U \subset Z$.

The following Corollary is a generalization [6, Lemma 1.5].

**Corollary 3.** Let $I \neq 0$ be an ideal of $R$ and $a \in R$. If $[[a, y]_{\alpha, \beta}, z]_{\alpha, \beta} \subset C_{\lambda, \mu}$ \forall $y, z \in I$ then $a \in C_{\alpha, \beta}$ or $R$ is commutative.

**Proof.** Using relation (C) and hypothesis we have $[[a, y]_{\alpha, \beta}, z]_{\alpha, \beta} = [[a, z]_{\alpha, \beta}, y]_{\alpha, \beta} + [a, [y, z]]_{\alpha, \beta}$ and so

$$[a, [y, z]]_{\alpha, \beta} \subset C_{\lambda, \mu} \text{ for all } y, z \in I.$$

Suppose that $R$ is not commutative. Since $R$ is a prime ring then $I \not\subset Z$. Hence there exist an element $y \in I$ such that $yr - ry \neq 0$ for all $r \in R$. Consider the mapping defined by $d(r) = [y, r], \forall r \in R$. Then $d$ is a nonzero derivation of $R$. Furthermore we have $[a, d(I)]_{\alpha, \beta} \subset C_{\lambda, \mu}$ by (2.6). This implies that $a \in C_{\alpha, \beta}$ or $R$ is commutative by Theorem 2. Considering our assumption we obtain that $a \in C_{\alpha, \beta}$.

**Lemma 5.** Let $U$ be a nonzero Lie ideal of $R$ and $a, b \in R$.

(i) If $[a, U]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then $a \in C_{\alpha, \beta}$ or $U \subset Z$.

(ii) If $I \neq 0$ is an ideal of $R$ such that $[a, [b, I]]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then $a \in C_{\alpha, \beta}$ or $b \in Z$.

**Proof.** (i) Since $U$ is a Lie ideal of $R$ then $[a, U]_{\alpha, \beta} \subset C_{\lambda, \mu}$ implies that $[a, U, R]_{\alpha, \beta} \subset C_{\lambda, \mu}$. Suppose that $U \not\subset Z$. Then there exist an element $v \in U$ such that $v \notin Z$. Let us consider the mapping defined by $d(r) = [v, r], \forall r \in R$. Then $d$ is a nonzero derivation of $R$ and $[a, d(R)]_{\sigma, \tau} \subset C_{\alpha, \beta}$. This implies that $a \in C_{\alpha, \beta}$ or $R$ is commutative by Theorem 2. If $R$ is commutative then we have $v \in Z$. This contradicts to our assumption. Hence it must be $a \in C_{\alpha, \beta}$ or $U \subset Z$.

(ii) The mapping defined by $d(r) = [b, r], \forall r \in R$ is a derivation. Then $[a, [b, I]]_{\alpha, \beta} \subset C_{\lambda, \mu}$ implies that $[a, d(I)]_{\alpha, \beta} \subset C_{\lambda, \mu}$. Hence we have $a \in C_{\alpha, \beta}$ or
$d = 0$ or $R$ is commutative by Theorem 2. If $d = 0$ then we have $b \in Z$. Thus we obtain that $a \in C_{\alpha, \beta}$ or $b \in Z$.

**Corollary 4.** [3, Theorem A] If $R$ is prime of characteristic $\neq 2$, $I \neq 0$ an ideal of $R$, and $a, b \in R$ such that $[a, [b, I]] \subset Z$ then either $a \in Z$ or $b \in Z$.

**Proof.** By Lemma 5(ii).

**Corollary 5.** [3, Corollary A] If $R$ is prime of characteristic $\neq 2$, $I \neq 0$ an ideal of $R$, and $a \in R$. If $[a, I] \subset Z$ then $a \in Z$.

**Proof.** Considering $\alpha = \beta = \lambda = \mu = 1$ in Lemma 5(i), (where $1 : R \rightarrow R$ is identity map) we obtain that $a \in Z$ or $I \subset Z$. Since $R$ is prime $I \subset Z$ implies that $R$ is commutative and so $a \in Z$.

**Theorem 3.** Let $U, V$ be two nonzero Lie ideals of $R$ and $d$ a nonzero $(\varphi, \theta)$-derivation of $R$.

(i) If $[U, U]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then $U \subset Z$.

(ii) If $[d(U), V]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then $V \subset Z$ or $R$ is commutative.

(iii) Let $I \neq 0$ be an ideal of $R$ and $d\varphi = \varphi d, d\theta = \theta d$. If $[[a, d(I)]_{\sigma, \tau}, U]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then $a \in C_{\sigma, \tau}$ or $U \subset Z$.

**Proof.** (i) If $[U, U]_{\alpha, \beta} \subset C_{\lambda, \mu}$ then we have $U \subset C_{\alpha, \beta}$ or $U \subset Z$ by Lemma 5(i). If $U \subset C_{\alpha, \beta}$ then $[U, R]_{\alpha, \beta} = 0$ and so $[[R, U], R]_{\alpha, \beta} = 0$. This implies that $[U, \beta(R)] = 0$ by Lemma 1. Hence $U \subset Z$ is obtained.

(ii) $[d(U), V]_{\alpha, \beta} \subset C_{\lambda, \mu}$ implies that $d(U) \subset C_{\alpha, \beta}$ or $V \subset Z$ by Lemma 5(i). If $d(U) \subset C_{\alpha, \beta}$ then $R$ is commutative by Lemma 4.

(iii) $[[a, d(I)]_{\sigma, \tau}, U]_{\alpha, \beta} \subset C_{\lambda, \mu}$ implies that $[a, d(I)]_{\sigma, \tau} \subset C_{\alpha, \beta}$ or $U \subset Z$ by Lemma 5(i). If $[a, d(I)]_{\sigma, \tau} \subset C_{\alpha, \beta}$ then $a \in C_{\sigma, \tau}$ or $U \subset Z$ by Theorem 2.

**Theorem 4.** Let $d : R \rightarrow R$ be a nonzero $(\sigma, \tau)$-derivation and $a \in R$. If $[d(I), a]_{\alpha, \beta} = 0$ then $a \in Z$ or $d\sigma^{-1}a \in Z$.

**Proof.** If $h : R \rightarrow R, h(r) = [r, a]_{\alpha, \beta}$ for all $r \in R$ then

$$h(rs) = h(r)s + rd_1(s) = d_2(r)s + rh(s) \text{ for all } r, s \in R.$$

where $d_1(r) = [r, \alpha(a)], \forall r \in R$ and $d_2(r) = [r, \beta(a)], \forall r \in R$. Thus we have $hd(I) = 0$ by hypothesis. Now for any $x, y \in I$ we have $0 = hd(xy) = h(d(x)\sigma(y) + \tau(x)d(y)) = hd(x)\sigma(y) + d(x)d_1\sigma(y) + d_2\tau(x)d(y) + \tau(x)hd(y) = d(x)d_1\sigma(y) + d_2\tau(x)d(y)$. This gives that

$$d(x)\sigma(y), [\alpha(a)] + [\tau(x), \beta(a)]d(y) = 0 \text{ for all } x, y \in I.$$

Replacing $y$ by $yr, r \in R$ in (2.7) and using (2.7) we get

$$d(x)\sigma(y)[\alpha(a)] + [\tau(x), \beta(a)]d(y) = 0 \text{ for all } x, y \in I, r \in R.$$

On the other hand, if we take $\sigma^{-1}a$ instead of $r$ in (2.8) we obtain that $[\tau(I), \beta(a)]d(\sigma^{-1}a) = 0$. Since $\tau(I)$ is a nonzero ideal of $R$ and $R$ is a
prime ring then we have $[\tau(I), \beta(a)] = 0$ or $d\sigma^{-1}\alpha(a) = 0$. This implies that $a \in Z$ or $d\sigma^{-1}\alpha(a) = 0$.

\[ \Box \]

\section*{References}


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