A Note on the Decomposition of the Inverse Limit of Finite Dimensional Lie Algebras

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Abstract

We extend the Levi-Malcev decomposition theorem of finite dimensional Lie algebras to the case of profinite dimensional Lie algebras. We prove that the inverse limit $L$ of a surjective inverse system of finite dimensional Lie algebras $L_i$, $i \in I$, where $I$ is a countable set with directed partial ordering, can be written as $L = R \oplus S$, where $R$ is a prosolvable ideal of $L$ and $S$ is a prosemisimple Lie subalgebra.

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1 Introduction

Most of the general theory on Lie algebras has been established for finite dimensional Lie algebras. However, little is known about the general theory of infinite dimensional Lie algebras.

An important class of such Lie algebras are the profinite dimensional Lie algebras $L = \lim L_i$ which are inverse limits of finite dimensional Lie algebras. Such Lie algebras appear as the Lie algebras of proaffine algebraic groups which play an important role in the representation theory of Lie groups. For more detailed information, the reader is referred to [3], [5], [6], [7], [8].

So it is of interest to extend the basic theory (found for example in [1], [2], [4]) concerning finite dimensional Lie algebras to the category of profinite dimensional Lie algebras.

In this paper, we generalize the Levi-Malcev decomposition theorem of finite dimensional Lie algebras to the case of profinite dimensional Lie algebras $L = \lim L_i$, $i \in I$, where $I$ is a countable set with directed partial ordering. We prove first that the inverse limit $L = \lim L_n$ ($n \in N$) of a surjective inverse
sequence of finite dimensional Lie algebras has a Levi-Malcev decomposition as $L = R \oplus S$, where $R$ is a prosolvable ideal of $L$ and $S$ is a prosemisimple Lie subalgebra and then we generalize the decomposition to the case of inverse limits of surjective inverse systems of finite dimensional Lie algebras over arbitrary countable sets with directed partial ordering.

2 Preliminaries

All Lie algebras in this paper are considered over a fixed algebraically closed field $K$ of characteristic 0.

**Definition 2.1** Let $I$ be a set with a partial ordering $\leq$. Suppose $I$ is directed upwards, i.e., for every $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let $S = \{S_i : i \in I\}$ be a family of sets such that for every pair $(i, j) \in I \times I$ with $j \geq i$, there is a map $\pi_{ji} : S_j \to S_i$ satisfying the following two conditions:

1. $\pi_{ii}$ is the identity map for every $i \in I$;
2. if $i \leq j \leq k$, then $\pi_{ki} = \pi_{ji} \circ \pi_{kj}$.

Let $\pi = \{\pi_{ji}, i, j \in I, j \geq i : \pi_{ji} : S_j \to S_i\}$

Then $(S, \pi)$ or $(S, \pi_{ji})_{i,j \in I}$ or simply $(S_i)$ is called an inverse system and the maps $\pi_{ji} : S_j \to S_i$ are called the transition maps of the inverse system.

The inverse limit of this system, denoted by $\lim \leftarrow S_i$, is the subset of the Cartesian product $\prod_{i \in I} S_i$ consisting of all elements $s = (s_i)_{i \in I}$ such that $\pi_{ji}(s_j) = s_i$ for every $j \geq i$. In other words, $\lim S_i$ is the set of all families of elements $(s_i)_{i \in I}$ which are compatible with the transition maps $\pi_{ji} : S_j \to S_i$.

**Remark 2.2** If $(S_i)$ is an inverse system, let $\pi_i : \lim S_i \to S_i$ be the canonical projection sending $(s_i)$ to $s_i$. Then $\pi_i = \pi_{ji} \circ \pi_j$ for every $j \geq i$.

**Remark 2.3** A surjective inverse system is an inverse system whose transition maps are surjective.

**Remark 2.4** An inverse sequence is an inverse system whose index set is $I = \mathbb{N}$ (directed by its natural order).

**Definition 2.5** Let $I$ be a directed poset (partially ordered set). A subset $J$ of $I$ is said to be cofinal in $I$ if for each $i \in I$ there exists $j \in J$ such that $i \leq j$. 
Remark 2.6 Let \((I, \leq)\) be a partially ordered set (directed upwards) which is infinite and countable. Then either \(I\) has a maximum or there exists a cofinal subset \(J \subset I\) such that \((J, \leq) \sim (\mathbb{N}, \leq)\) i.e. the bijection between \(J\) and \(\mathbb{N}\) preserves the order.

Proof. Suppose \(I\) does not have a maximum element and let us construct a cofinal subset \(J\) of \(I\) such that \(J \sim \mathbb{N}\). This is done as follows: Suppose \(I = (a_1, a_2, \ldots)\). Let \(\alpha_1 = a_1\). Given \(\alpha_n\), choose \(\alpha_{n+1}\) to be \(\text{Max}(\alpha_n, a_{n+1})\), where

\[
\text{Max}(a_i, a_j) = \begin{cases} 
  a_i & \text{if } a_i \geq a_j; \\
  a_j & \text{if } a_j \geq a_i; \\
  a_k & \text{where } a_k \geq a_i \text{ and } a_k \geq a_j \text{ if } a_i \text{ and } a_j \text{ are not comparable}; \\
  a_k & \text{exists since } I \text{ is directed upwards}.
\end{cases}
\]

Let \(J = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)\). Then \(J\) is a cofinal subset of \(I\), and \(J \sim \mathbb{N}\).

Proposition 2.7 If \(X = \varprojlim X_i, i \in I\), is an inverse limit of sets over a directed (upwards) poset \(I\) and \(J\) is a cofinal subset of \(I\), then a compatible family \((x_j)_{j \in J} \in \varprojlim X_j\) can be completed to a compatible family \((x_i)_{i \in I} \in \varprojlim X_i\) in a unique way.

Proof. For \(i \in I\), there exists \(j \in J\) such that \(j \geq i\). Let \((x_j)_{j \in J}\) be a compatible family of \(\varprojlim X_j\). Define \(x_i\) as \(x_i = \pi_{ji}(x_j)\). This is well defined, since, suppose there exists also \(j_2 \in J\) such that \(j_2 \geq i\). Then since \(J\) is directed upwards, there exists \(j_3 \in J\) such that \(j_3 \geq j_2 \) and \(j_3 \geq j\).

\[
\pi_{j_3, i}(x_{j_3}) = \pi_{j_i} \circ \pi_{j_3, j}(x_{j_3}) = \pi_{j_i}(x_j) = x_i.
\]

Thus \(x_i = \pi_{j_3, i}(x_{j_3}) = \pi_{j_2, i}(x_{j_2}) = \pi_{j_2, j}(x_{j_2}) = \pi_{j_2, i}(x_{j_2}) = \pi_{j_1, i}(x_{j_1})\). Hence \(x_i = \pi_{j_3, i}(x_{j_3}) = \pi_{j_2, i}(x_{j_2}) = \pi_{j_1, i}(x_{j_1})\), and in this way a compatible family \((x_j)_{j \in J}\) can be completed to a compatible family \((x_i)_{i \in I}\).

Theorems 2.8 and 2.9 state important results related to finite dimensional Lie algebras. Proofs of these theorems can be found in [2].

Theorem 2.8 Let \(L\) be a finite dimensional Lie algebra. Then

1. \(L = R \oplus S\), where \(R\) is the radical of \(L\) and \(S\) is a Levi subalgebra of \(L\) (Levi-Malcev Decomposition).

2. Every ideal \(A\) of \(L\) can be written as \(A = A \cap R \oplus A \cap S\).

3. For a subalgebra \(S\) of \(L\) to be a Levi subalgebra, it is necessary and sufficient that \(S\) is a maximal semisimple subalgebra of \(L\).

Theorem 2.9 Let \(L\) be a finite dimensional Lie algebra. Then
1. If $L$ is solvable, then so are all subalgebras and homomorphic images of $L$.

2. If $I$ is a solvable ideal of $L$ such that $L/I$ is solvable, then $L$ itself is solvable.

3. If $I$ and $J$ are solvable ideals of $L$, then so is $I + J$.

Below are some well known propositions concerning finite dimensional Lie algebras that are used in this paper.

**Proposition 2.10** Let $I$ be a solvable ideal of a Lie algebra $L$ such that $L/I$ is semisimple. Then $I$ is the radical of $L$.

*Proof.* $I$ is a solvable ideal of $L$, thus $(I + \text{Rad } L)/I$ is a solvable ideal of $L/I$. But $L/I$ is semisimple, therefore $(I + \text{Rad } L)/I$ is 0, i.e. $I + \text{Rad } L = I$ and $I = \text{Rad } L$.

**Proposition 2.11** Let $f: L \to L'$ be a surjective Lie algebra homomorphism. Then $f(\text{Rad } L) = \text{Rad } L'$.

*Proof.* Let $\vartheta: L/\text{Rad } L \to L'/f(\text{Rad } L)$ be given by $\vartheta(l + \text{Rad } L) = f(l) + f(\text{Rad } L)$. Then $\vartheta$ is a well defined surjective Lie algebra homomorphism.

$(L/\text{Rad } L)/\text{Ker } \vartheta \cong L'/f(\text{Rad } L)$, but $(L/\text{Rad } L)/\text{Ker } \vartheta$ is semisimple because $L/\text{Rad } L$ is semisimple, thus $L'/f(\text{Rad } L)$ is semisimple, and by the previous proposition $f(\text{Rad } L) = \text{Rad } L'$.

**Proposition 2.12** Let $f: L \to L'$ be a surjective Lie algebra homomorphism, and let $S$ be a maximal semisimple Lie subalgebra of $L$. Then $f(S)$ is a maximal semisimple Lie subalgebra of $L'$.

*Proof.* $L = \text{Rad } L \oplus S, L' = f(L) = f(\text{Rad } L) + f(S) = \text{Rad } L' \oplus f(S)$. But $\text{Rad } L' \cap f(S)$ is 0. Thus $L' = \text{Rad } L' \oplus f(S)$ and $f(S)$ is a maximal semisimple Lie subalgebra of $L'$.

**Proposition 2.13** Let $f: L \to L'$ be a surjective Lie algebra homomorphism. If $S'$ is a maximal semisimple Lie subalgebra of $L'$, then there exists a maximal semisimple Lie subalgebra $S$ of $L$ such that $f(S) = S'$.

*Proof.* $f^{-1}(S') = \text{Rad } [f^{-1}(S')] \oplus S$, where $S$ is a maximal semisimple Lie subalgebra of $f^{-1}(S')$. Let $M$ be a maximal semisimple Lie subalgebra of $L$ containing $S$, i.e. $S \subset M$. Then $f(S) \subset f(M)$.

But $f(S) = S'$, since $f(\text{Rad } [f^{-1}(S')]) = 0$. Thus $S' \subset f(M)$ and $S'$ is a maximal semisimple Lie subalgebra of $L'$, hence $f(M) = S'$, i.e. $M \subset f^{-1}(S')$. But $S$ is a maximal semisimple Lie algebra of $f^{-1}(S')$, therefore $M = S$. 

3 Levi-Malcev decomposition

This section contains the new results. We prove that the inverse limit of a surjective inverse system of finite dimensional Lie algebras, \( L = \lim_{i \in I} L_i \), where \( I \) is a countable set with directed partial ordering, can be written as the direct sum of a prosolvable ideal and a prosemisimple subalgebra of \( L \).

**Theorem 3.1** Let \( L \) be the inverse limit of a surjective inverse sequence of finite dimensional Lie algebras. Then \( L = R \oplus S \), where \( R = \lim_{i \in I} \text{Rad} L_i \) and \( S = \lim_{i \in I} S_i \) for a compatible family of Levi subalgebras \( S_i \) of \( L_i \).

**Proof.** Let \( L_0 = \text{Rad} L_0 \oplus S_0 \) be a Levi-Malcev decomposition of \( L_0 \). By Proposition 2.13, there exists a maximal semisimple Lie algebra \( S_1 \) of \( L_1 \) such that \( \pi_1,0(S_1) = S_0 \). Similarly, there exists a maximal semisimple Lie algebra \( S_2 \) of \( L_2 \) such that \( \pi_2,1(S_2) = S_1 \) and so on we construct a compatible family of \( S_n \). Also, the \( \text{Rad} L_n \) form a compatible subinverse system of \( L \) because the radical is unique.

Let \( R = \lim_{i \in I} \text{Rad} L_i \) and \( S = \lim_{i \in I} S_i \). Then, \( L = R \oplus S \):

For an arbitrary element \( l \in L \), \( l = (l_n)_{n \in \mathbb{N}} = (r_n + s_n) \), where \( r_n \in \text{Rad} L_n \) and \( s_n \in S_n \). If \( l_{n+1} = r_{n+1} + s_{n+1} \), \( l_n = r_n + s_n \) and \( l_{n-1} = r_{n-1} + s_{n-1}, \) where \( s_{n-1}, s_n \) and \( s_{n+1} \) belong respectively to the elements \( S_{n-1}, S_n \) and \( S_{n+1} \) of the compatible family \( \{S_n\} \).

Then, because the \( l_n \) are compatible, \( \pi_{n+1,n-1}(l_{n+1}) = l_{n-1} \), thus

\[ \pi_{n+1,n-1}(r_{n+1} + s_{n+1}) = r_{n-1} + s_{n-1}, \text{ i.e.,} \]

\[ \pi_{n+1,n-1}(r_{n+1}) + \pi_{n+1,n-1}(s_{n+1}) = r_{n-1} + s_{n-1}. \]

But since \( \text{Rad} L_{n-1} \cap S_{n-1} = 0 \), we get

\[ \pi_{n+1,n-1}(r_{n+1}) = r_{n-1} - \pi_{n+1,n-1}(s_{n+1}) = 0. \]

Thus, \( \pi_{n+1,n-1}(r_{n+1}) = r_{n-1} \) and \( \pi_{n+1,n-1}(s_{n+1}) = s_{n-1}. \)

Thus the \( r_n \) and the \( s_n \) are compatible.

Hence \( l = (r_n) + (s_n) \in R + S \).

But \( R \cap S = 0 \) because \( \text{Rad} L_n \cap S_n = 0 \) \( \forall \ n \in \mathbb{N}. \)

Thus, \( L = R \oplus S \).

We will call \( S \) a prolevi subalgebra of \( L \) and \( R \) the proradical of \( L \).

From the above theorem the following results follow directly:

**Remark 3.2** Let \( L = R \oplus S \) be the inverse limit of a surjective inverse system of finite dimensional Lie algebras, where \( R \) is the proradical of \( L \) and \( S \) is a prolevi subalgebra of \( L \). Then

(i) \( R \) is the largest prosolvable ideal of \( L \).

(ii) \( S \) is a maximal prosemisimple subalgebra of \( L \).

**Corollary 3.3** Let \( L = R \oplus S \) be the inverse limit of a surjective inverse sequence of finite dimensional Lie algebras and let \( A \) be a closed ideal of \( L \). Then \( A = A \cap R \oplus A \cap S \).
Proof. Let $S_n$ and $A_n$ be the $n^{th}$ projections of $S$ and $A$ respectively. Then $A_n = A_n \cap \text{Rad } L_n \oplus A_n \cap S_n$. Because $A$ is closed, $A = \lim A_n$. Thus, $A = \lim (A_n \cap \text{Rad } L_n \oplus A_n \cap S_n)$.

For an arbitrary element $a$ in $A$, $a = (a_n) = (r_n + s_n)$, where $r_n \in A_n \cap \text{Rad } L_n$ and $s_n \in A_n \cap S_n$. If $a_{n+1} = r_{n+1} + s_{n+1}$, $a_n = r_n + s_n$ and $a_{n-1} = r_{n-1} + s_{n-1}$, where $s_{n-1}$, $s_n$ and $s_{n+1}$ belong respectively to the elements $A_{n-1} \cap S_{n-1}$, $A_n \cap S_n$ and $A_{n+1} \cap S_{n+1}$ of the compatible family $\{A_n \cap S_n\}$. Then, because the $a_n$ are compatible, $\pi_{n, n-1}(a_{n+1}) = a_{n-1}$, thus

$$\pi_{n+1, n-1}(r_{n+1} + s_{n+1}) = r_{n-1} + s_{n-1},$$

$$\pi_{n+1, n-1}(r_{n+1}) + \pi_{n+1, n-1}(s_{n+1}) = r_{n-1} + s_{n-1}.$$ But since $(A_{n-1} \cap \text{Rad } L_{n-1}) \cap (A_{n-1} \cap S_{n-1}) = 0$, we get

$$\pi_{n+1, n-1}(r_{n+1}) - r_{n-1} = s_{n-1} - \pi_{n+1, n-1}(s_{n+1}) = 0.$$ Thus,

$$\pi_{n+1, n-1}(r_{n+1}) = r_{n-1}$$ and $\pi_{n+1, n-1}(s_{n+1}) = s_{n-1}$.

So the $r_n$ and the $s_n$ are compatible. Hence $a = (r_n) + (s_n)$.

Thus,

$$A = \lim (A_n \cap \text{Rad } L_n \oplus A_n \cap S_n) = \lim (A_n \cap \text{Rad } L_n) \oplus \lim (A_n \cap S_n).$$

But $A_n \cap \text{Rad } L_n \subset A_n$ and $A_n \cap \text{Rad } L_n \subset \text{Rad } L_n$, for all $n$, thus

$$\lim (A_n \cap \text{Rad } L_n) \subset \lim A_n = A$$

and $\lim (A_n \cap \text{Rad } L_n) \subset \lim \text{Rad } L_n = R$, hence

$$\lim (A_n \cap \text{Rad } L_n) \subset A \cap R.$$ Similarly, $\lim (A_n \cap S_n) \subset A \cap S$. Thus, $A = A \cap R \oplus A \cap S$.

Remark 3.4 According to Remark 2.6, $\mathbb{N}$ is cofinal in any countable set $I$ with directed partial ordering, and thus Theorem 3.1 and Corollary 3.3 can be generalized as follows:

Theorem 3.5 Let $L = \lim L_i$, $i \in I$, be the inverse limit of a surjective inverse system of finite dimensional Lie algebras, where $I$ is a countable set with directed partial ordering, then $L = R \oplus S$, where $R = \lim \text{Rad } L_i$ and $S = \lim S_i$ for a compatible family of Levi subalgebras $S_i$ of $L_i$.

Corollary 3.6 Let $L = R \oplus S$ be the inverse limit of a surjective inverse system of finite dimensional Lie algebras $L = \lim L_i$, $i \in I$, where $I$ is a countable set with directed partial ordering, and let $A$ be a closed ideal of $L$. Then $A = A \cap R \oplus A \cap S$.

References


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