

A Note on the Inverse Limit of Finite Dimensional Vector Spaces

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Abstract

In this paper, we solve some basic problems concerning inverse limits of finite dimensional vector spaces. For example, we prove directly that the inverse limit of an inverse system of vector spaces, which may fail to be compact, is always linearly compact. We show also that if $V = \varprojlim V_i$ is an inverse system of finite dimensional vector spaces and $\{f_i : V_i \rightarrow V_i\}$ is a family of diagonalizable linear operators which are compatible with respect to transition maps and λ is an eigenvalue of some f_i , then λ is an eigenvalue of $f = \varprojlim f_i$.

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1 Introduction

This paper starts with a clear concise introduction to some of the basic facts needed from the theory of inverse limits that includes definitions, examples, and some basic theorems of the theory of inverse limits. Such an introduction is given to provide a convenient repository for all readers. Other introductions can be found in [1], [2], [3], [6], [9]. Readers who are interested in inverse limits of algebraic groups or pro-affine algebraic groups can refer, for example, to [4], [5], [7], [8]. In section 3 we investigate the inverse limit of a Hochschild-Mostow inverse system, i.e. $\varprojlim X_i$, where each X_i is a non-empty set that can be equipped with a compact T_1 topology such that each transition map is a closed continuous map. Section 4 contains the new results. We study some important problems concerning inverse limits of finite dimensional vector spaces and Hochschild-Mostow inverse systems. For example, we show that if (V_i, π_{ji}) is a Hochschild-Mostow inverse system, then each projection map π_i

: $\varprojlim V_i \rightarrow V_i$ is closed. We prove also that the inverse limit of an inverse system of vector spaces is always linearly compact. Moreover, we prove that if (V_i) is a surjective inverse system of finite dimensional vector spaces and $\{f_i : V_i \rightarrow V_i\}$ is a family of diagonalizable linear operators which are compatible with respect to transition maps. Then, if λ is an eigenvalue of some f_i , then λ is an eigenvalue of $f = \varprojlim f_i$.

2 Preliminary Notes

Definition 2.1 Let I be a set with a partial ordering \leq . Suppose I is directed upwards, i.e., for every $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let $S = \{S_i : i \in I\}$ be a family of sets such that for every pair $(i, j) \in I \times I$ with $j \geq i$, there is a map $\pi_{ji} : S_j \rightarrow S_i$ satisfying the following two conditions:

1. π_{ii} is the identity map for every $i \in I$;
2. if $i \leq j \leq k$, then $\pi_{ki} = \pi_{ji} \circ \pi_{kj}$.

Let $\pi = \{\pi_{ji}, i, j \in I, j \geq i : \pi_{ji} : S_j \rightarrow S_i\}$

Then (S, π) or $(S_i, \pi_{ji})_{i, j \in I}$ or simply (S_i) is called an inverse system and the maps $\pi_{ji} : S_j \rightarrow S_i$ are called the transition maps of the inverse system.

The inverse limit of this system, denoted by $\varprojlim S_i$, is the subset of the Cartesian product $\prod_{i \in I} S_i$ consisting of all elements $s = (s_i)_{i \in I}$ such that $\pi_{ji}(s_j) = s_i$ for every $j \geq i$. In other words, $\varprojlim S_i$ is the set of all families of elements $(s_i)_{i \in I}$ which are compatible with the transition maps $\pi_{ji} : S_j \rightarrow S_i$.

Remark 2.2 If (S_i) is an inverse system, let $\pi_i : \varprojlim S_i \rightarrow S_i$ be the canonical projection sending (s_i) to s_i . Then $\pi_i = \pi_{ji} \circ \pi_j$ for every $j \geq i$.

Remark 2.3 A surjective inverse system is an inverse system whose transition maps are surjective.

Remark 2.4 An inverse sequence is an inverse system whose index set is $I = \mathbb{N}$ (directed by its natural order).

In general there are two fundamental problems:

If (S_i) is an inverse system of non-empty sets, then

1. $\varprojlim S_i$ may be empty.
2. If each transition map is surjective, then the canonical projection $\pi_i : \varprojlim S_i \rightarrow S_i$ may fail to be surjective. i.e, each s_i in S_i may fail to be part of a compatible family (s_i) in $\varprojlim S_i$

Definition 2.5 A collection of sets (A_i) is called a subinverse system of an inverse system (G_i) , and we write $(A_i) \subset (G_i)$ if $A_i \subset G_i$ for all $i \in I$ and $\pi_{ji}(A_j) \subseteq A_i$ for all $j \geq i$.

Example 2.6 Let (S, π) be an inverse sequence where $S_n = \mathbb{N}$ for every integer n . If each transition map is the identity, then $\varprojlim S_n$ is bijective to \mathbb{N} .

Example 2.7 Let (S, π) be an inverse sequence where $S_n = \mathbb{N}$ for every integer n . If $\pi_{n+1,n}(x) = x + 1$, then $\varprojlim S_n = \emptyset$.

Proof. By contradiction, suppose that $\varprojlim S_n \neq \emptyset$ and let (s) be an element of $\varprojlim S_n$. Then, $(s) = (s_0, s_1, s_2, \dots)$, where $s_1 = s_0 - 1$, $s_2 = s_0 - 2$, and so on. Thus $s_{s_0} = s_0 - s_0 = 0$, and $s_{s_0+1} = -1 \notin \mathbb{N}$.

Thus, $\varprojlim S_n$ is empty.

Example 2.8 Directed unions yield inverse systems:

Let $A = \cup A_i$ be a directed union of sets (directed by inclusion). That is, for each A_i, A_j , there exists A_k such that $A_i \subset A_k$ and $A_j \subset A_k$. Let K be any fixed set and let $\text{Fun}(A_i, K)$ be the set of all maps of A_i into K . If $A_i \subset A_j$, let $\pi_{ji} : \text{Fun}(A_j, K) \rightarrow \text{Fun}(A_i, K)$ be the restriction map, and let

$$F = \{\text{Fun}(A_i, K), i \in I\} \text{ and } \pi = \{\pi_{ji}, i, j \in I, j \geq i (\pi_{ji} : \text{Fun}(A_j, K) \rightarrow \text{Fun}(A_i, K))\}.$$

Then (F, π) is an inverse system. Moreover, its inverse limit is bijective to $\text{Fun}(A, K)$.

Proof. (F, π) is clearly an inverse system. Moreover, define

$h : \text{Fun}(A, K) \rightarrow \varprojlim \text{Fun}(A_i, K)$ by $h(f) = (f_i)$ where f_i is the restriction of f to A_i , then h is one to one. h is also onto : let $(f_i) \in \varprojlim \text{Fun}(A_i, K)$, so $\pi_{ji}(f_j) = f_i$ for all $j \geq i$ ($A_i \subset A_j$).

Define $f : A \rightarrow K$ by $f(a) = f_i(a)$ where $a \in A_i$.

Then f is well defined because A is directed by inclusion, and for all $a \in A_i$, $f(a) = f_i(a)$. So f_i is the restriction of f to A_i , and $h(f) = (f_i)$.

Example 2.9 (Waterhouse). In the above example, let A be any uncountable directed set, and let $\{A_i\}$ be the family of all finite subsets of A , then $A = \cup A_i$. Let $K = \mathbb{N}$, and let $\text{Fun}(A_i, \mathbb{N})$ be the set of all 1-1 maps of A_i into \mathbb{N} . If $A_i \subset A_j$, let $\pi_{ji} : \text{Fun}(A_j, \mathbb{N}) \rightarrow \text{Fun}(A_i, \mathbb{N})$ be the restriction map and let $F = \{\text{Fun}(A_i, \mathbb{N}), i \in I\}$ and let $\pi = \{\pi_{ji}, i, j \in I, j \geq i (\pi_{ji} : \text{Fun}(A_j, \mathbb{N}) \rightarrow \text{Fun}(A_i, \mathbb{N}))\}$. Then, according to the previous example the inverse limit of the surjective inverse system (F, π) is bijective to the set of all 1-1 maps of A into \mathbb{N} which is empty since an uncountable set cannot be embedded in \mathbb{N} .

Example 2.10 Let $\mathbb{R}_n[x]$ be the set of all polynomials of degree at most n with coefficients in \mathbb{R} , and consider the inverse system

$$\begin{aligned} \mathbb{R} &\leftarrow \mathbb{R}_1[x] \leftarrow \mathbb{R}_2[x] \leftarrow \dots \text{with} \\ \pi_{n,n-1}(a_0 + a_1x + \dots + a_nx^n) &= a_0 + a_1x + \dots + a_{n-1}x^{n-1}. \text{ Then} \\ \varprojlim \mathbb{R}_n[x] &= \mathbb{R}[[x]], \text{ the power series on } x. \end{aligned}$$

Example 2.11 Let p be a prime number. Consider the following sequence $pZ \supset p^2Z \supset p^3Z \supset \dots$. This descending sequence yields an inverse system of rings $Z/pZ \leftarrow Z/p^2Z \leftarrow Z/p^3Z \leftarrow \dots$ whose transition maps are given by: $x + p^jZ \mapsto x + p^iZ$.

Then $\varprojlim Z/p^nZ$ is the ring of p -adic integers.

Definition 2.12 Let (S_i, π_{ji}) and (X_i, π'_{ji}) be two inverse systems over the same index set I . Suppose that for every $i \in I$ we are given maps $f_i : S_i \rightarrow X_i$ such that $\pi'_{ji} \circ f_j = f_i \circ \pi_{ji}$ for every $i \leq j$. That is, each such diagram

$$\begin{array}{ccc} S_j & \xrightarrow{f_j} & X_j \\ \pi_{ji} \downarrow & & \pi'_{ji} \downarrow \\ S_i & \xrightarrow{f_i} & X_i \end{array}$$

commutes. Then there exists a unique map $f : \varprojlim S_i \rightarrow \varprojlim X_i$ such that $\pi_i \circ f = f_i \circ \pi_i$ for every $i \in I$. We write $f = \varprojlim f_i$.

Remark 2.13 Let (G, π) be an inverse system of groups where each π_{ji} is a group homomorphism. Then

- (i) $\varprojlim G_i$ is a group
- (ii) each projection map $\pi_i : \varprojlim G_i \rightarrow G_i$ is a group homomorphism.

A similar remark holds for rings, modules and vector spaces.

Remark 2.14 Let (S, π) be a surjective inverse sequence. Then $\varprojlim S_i \neq \emptyset$ and each projection $\pi_i : \varprojlim S_i \rightarrow S_i$ is surjective.

Remark 2.15 Universal Property. Let (X, π) be an inverse system of sets and let $\pi_i : \varprojlim X_i \rightarrow X_i$ be the canonical projection. Then $\varprojlim X_i$ has the following universal property. For every set Y with projections $p_i : Y \rightarrow X_i$ such that $\pi_{ji} \circ p_j = p_i$ for each $i \leq j$, there exists a unique map $f : Y \rightarrow \varprojlim X_i$ such that $\pi_i \circ f = p_i$ for every $i \in I$, where f is defined as follows $f(y) = (p_i(y))$

Theorem 2.16 The following properties of a topological space X are equivalent:

- (i) If $\{U_i\}$ is any open covering of X , then some finite subcollection of $\{U_i\}$ is already a covering.
- (ii) If $\{F_i\}$ is a collection of closed sets with the finite intersection property, the intersection of the whole collection is non-empty.

Definition 2.17 A topological space X satisfying any of the equivalent properties in the above theorem is said to be compact.

Definition 2.18 Inverse limit topology Let (S_i, π_{ji}) be an inverse system where each set S_i is a topological space and each transition map is continuous. Then the inverse limit topology on $\varprojlim S_i$ is the induced topology inherited from the product topology on $\prod_{i \in I} S_i$ which has a basis consisting of the sets of the form $\prod_{i \in I_i} U_i$, with U_i is an open subset of S_i for every $i \in I$ and $U_i = S_i$ for all but a finitely many $i \in I$.

Remark 2.19 The inverse limit topology is the smallest topology that makes each projection map $\pi_i : \varprojlim S_i \rightarrow S_i$ continuous. A basis for the topology of $S = \varprojlim S_i$ is given by the sets $\pi_i^{-1}(U_i)$, as i varies, where U_i is open in S_i .

Remark 2.20 Let A be a subset of $\varprojlim S_i$ and let A_i be the image of A under the canonical projection $\pi_i : \varprojlim S_i \rightarrow S_i$. Then the closure of A is given by $\overline{A} = \varprojlim \overline{A_i}$. In particular, if A is closed in $\varprojlim S_i$, then $A = \overline{A} = \varprojlim \overline{A_i} = \varprojlim A_i = \cap(A + \text{Ker } \pi_i)$. Moreover, if $A = \varprojlim C_i$ where (C_i) is a closed inverse subsystem of (S_i) , then A is closed in $\varprojlim S_i$. If each S_i is given the discrete topology, the resulting inverse limit topology on $\varprojlim S_i$ is called the prodiscrete topology. Let (S_i, π_{ji}) and (S_i, π'_{ji}) be two inverse systems over the same index set I . If for every $i \in I$, the map $f_i : S_i \rightarrow S'_i$ is continuous, then $\varprojlim f_i : \varprojlim S_i \rightarrow \varprojlim S'_i$ is also continuous. Moreover, if each S_i is a topological group and each transition map is a continuous homomorphism, then $\varprojlim S_i$ is a topological group.

Definition 2.21 (The Coset Topology on algebraic groups) Let G be an algebraic group (over an algebraically closed field). The closed sets in the coset topology are the subsets of G that are finite unions of cosets xH , where x ranges over G and H ranges over the set of closed algebraic subgroups of G .

Definition 2.22 (The Coset Topology on Vector Spaces) Let V be a finite dimensional vector space over an arbitrary field F . There is a topology on V whose closed sets are the finite unions of sets of the form $x + W$, where x ranges over V and W ranges over the subspaces of V .

Remark 2.23 The vector space V equipped with the coset topology is a compact T_1 space. Also, every linear transformation from V into another such space is continuous and sends closed sets onto closed sets.

3 Hochschild-Mostow Inverse Systems

Definition 3.1 (*essential elements and perfect inverse systems*) Let (X, π) be an inverse system and let $x_i \in X_i$. Then

(i) x_i is called an essential element if $x_i \in \text{Image}(\pi_{ji})$ for all $j \geq i$ in I . That is, $x_i \in \bigcap_{j \geq i} \pi_{ji}(X_j)$.

(ii) x_i is called super essential element if x_i can be completed to a full compatible family. That is, there exists $x = (x_i) \in \varprojlim X_i$ where x_i is a component of x .

(iii) (X, π) is called a perfect inverse system if for all $x_i \in X_i$, x_i is a component of a compatible family $(x_i)_{i \in I} \in \varprojlim X_i$, that is, each element $x_i \in X_i$ is a super essential element, so each $\pi_i: \varprojlim X_i \rightarrow X_i$ is surjective.

Remark 3.2 If x_i is a super essential element, then it is an essential element.

Proof. Let $j \geq i$, and let $x = (x_i) \in \varprojlim X_i$ be a compatible family with x_i being one of its components. Then there exists x_j component of x such that $\pi_{ji}(x_j) = x_i$, so $x_i \in \pi_{ji}(X_j)$.

Note 3.3 The converse is not true in general. Take for instance the Waterhouse example. It provides an example where we have a surjective inverse system with empty inverse limit. Thus all the elements are essential, but, none is a part of a compatible family in $\varprojlim X_i$.

However, the converse is true when:

(i) (X, π) is a surjective inverse sequence.

(ii) In the standard limit theorem situation (Theorem 3.6).

(iii) In the projective limit theorem situation (Theorem 3.13).

Remark 3.4 In a surjective inverse system, all the elements are essential.

Remark 3.5 If (X, π) is a perfect inverse system then it is a surjective inverse system, while the converse is not true by the Waterhouse Example.

Theorem 3.6 (*The standard limit theorem*) Let (S, π) be an inverse system (over any directed set I) of non-empty compact Hausdorff spaces with continuous transition maps π_{ji} . Then

(i) $S = \varprojlim S_i$ is a non-empty compact space.

(ii) If each π_{ji} is surjective, then each $\pi_i: \varprojlim S_i \rightarrow S_i$ is surjective.

(iii) If $s_i \in S_i$ is an essential element, then s_i is super essential.

Corollary 3.7 *Let (S, π) be an inverse system over any directed set I of non-empty finite sets. Then $\varprojlim S_i$ is not empty, and if each π_{ji} is surjective, then each $\pi_i: \varprojlim S_i \rightarrow S_i$ is surjective.*

Definition 3.8 *(Hochschild-Mostow inverse system) An inverse system (X, π) of non-empty sets is called a Hochschild-Mostow inverse system, or a super inverse system, or a compact system if each X_i can be equipped with a compact T_1 topology such that each transition map is a closed continuous map.*

Example 3.9 *Every inverse system of non-empty finite sets is a Hochschild-Mostow inverse system by taking the discrete topology on each given set.*

Example 3.10 *Every inverse system of non-empty compact Hausdorff (T_2) spaces (with continuous transition maps) is a Hochschild-Mostow inverse system. This is because every continuous map between compact Hausdorff spaces is a closed map.*

Example 3.11 *Every inverse system of finite dimensional vector spaces over an algebraically closed field (with linear transition maps) is a Hochschild-Mostow inverse system provided we give each vector space its coset topology.*

Example 3.12 *Every inverse system of linear algebraic groups over an algebraically closed field (whose transition maps are rational homomorphisms) is a Hochschild-Mostow inverse system provided we give each algebraic group its coset topology.*

Theorem 3.13 *(The Projective Limit Theorem) Let (V_i, π_{ji}) be a super inverse system consisting of non-empty compact T_1 spaces and continuous closed maps $\pi_{ji}: V_j \rightarrow V_i$. Then*

- (i) $\varprojlim V_i \neq \emptyset$.
- (ii) If each π_{ji} is surjective, then each canonical map projection $\pi_i: \varprojlim V_i \rightarrow V_i$ is surjective.
- (iii) If $v_i \in V_i$ is an essential element, then v_i is super essential.

Remark 3.14 *Let (V_i, π_{ji}) be a compact inverse system and let (A_i) be an subinverse system of (V_i) such that each A_i is a non-empty closed subset of V_i . Then (A_i) is also a compact inverse system.*

Proposition 3.15 *Let $f: X \rightarrow Y$ be a continuous map of topological spaces. If X is compact, Y is a T_1 space, $\{A_i\}$ is a chain of closed subsets of X , then $f(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f(A_i)$.*

Proof. It is obvious that $f(\cap_{i \in I} A_i) \subseteq \cap_{i \in I} f(A_i)$. On the other hand, let $y \in \cap_{i \in I} f(A_i)$ so $y = f(x_i)$ with $x_i \in A_i$ for all $i \in I$. Then, $y = f(x)$, where $x \in A_i$ for all $i \in I$. Consider the collection $\{f^{-1}(y) \cap A_i\}_{i \in I}$; Since $\{y\}$ is closed in the T_1 space Y and f is continuous then $f^{-1}(y)$ is closed in X . Hence $f^{-1}(y) \cap A_i$ is closed in X for all $i \in I$. Also, $x_i \in f^{-1}(y) \cap A_i$ for all $i \in I$. Since $\{A_i\}$ is a chain, $\{f^{-1}(y) \cap A_i\}$ is a chain of non-empty closed subsets of X . Take a finite subset $\{f^{-1}(y) \cap A_{ij}\}_{j=1,2,\dots,n}$ of $\{f^{-1}(y) \cap A_i\}$. Because we have a chain, all elements are comparable, so there exists a smallest $f^{-1}(y) \cap A_{il}$ for some $l \in \{1, 2, \dots, n\}$. So $\bigcap_{j=1}^n (f^{-1}(y) \cap A_{ij}) = f^{-1}(y) \cap A_{il}$ is non-empty, so the finite intersection property holds for the collection $\{f^{-1}(y) \cap A_i\}$, but X is compact, then the intersection of the whole collection is non-empty, i.e. there exists $x \in f^{-1}(y) \cap A_i$ for all $i \in I$ such that $y = f(x)$ and $x \in \cap_{i \in I} A_i$. Thus $f(x) \in f(\cap_{i \in I} A_i)$ which implies that $y \in f(\cap_{i \in I} A_i)$ and $\cap_{i \in I} f(A_i) \subseteq f(\cap_{i \in I} A_i)$.

4 Inverse Limits Of Vector Spaces

Lemma 4.1 *Let (V_i, π_{ji}) be a Hochschild-Mostow inverse system and let $V = \varprojlim V_i$ (with its inverse limit topology). Then each projection $\pi_i : V \rightarrow V_i$ is a closed map.*

Proof. Let A be a closed subset of V . Then $A = \varprojlim A_i = \varprojlim \overline{A_i}$ where $A_i = \pi_i(A)$. Because the π_{ji} are continuous, $\pi_{ji}(\overline{A_j}) \subset \overline{\pi_{ji}(A_j)} = \overline{A_i}$ for all $j \geq i$, but the π_{ji} are also closed, so $\pi_{ji}(\overline{A_j}) = \overline{A_i}$ and consequently $(\overline{A_i})$ is a surjective Hochschild-Mostow subinverse system of (V_i) . Now, by the Projective limit theorem, each projection $\pi_i : \varprojlim \overline{A_i} \rightarrow \overline{A_i}$ is surjective. But $\varprojlim \overline{A_i} = A$ hence $\pi_i(A) = \pi_i(\varprojlim \overline{A_i}) = \overline{A_i}$. Thus, π_i is a closed map.

Theorem 4.2 *Let (V_i, π_{ji}) be a Hochschild-Mostow inverse system. Then, $\varprojlim V_i$ (with its inverse limit topology) is compact.*

Proof. This theorem was proved by Stone in [10]. But here we give the following different proof: Let $\{F_\alpha\}_{\alpha \in A}$ be a family of non-empty closed subsets of $\varprojlim V_i$ with the finite intersection property. Then $\cap_{\alpha \in A} F_\alpha \neq \emptyset$: $\pi_i(F_\alpha)$ is closed for all $\alpha \in A$ since each π_i is a closed map. Let $X_i = \cap_{\alpha \in A} \pi_i(F_\alpha)$. Since each V_i is compact and each X_i is a closed subset of V_i , then each X_i is a non-empty compact T_1 space. Hence (X_i) is a Hochschild-Mostow inverse subsystem of (V_i) , and the Projective limit theorem implies that $\varprojlim X_i \neq \emptyset$. Let $(x) \in \varprojlim X_i$, and $x_i = \pi_i((x)) \in X_i$. $x_i \in \pi_i(F_\alpha)$ for all $i \in I$, $\alpha \in A$, hence $(x) \in \varprojlim \pi_i(F_\alpha) = F_\alpha$ for all $\alpha \in A$ (since each F_α is closed). So $(x) \in \cap_{\alpha \in A} (F_\alpha)$, and thus the intersection of the whole collection is non-empty, and consequently $\varprojlim V_i$ is compact.

Remark 4.3 *One might expect that, we could prove*

“Let I be a chain, and let $(X_i, \pi_{ji})_{i \in I}$ be a surjective inverse system of compact T_1 spaces. Then $\varprojlim X_i$ is compact”. This, however, is false as the following example shows.

Example 4.4 *For $n \in \mathbb{N}$, define $X_n = \mathbb{N}$. Give each X_n the following topology: Each of the points $1, 2, \dots, n$ is isolated, and for each $m > n$, a neighborhood base at m consists of the cofinite subsets of \mathbb{N} that contain m . This is easily seen to give a compact T_1 topology. Define $\pi_{n+1, n}: X_{n+1} \rightarrow X_n$ to be the identity map. This is continuous, because the only point at which the topologies of X_n and X_{n+1} differ is $n + 1$, which is isolated in X_{n+1} . The inverse limit $\varprojlim X_i$ is the subset of $\mathbb{N}^{\mathbb{N}}$ consisting of the points all of whose coordinates are equal. This is an infinite discrete set, since, given $p = (m, m, \dots)$ of $\varprojlim X_i$, take $n > m$, then $\pi_n^{-1}(m)$ is a neighborhood of p in $\mathbb{N}^{\mathbb{N}}$ that contains no other point of $\varprojlim X_i$, where π_n is the projection map. Thus $\varprojlim X_i$ is not compact.*

Proposition 4.5 *Let $V = \varprojlim V_i$, where $(V_i, \pi_{ji})_{i, j \in I}$ is an inverse system of vector spaces over an algebraically closed field such that each $\pi_{ji}: V_j \rightarrow V_i$, for $j \geq i$ is a linear transformation and let $\pi_i: V \rightarrow V_i$ be the canonical projection. Then*

1. *if each π_{ji} is surjective, then so is each π_i .*
2. *if $(A_i)_{i \in I}$ is a subinverse system of $(V_i)_{i \in I}$, where each V_i is a closed non-empty subset of V_i , then $\varprojlim A_i$ is non-empty.*

Proof. This follows directly from the Projective limit Theorem and from Remark 3.14.

Theorem 4.6 *Let $V = \varprojlim V_i$, where $(V_i, \pi_{ji})_{i, j \in I}$ is an inverse system of finite-dimensional vector spaces with linear transition maps over an algebraically closed field. Then V is linearly compact in the sense that any family of left cosets of closed subspaces of V with the finite intersection property has non-empty intersection (where V is given the pro-discrete topology).*

Proof. This can be obtained from Theorem 4.2, but we will give a different direct proof. Let $\{S^t\}_{t \in T}$ be a family of closed left cosets of V with the finite intersection property. By Lemma 4.1 the projections π_i map closed subspaces to closed subspaces. $\{\pi_i(S^t) = (S^t)_i\}$ is a family of closed left cosets with the finite intersection property. Let $X_i = \bigcap_t (S^t)_i$. Since V_i is compact, X_i is a non-empty closed subset of V_i . Thus, $(X_i)_{i \in I}$ is a sub-inverse system of closed subsets of $(V_i)_{i \in I}$. Hence, $X = \varprojlim X_i$ is non-empty by Proposition 4.5. Let $x \in X$. For a given t , $\pi_i(x) = x_i \in (S^t)_i$ for every $i \in I$ and hence $x \in \varprojlim (S^t)_i$. Since S^t is closed in V , $S^t = \varprojlim (S^t)_i$. Thus, for every $t \in T$ $x \in S^t$, thus $\bigcap_t (S^t)_i$ is non-empty.

Theorem 4.7 *Let (V_i) be a surjective inverse system of finite dimensional vector spaces over an algebraically closed field. Let $\{f_i : V_i \rightarrow V_i\}$ be a family of diagonalizable linear operators which are compatible with respect to transition maps. If λ is an eigenvalue of some f_i , then λ is an eigenvalue of $f = \varprojlim f_i$.*

Proof. For all $j \geq i$, let $\{E_\lambda^j = x \in V_j : f_j(x) = \lambda x\}$. $E_\lambda^j \neq 0$, $\pi_{ji}(E_\lambda^j) = E_\lambda^i$ and $\pi_{kj}(E_\lambda^k) = E_\lambda^j$ for all $k \geq j \geq i$, thus $\{E_\lambda^j : j \geq i\}$ form a surjective inverse system of finite dimensional vector spaces. Let $E = \varprojlim E_\lambda^j$. Then the projection map: $\pi_i : E \rightarrow E_i$ is surjective by the Projective Limit theorem, and thus $E \neq 0$. Let $e = (e_i)$ be a non zero element of $E = \varprojlim E_\lambda^j$. Then $f(e) = f((e_j)) = (f_j(e_j)) = (\lambda e_j) = \lambda e$. Thus λ is an eigenvalue of f .

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