On $\sigma(n) \equiv 0 \mod n$, $n\sigma(n) \equiv 2 \mod \varphi(n)$ and $\varphi(N)|N - 1$

Cheng Linfeng

College of Science, China University of Mining and Technology
Xuzhou, Jiangsu, 221008, P.R. China
yxjcumt@126.com

Abstract

For any positive integer $n$, let $\varphi(n)$ denote the Euler function, $\sigma(n)$ denote the sum of the positive divisors of $n$. In this paper we give the following results: (1) There are no squarefree composite $n$ with $\sigma(n) \equiv 0 \mod n$ except for $n = 6$; (2) There are no squarefree composite $n$ with $n\sigma(n) \equiv 2 \mod \varphi(n)$ except for $n = 6$ or $n = 22$; (3) Let $n$ and $t$ be two positive integers with $t \geq 2$ and $\varphi(n)|n - 1$. If $n$ has exactly $t$ prime factors $p_1 < p_2 < \cdots < p_t$, then $p_1 < 2t$, $p_j < (\frac{3}{2})^t(t - j + 1)(p_1 - 1)\cdots(p_{j-1} - 1)$.

Mathematics Subject Classification: 11A07; 11B25

Keywords: Squarefree, Euler function

1 Intruduction

For any positive integer $n$, let $\varphi(n)$ denote the Euler function and $\sigma(n)$ denote the sum of the positive divisors of $n$. If $\sigma(n) \equiv 0 \mod n$ or $\sigma(n) = kn$, then $n$ is said to be a (multiply) perfect number. Several partial results about (multiply) perfect number can be found in [1]. In this paper we prove that:

**Theorem 1.1** There are no squarefree composite $n$ with $\sigma(n) \equiv 0 \mod n$ except for $n = 6$.

Obviously, if $n$ is prime, then $n\sigma(n) \equiv 2 \mod \varphi(n)$. It is also true when $n$ is 4 or 6 or 22. Is this true for any other composite $n$? The question was posed in [1.B37]. It is easy to see that if such $n > 4$ exists, then $n$ is squarefree. About this question, we prove that:

**Theorem 1.2** There are no squarefree composite $n$ with $n\sigma(n) \equiv 2 \mod \varphi(n)$ except for $n = 6$ or $n = 22$. 
If $n$ is prime, then $\varphi(n)|n-1$. D.H. LEHMER conjectured that $\varphi(n)|n-1$ if and only if $n$ is prime [1]. Is this true for any composite $n$? This is still an open problem. Several partial results can be found in [1], [2].

Since $\varphi(n)|n-1$, we can find an integer $k \geq 2$ such that

$$n - 1 = k\varphi(n)$$

So we have $(n, \varphi(n)) = 1$. We see that $n$ must be squarefree and odd. In this note, we prove that

**Theorem 1.3** Let $n$ and $t$ be two positive integers with $t \geq 2$ and $\varphi(n)|n-1$. If $n$ has exactly $t$ prime factors $p_1 < p_2 < \cdots < p_t$, then $p_1 < 2t$, $p_j < (\frac{3}{2})^t(t-j+1)(p_1-1)\cdots(p_{j-1}-1)$.

**Remark** In fact, by use this result we can easily prove that there are no integers $n$ with $2 \leq t \leq 3$ and $\varphi(n)|n-1$.

## 2 Proof of the Theorems

In this section, we will complete the proofs of the three Theorems.

**Proof of Theorem 1.1**

Let $n$ be squarefree and has exactly $t$ prime factors $p_1 < p_2 < \cdots < p_t$ with $t \geq 2$, then

$$n = p_1 p_2 \cdots p_t$$

$$\sigma(n) = (p_1 + 1)(p_2 + 1)\cdots(p_t + 1)$$

By $\sigma(n) \equiv 0 \mod n$ we have

$$p_1 p_2 \cdots p_t | (p_1 + 1)(p_2 + 1)\cdots(p_t + 1)$$

So there exists a positive integer $k \geq 2$ such that

$$(p_1 + 1)(p_2 + 1)\cdots(p_t + 1) = kp_1 p_2 \cdots p_t \quad (1)$$

Then we have

$$k = \left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2}\right)\cdots\left(1 + \frac{1}{p_t}\right) < \left(1 + \frac{1}{p_1}\right)^t \leq \left(\frac{3}{2}\right)^t \quad (2)$$

If $p_1 \geq 3$, from (1) we can get $2^t|k$, a contradiction with (2). So we have $p_1 = 2$. Hence, by (1) we get

$$3(p_2 + 1)\cdots(p_t + 1) = 2k p_2 \cdots p_t \quad (3)$$

It is easy to see that $2^{t-2}|k$, so $2^{t-2} \leq k$. Combining this inequality with (2), we have

$$2^{t-2} < \left(\frac{3}{2}\right)^t$$

Obviously, this is impossible for $t \geq 5$. Hence, $t \leq 4$. 

On $\sigma(n) \equiv 0 \mod n$, $n\sigma(n) \equiv 2 \mod \varphi(n)$ and $\varphi(N)|N-1$ 615

Now we consider the case $t = 2, 3, 4$ respectively. If $t = 2$, from (3) we obtain $3(p_2 + 1) = 2kp_2$, which has the only solution $k = 2$ and $p_2 = 3$, so we have $n = 6$; If $t = 3$, from (3) we obtain $3(p_2 + 1)(p_3 + 1) = 2kp_2p_3$. Hence

$$3(1 + \frac{1}{p_2})(1 + \frac{1}{p_3}) = 2k < 3(1 + \frac{1}{p_2})^2 < \frac{46}{3} < 6$$

So we have $k = 2$ and $3(p_2 + 1)(p_3 + 1) = 4p_2p_3$, which we can get $p_2 = 3$.

Then we have $12(p_3 + 1) = 12p_3$, a contradiction. Similarly to the case $t = 3$, we can prove that there are no integers $n$ with $\sigma(n) \equiv 0 \mod n$ when $t = 4$, which complete the proof of theorem 1.

**Proof of Theorem 1.2**

Since $n\sigma(n) \equiv 2 \mod \varphi(n)$, then we have

$$(p_1 - 1)(p_2 - 1) \cdots (p_t - 1)p_1p_2 \cdots p_t(p_1 + 1)(p_2 + 1) \cdots (p_t + 1) - 2$$

(4)

If $p_1 \geq 3$, from (4) we can get $2^t|2$, a contradiction with $t \geq 2$. So we have $p_1 = 2$. By (4), we can obtain

$$(p_2 - 1) \cdots (p_t - 1)6p_2 \cdots p_t(p_2 + 1) \cdots (p_t + 1) - 2$$

Hence, $2^{t-1}|2$, which is true if and only if $t = 2$.

So there exists a positive integer $k \geq 2$ such that

$$6p_2(p_2 + 1) - 2 = k(p_2 - 1)$$

So the question turn to be whether the equation

$$6x^2 + (6 - k)x + k - 2 = 0$$

(5)

has the prime solution. We know that the possible solution may be $k - 2$ or its prime factors. It is clear that $k - 2$ is not the solution of equation (5), so we may assume that

$$k - 2 = pT$$

(6)

where $p$ is a prime and $T$ is a positive integer with $T \geq 2$. If $p$ is a solution of equation (5), then we have

$$6x^2 + (6 - k)x + k - 2 = (x - p)(6x - T)$$

Hence

$$6p + T = k - 6$$

Combining the above equality with (6), we can get

$$p(T - 6) = T + 4$$

Which is true if and only if $T = 7$ or $T = 11$. So $p = 3$ or $p = 11$. It follows that $n = 6$ or $n = 22$.

**Proof of Theorem 1.3**

By $n - 1 = k\varphi(n)$ we have

$$p_1p_2 \cdots p_t - 1 = k(p_1 - 1)(p_2 - 1) \cdots (p_t - 1)$$

Then
By (7) we have

\[ k\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_t}\right) + \frac{1}{p_1 p_2 \cdots p_t} = 1 \quad (7) \]

So

\[ 1 > k\left(1 - \frac{1}{p_1}\right)^t \geq k\left(1 - \frac{t}{p_1}\right) \quad (8) \]

The last inequality is based on the fact that \((1 - x)^\alpha \geq 1 - \alpha x\) for \(0 < x < 1\) and \(\alpha \geq 1\). Hence

\[ p_1 < \frac{k}{k-1}t \leq 2t \]

For \(2 \leq j \leq t\), we have

\[ 1 = k\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_j}\right) + \frac{1}{p_1 p_2 \cdots p_j} \]

\[ \leq k\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_{j-1}}\right) \left(1 - \frac{1}{p_j}\right) + \frac{1}{p_1 p_2 \cdots p_j} \]

\[ < k\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_{j-1}}\right) \]

It is that

\[ k\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_{j-1}}\right) - 1 \geq 0 \]

Since the left side of the above inequality is a positive rational number, it is at least as large as \(\frac{1}{p_1 p_2 \cdots p_{j-1}}\).

Thus

\[ k\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_{j-1}}\right) - 1 \geq \frac{1}{p_1 p_2 \cdots p_{j-1}} \]

Hence

\[ 1 \leq k\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_{j-1}}\right) = \frac{1}{p_1 p_2 \cdots p_{j-1}} \quad (9) \]

By (7) we have

\[ 1 = k\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_j}\right) + \frac{1}{p_1 p_2 \cdots p_j} \]

\[ > k\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_{j-1}}\right) \left(1 - \frac{1}{p_j}\right) \left(t - \frac{j}{p_j}\right) \]

\[ \geq k\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_{j-1}}\right) \left(1 - \frac{t - j + 1}{p_j}\right) \]

Combining the above inequality with (8), we have

\[ k\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_{j-1}}\right) \left(1 - \frac{t - j + 1}{p_j}\right) < k\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_{j-1}}\right) - \frac{1}{p_1 p_2 \cdots p_{j-1}} \]

So we have

\[ p_j < k(t - j + 1)(p_1 - 1)(p_2 - 1) \cdots (p_{j-1} - 1) \]

Since \(n - 1 = k\varphi(n)\) and \(\sum_{d|n} \varphi(d) = n\), then

\[ k = \sum_{d|n,d|\varphi(d)} \frac{1}{\varphi(d)} < \left(\frac{3}{2}\right)^t \]

Which complete the proof.

**Acknowledgement:** we would like to thank the referee for his/her suggestions.
On \( \sigma(n) \equiv 0 \mod n, n\sigma(n) \equiv 2 \mod \varphi(n) \) and \( \varphi(N)|N-1 \)

References


Received: December, 2008