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Concept Lattices and Co-covering Graphs[1]

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Abstract

This paper provides the definitions of co-basis and co-covering graph. Using these new notations, it deals with the relationship between concept lattices and co-covering graphs. Afterwards, it solves the relation between complete lattices and co-covering graphs, and further, obtains that for a complete lattice, there is a concept lattice isomorphic to it.

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1 Introduction

A.Berry *et.al.* pointed out [1] that concept lattice theory, though studied by mathematicians as far back as the Nineteenth Century, was made popular and developed by Wille and his team, and remains one of the important current trends of research in Data Mining and Artificial Intelligence: concept lattices are used in fields as varied as the discovery of association rules in Data Bases, the generation of frequent item sets, machine learning, software engineering and the reorganization of object hierarchies.

Additionally, [2] said that formal concept analysis includes a variety of construction methods. Many of these constructions are lattice-theoretic in nature and origin.

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All of references tell that concept lattice theory is the essential of formal concept analysis. It is just as B.Ganter and R.Wille said [5] that formal concept analysis can be regarded chiefly as a branch of applied lattice theory.

The beyond implies that it is of primary importance to find out the properties and constructions of concept lattice in lattice theory.

The main purpose of this paper is to solve with the relationships between concept lattices and co-covering graphs. Simultaneously, it presents a new way to discuss the relation between complete lattices and concept lattices. All these are based on lattice theory.

We announce that in this paper, some of conclusions are obtained by other authors before, but here we just try to provide approaches different from the others to earn the same conclusions. We notice that:

(1) [11] discusses Galois connections. We know that concept lattices have some relations with Galois connections. It seems that our paper is not worth to read. Actually, [11] is strongly for its topology applications, and does not talk about other applications. From this point, we also say that this paper shows an application of Galois connections to the theory of concept lattice.

(2) We exactly get some hints from [11], and we obtain our ideas and helps from [6,11] and all other our references to prove and express our results. All these proofs here are directly and easy to deal with the properties for concept lattices. Additionally, the approaches here are perhaps useful for the research in the future.

(3) [6] provides some algorithms for building lattices. Using our results here, we will directly find an algorithm from [6] to calculate its concept lattice for a given context. From this point, we see that our results here are the preparation and foundation to generate concept lattices using the algorithms in [6]. We will not describe to choose which algorithms in [6] is better to generate its concept lattice for a given context in this paper. In fact, it is not difficult to do this work.

Based on these three points, we see that our paper has some value to be read, and the results here could provides their lights on our readers.

Our process is firstly to present the definition of the co-covering graph generated by a co-basis related to a context. Afterwards, it sets up the relationship between the concept lattice of a context and the co-covering graph related to the context, and the following is to deal with the relation between complete lattices and concept lattices. At the final part, we present an example to show that our method is feasible.

To do this, we assume that the reader is familiar with standard definitions for partially ordered sets especially lattices. For the definitions and proofs of poset theory not given here, we refer to [3].

2 Preliminaries

Although some of the definitions appearing throughout this section do not require that the sets involved be finite, we make a standing assumption that all sets under consideration in this paper are finite.

For the knowledge of concept lattices, we recall back a few as follows. For more message, we will get from [4,5].

Definition 1 [4,& 5,pp.17-18] A triple (O, P, I) is called a *context*, if O and P are sets and $I \subseteq O \times P$ is a binary relation between O and P. We call the elements of O *objects*, those of P *attributes*, and I the *incidence* of the context (O, P, I). For $A \subseteq O$, we define

$A' = \{p \in P | (o, p) \in I$, for all $o \in A\}$, and dually, for $B \subseteq P$,

$B' = \{o \in O | (o, p) \in I$, for all $p \in B\}$.

A pair (A, B) is a *concept* of (O, P, I) if and only if

$A \subseteq O, B \subseteq P, A' = B$, and $B' = A$.

A is called the *extent* and B the *intent* of the concept (A, B).

Lemma 1[4,&5,pp.19-20] Let $R = (O, P, I)$ be a context. For all $A_j, A \subseteq O$ and all $B_j, B \subseteq P, (j \in J)$, it has the following rules:

(1) I) $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$; I') $B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$;

II) $A \subseteq A''$ and $A' = A'''$; II") $B \subseteq B''$ and $B' = B'''$;

III) $A \subseteq B' \Leftrightarrow B \subseteq A'$;

IV) $(\bigcup\limits_{j \in J} A_j)' = \bigcap\limits_{j \in J} A_j'$ and $(\bigcup\limits_{j \in J} B_j)' = \bigcap\limits_{j \in J} B_j'$;

V) The concepts of R are ordered by the following relation,

$(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2$ $(\Leftrightarrow B_2 \subseteq B_1)$.

The ordered set of all concepts of R is denoted by $Gal(R)$ and it is called the *concept lattice* of R.

(2) $Gal(R)$ is a complete lattice in which infimum and supremum are given by

$\bigwedge\limits_{t \in T} (A_t, B_t) = (\bigcap\limits_{t \in T} A_t, (\bigcup\limits_{t \in T} B_t)'')$, $\bigvee\limits_{t \in T} (A_t, B_t) = ((\bigcup\limits_{t \in T} A_t)'', \bigcap\limits_{t \in T} B_t)$.

(3) Concept lattices can be visualized with the usual lattice diagrams.

We write $(o, p) \in I$ as oIp.

We present some definitions as follows. They are useful in the following sections.

Definition 2 Let $R = (O, P, I)$ be a context and $x \in O$. We define $\uparrow x = \{y \in P | xIy\}$. $\mathcal{C} = \{\uparrow x | x \in O\}$ is called the *co-basis related to* R. We denote by $\mathcal{T}_{\mathcal{C}}$ the family generated by intersection from the co-bases $\mathcal{C}$, i.e. $\mathcal{T}_{\mathcal{C}} = \{\bigcap\limits_{B \in J} B | J \subseteq \mathcal{C}\}$.

Lemma 2 (1) The family $\mathcal{T}_\mathcal{C}$ is said closed under intersection.

(2) When ordered by inverse inclusion $\subseteq_c$(i.e. $A \subseteq_c B \Leftrightarrow$ "$\forall y \in B \Rightarrow y \in A$"), $\mathcal{T}_\mathcal{C}$ is an inf-sublattice of the powerset lattice 2^X and thus a complete lattice. $(\mathcal{T}_\mathcal{C}, \subseteq_c)$ is called the *co-covering graph related to* R.

Proof Routine verification.

3 Main Results

We present theorems about the relation between concept lattices and co-covering graphs, besides, about the relation between concept lattices and complete lattices.

In light of Definition 1, if $(A, B) \in Gal(R)$, then A is uniquely determined by B. This hints that we only need to find out all the intents' properties when we consider $Gal(R)$. This idea will be through this section.

Theorem 1 Let R be a context and $\mathcal{C}$ be the co-basis related to R. Then $Gal(R)$ is isomorphic to $(\mathcal{T}_\mathcal{C}, \subseteq_c)$, i.e. $Gal(R) \cong (\mathcal{T}_\mathcal{C}, \subseteq_c)$.

To carry out the proof of Theorem 1, we firstly provide some lemmas.

Lemma 3 Let $R = (O, P, I)$ be a context. Then the following statements are true.

(1) $((\uparrow a)', \uparrow a)$ is a concept of R for any $a \in O$.

(2) If $(A, B) \in Gal(R)$, then there is $(A, B) = (\bigcup\limits_{a \in A} (\uparrow a)', \bigcap\limits_{a \in A} \uparrow a)$.

(3) $(A, B) \in Gal(R)$ hints that there are $a_i \in O, (i \in I_B)$ satisfying $B = \bigcap\limits_{i \in I_B} \uparrow a_i$.

Proof (1) Because $(\uparrow a)' = \{x \in O | xIb \text{ for any } b \in \uparrow a\}$ and $(\uparrow a)'' = \{y \in P | xIy \text{ for any } x \in (\uparrow a)'\}$. Obviously, $a \in (\uparrow a)'$. Furthermore, for every $y \in (\uparrow a)''$, it should have aIy, and so $y \in \uparrow a$, and hence $(\uparrow a)'' \subseteq \uparrow a$. This result combining with Lemma 1 II) will induce $(\uparrow a)'' = \uparrow a$. Therefore, $((\uparrow a)', \uparrow a)$ is a concept of R.

(2) Let $a \in A$. We see that $B = A' = \{b \in P | xIb \text{ for any } x \in A\}$ is right, especially, aIb holds for any $b \in B$. This hints $B \subseteq \uparrow a$. Thus, $(A, B) \supseteq ((\uparrow a)', \uparrow a)$ is correct.

Furthermore, $\bigvee\limits_{a \in A} ((\uparrow a)', \uparrow a) \subseteq (A, B)$, i.e. $((\bigcup\limits_{a \in A} (\uparrow a)')'', \bigcap\limits_{a \in A} \uparrow a) =$ $((\bigcap\limits_{a \in A} (\uparrow a)'')', \bigcap\limits_{a \in A} \uparrow a) = ((\bigcap\limits_{a \in A} \uparrow a)', \bigcap\limits_{a \in A} \uparrow a) = (\bigcup\limits_{a \in A} (\uparrow a)', \bigcap\limits_{a \in A} \uparrow a) \subseteq (A, B)$.

In addition, for any $a \in A$, it is evident $a \in (\uparrow a)'$, and so $A \subseteq \bigcup\limits_{a \in A} (\uparrow a)'$. Then by Definition 1, $(A, B) \subseteq (\bigcup\limits_{a \in A} (\uparrow a)', \bigcap\limits_{a \in A} \uparrow a)$.

Therefore, $(A, B) = (\bigcup\limits_{a \in A} (\uparrow a)', \bigcap\limits_{a \in A} \uparrow a)$.

(3) Let $A = \{a_i | i \in I_B\}$. Then it is easy to obtain $B = \bigcap_{i \in I_B} \uparrow a_i$ by (2).

Proof of Theorem 1 Using two steps to carry out the proof.

Step 1. By Lemma 3, $(A, B) \in Gal(R)$ implies $B = \bigcap_{a_i \in A} \uparrow a_i$. We assert that for any $a \in O \setminus A$, it has $B \cap \uparrow a \neq B$.

Otherwise, $B = B \cap \uparrow a = (\bigcap_{i \in I_B} \uparrow a_i) \cap \uparrow a$ holds for some $a \in O \setminus A$ where $A = \{a_i | i \in I_B\}$. $\uparrow a = \{y \in P | aIy\}$ leads to aIb for any $b \in B$. But $A = B' = \{y \in O | yIb$ for any $b \in B\}$ is true . Hence, $a \in B'$, i.e. $a \in A$ holds, a contradiction to $a \in O \setminus A$.

Conversely, it could receive that if $B = \bigcap_{a_i \in E \subseteq O} \uparrow a_i$ and for any $a \in O \setminus E$, it has $B \cap \uparrow a \neq B$, then $(E, B) \in Gal(R)$.

This is because $B'' = \{y \in P | xIy$ for any $x \in B'\} = \bigcap_{x \in B'} \uparrow x \supseteq B$. By Lemma 1 and the given, it has $B'' = (\bigcap_{a_i \in E} \uparrow a_i) \cap (\bigcap_{x \in B' \setminus E} \uparrow x)$.

Suppose $B \subset B''$. This implies that $B' \setminus E \neq \emptyset$ is set up. But in view of the given assumption, it gets $B' \setminus E = \emptyset$. This follows a contradiction. Therefore, $B'' = B$ holds. Certainly, $B' = E$ is right. In other words, it should have $(E, B) \in Gal(R)$.

Step 2. Define $f : Gal(R) \to (\mathcal{T}_\mathcal{C}, \subseteq_c)$ as

$(A, B) = (A, \bigcap_{i \in I_B} \uparrow a_i) \mapsto T_B = \bigcap_{i \in I_B} \uparrow a_i$, where $A = \{a_i | i \in I_B\}$.

According to Lemma 3 and Step 1, we know that I_B is uniquely decided by B, or equivalently to say, I_B is born from A. Hence, f is effective. Considering with Step 1, it is easy to prove that f is a bijection.

Let $(A_j, B_j) \in Gal(R)$ and $A_j = \{a_i | i \in I_{B_j}\}, (j = 1, 2)$. Lemma 1 tells us $(A_1, B_1) \wedge (A_2, B_2) = (A_1 \cap A_2, (B_1 \cup B_2)'')$ and $(A_1, B_1) \vee (A_2, B_2) = ((A_1 \cup A_2)'', B_1 \cap B_2)$.

Owing to the results in Section 2 and Step 1,

$(B_1 \cup B_2)'' = (B_1' \cap B_2')' = (A_1 \cap A_2)' = A_1' \cup A_2' = B_1 \cup B_2 = (\bigcap_{i \in I_{B_1}} \uparrow a_i) \cup (\bigcap_{i \in I_{B_2}} \uparrow a_i)$.

Besides, in view of Step 1, $B_1 \cap B_2 = (\bigcap_{i \in I_{B_1}} \uparrow a_i) \cap (\bigcap_{i \in I_{B_2}} \uparrow a_i) = \bigcap_{i \in I_{B_1} \cap I_{B_2}} \uparrow a_i$.

Posit $T_j = f(A_j, B_j) = \bigcap_{i \in I_{B_1}} \uparrow a_i, (j = 1, 2)$. Therefore,

$T_1 \vee T_2 = (\bigcap_{i \in I_{B_1}} \uparrow a_i) \vee (\bigcap_{i \in I_{B_2}} \uparrow a_i) = T_1 \cap T_2 = \bigcap_{i \in I_{B_1} \cap I_{B_2}} \uparrow a_i$,

$T_1 \wedge T_2 = (\bigcap_{i \in I_{B_1}} \uparrow a_i) \wedge (\bigcap_{i \in I_{B_2}} \uparrow a_i) = (\bigcap_{i \in I_{B_1}} \uparrow a_i) \cup (\bigcap_{i \in I_{B_2}} \uparrow a_i)$.

This means $f((A_1, B_1) \wedge (A_2, B_2)) = B_1 \cup B_2 = (\bigcap_{i \in I_{B_1}} \uparrow a_i) \cup (\bigcap_{i \in I_{B_2}} \uparrow a_i) = T_1 \wedge T_2$;

$f((A_1, B_1) \vee (A_2, B_2)) = B_1 \cap B_2 = T_1 \cap T_2 = T_1 \vee T_2$.
Hence $Gal(R) \cong (\mathcal{T}_\mathcal{C}, \subseteq_c)$.

After we analyze for Theorem 1 and other researchers' consequences, we have the following viewpoints.

(1) [6] pointed out $Gal(R) \cong (\mathcal{F}_\mathcal{B}, \subseteq)$ where $\mathcal{B} = \{O \setminus \downarrow m | m \in P\}$ and $(\mathcal{F}_\mathcal{B}, \subseteq)$ is the covering graph. Theorem 1 says $Gal(R) \cong (\mathcal{T}_\mathcal{C}, \subseteq_c)$. The two isomorphisms taken together hints that $(\mathcal{T}_\mathcal{C}, \subseteq_c)$ must not be the dual of $(\mathcal{F}_\mathcal{B}, \subseteq)$. Otherwise, every concept lattice is self-dual. But many examples such as [5,p.20,Example 2] and [10] show the wrong of this.

This consequence also expresses that the research of Theorem 1 is valuable.

(2) For $(A, B) \in Gal(R)$, by Definition 1, we see that A is determined by B uniquely. That is, if we find out all the intents, the extents are yielded out at the same time. Then we will obtain all the concepts of R.

(3) The proof of Theorem 1 states clearly that the intent of a concept is decided by an element in $\mathcal{T}_\mathcal{C}$. However, $\mathcal{T}_\mathcal{C}$ was born from $\mathcal{C}$. Therefore, we reasonably say that the intent of a concept is decided by $\mathcal{C}$. Further, a concept is decided by $\mathcal{C}$.

(4) [7] said that the growth of bioinformatics has results in datasets with new characteristics. These datasets typically contain a large number of columns and a small number of rows. For example, many gene expression datasets must contain 10,000-100,000 columns but only 100-1000 rows. Such datasets pose a great challenge for existing frequent pattern discovery algorithms, since they have an exponential dependence on the average row length.

The definition of co-basis $\mathcal{C}$ determined by R makes clearly that $\mathcal{C}$ is yielded out of P. When the quantity of P is larger, the method here is much more suitable to produce $Gal(R)$. The reason is that every extent A satisfies $A = B'$, and B is found out by Theorem 1 directly from $\mathcal{C}$. In addition, Theorem 1 makes the relationship among concepts be bright.

(5) B.Ganter and R.Wille said [4] that the context is not explicitly given but can easily be read off from the lattice diagram. Additionally, they pointed out [4] that much of the mathematics requires for the applications comes from lattice theory.

It is important to search the Hasse diagram of a concept lattice(seen [8]). Our Theorem 1 will find out the Hasse diagram of $Gal(R)$. We predict that Theorem 1 will be useful for now and future study on not only concept lattice theory but also lattice theory.

On the other hand, these languages also express that $Gal(R) \cong (\mathcal{T}_\mathcal{C}, \subseteq_c)$ is primary importance to research on the properties of $Gal(R)$.

The above viewpoints reveal the value of Theorem 1 to concept lattice theory.

The beyond discussion in this Section could be generalized as follows. At first, we generalize Definition 2 to Definition 3.

Definition 3 (1) Let X be a set. A *co-basis* $\mathcal{C}$ is a set of subsets of X. We denote by $\mathcal{T}_{\mathcal{C}}$ the family generated by intersection from the co-bases $\mathcal{C}$, i.e. $\mathcal{T}_{\mathcal{C}} = \{\bigcap\limits_{B\in J} B | J \subseteq \mathcal{C}\}$.

(2) For an ordered set $P = (X, \leq)$, we define $\uparrow x = \{y \in X | x \leq y\}$.

Next we deal with the properties of $\mathcal{T}_{\mathcal{C}}$. We will introduce the definition of co-covering graph from the following lemma.

Lemma 4 (1) The family $\mathcal{T}_{\mathcal{C}}$ is said closed under intersection.

(2) When ordered by inverse inclusion $\subseteq_c$(i.e. $A \subseteq_c B \Leftrightarrow$ "$\forall y \in B \Rightarrow y \in A$"), $\mathcal{T}_{\mathcal{C}}$ is an inf-sublattice of the powerset lattice 2^X and thus the complete lattice. $(\mathcal{T}_{\mathcal{C}}, \subseteq_c)$ is called the *co-covering graph* of the family $\mathcal{T}_{\mathcal{C}}$ generated by $\mathcal{C}$, when ordered by inverse inclusion.

Proof Routine verification.

The following is to deal with the relation between a complete lattice and $\mathcal{T}_{\mathcal{C}}$.

Theorem 2 Let L be a complete lattice. Then there is $\mathcal{C}$ as a co-basis corresponding with L and $(\mathcal{T}_{\mathcal{C}}, \subseteq) \cong L$, where $\mathcal{T}_{\mathcal{C}} = \{\bigcap\limits_{C\in J} C | J \subseteq \mathcal{C}\}$.

Proof The complete property of L makes L have the biggest element 1_L and the smallest element 0_L. Let $[0_L, x] = \{y \in L | 0_L \leq y \leq x\}, \mathcal{D} = \{x \in L | x$ is covered by one and only one element in $L\}, \mathcal{C} = \{X \subseteq L | X = [0_L, x], x \in \mathcal{D}\}$, and $\mathcal{T}_{\mathcal{C}} = \{\bigcap\limits_{C\in J} C | J \subseteq \mathcal{C}\}$. Let h be the height function of L.

For any $x \in L$, it is evident $x = \bigwedge\limits_{x\leq y\in L} y = \bigwedge\limits_{\alpha\in\mathcal{A}} y_\alpha = \bigwedge\limits_{\beta\in\mathcal{B}} y_\beta$ where $\mathcal{A} = \{\alpha | x \leq y_\alpha\}$ and $\mathcal{B} = \{\beta | x \leq y_\beta \in \mathcal{D}\}$. It follows $[0_L, x] = \bigcap\limits_{x\leq y\in L} y = \bigcap\limits_{\beta\in\mathcal{B}} y_\beta$.

When $h(1_L) = 1$. This means $L = \{1_L\}$, and so the need result is effective evidently.

When $h(1_L) > 1$. This hints $0_L \neq 1_L$ and $\mathcal{D} \neq \emptyset$, further, $\mathcal{C} \neq \emptyset$ and $\mathcal{T}_{\mathcal{C}} \neq \emptyset$. Define

$f : L \to (\mathcal{T}_{\mathcal{C}}, \subseteq)$ as $y = \bigwedge\limits_{\beta\in\mathcal{B}_y} y_\beta \mapsto \bigcap\limits_{\beta\in\mathcal{B}_y} [0_L, y_\beta]$, where $\mathcal{B}_y = \{\beta | y \leq y_\beta \in \mathcal{D}\}$.

Then for any $y_1, y_2 \in L$ and $y_1 \neq y_2$, it has $[0_L, y_1 \wedge y_2] = [0_L, y_1] \cap [0_L, y_2]$. Furthermore, it has $[0_L, y_1 \wedge y_2] = \bigcap\limits_{\beta\in\mathcal{B}_{12}} [0_L, y_\beta]$ where $\mathcal{B}_{12} = \{\beta | y_1 \wedge y_2 \leq y_\beta \in \mathcal{D}\} = \{\beta_1, \ldots, \beta_n\}$, besides, $y_1 = \bigwedge\limits_{j=1}^{k} y_{\beta_j}$ and $y_2 = \bigwedge\limits_{j=k+1}^{n} y_{\beta_j}$. Hence according to the properties of lattice theory,

$$\begin{aligned} f(y_1 \wedge y_2) &= \bigcap_{\beta \in \mathcal{B}_{12}} [0_L, y_\beta] = [0_L, y_{\beta_1}] \cap \ldots \cap [0_L, y_{\beta_n}] \\ &= [0_L, y_{\beta_1}] \cap \ldots \cap [0_L, y_{\beta_k}] \cap [0_L, y_{\beta_{k+1}}] \cap \ldots \cap [0_L, y_{\beta_n}] \\ &= \bigcap_{j=1}^{k} [0_L, y_{\beta_j}] \cap \bigcap_{j=k+1}^{n} [0_L, y_{\beta_j}] = f(y_1) \wedge f(y_2). \end{aligned}$$

Moreover, f is $\wedge$-morphism and a bijection. Therefore, by [3,& 5,p.7], f is an isomorphism, and so L is isomorphic to $(\mathcal{T}_{\mathcal{C}}, \subseteq)$.

One of our main purpose in this paper is to find out the relation between a concept lattice and a complete lattice. The following theorem is precisely to conduct this purpose.

Theorem 3 Let L be a complete lattice. Then there is a context R suiting $L \cong Gal(R)$.

Proof Let $R = (O, P, I) = (L, L, I)$ where $xIy \Leftrightarrow x \leq_c y \Leftrightarrow x \leq y$ for $x, y \in L$. Denote $\vee_R$ to be the union in $(\mathcal{T}_{\mathcal{C}_R}, \subseteq_c)$.

Reviewing Lemma 3, the co-basis $\mathcal{C}_R$ related to R is $\{\uparrow a | a \in O\}$, where $\uparrow a = \{y \in P | aIy\} = \{y \in L | a \leq_c y\}$.

Moreover, the co-covering graph is $(\mathcal{T}_{\mathcal{C}_R}, \subseteq_c) = (\{\bigcap_{C \in J} C | J \subseteq \mathcal{C}_R\}, \subseteq_c)$.

$\uparrow x \cap \uparrow y = \{z \in L | x \leq_c z \text{ and } y \leq_c z\} = \{z \in L | x \vee y \leq_c z\} = \uparrow (x \vee y)$ holds. Because L is finite, we will induce

$\bigcap_{C \in J} C = \bigcap_{C = \uparrow a, C \in J} \uparrow a = \uparrow d$ for $d = \vee a \in L$ which $\uparrow a = C \in J \subseteq \mathcal{C}$.

Define $g : L \rightarrow (\mathcal{T}_{\mathcal{C}_R}, \subseteq_c)$ as $x \mapsto \uparrow x$ for any $x \in L$. Let $x, y \in L$ and $x \neq y$. Obviously, in view of lattice theory, $\uparrow x \neq \uparrow y$ is true, i.e. $g(x) \neq g(y)$. It is also true for the converse part, i.e. $\uparrow x \neq \uparrow y$ means $x \neq y$. It follows that f is a bijection.

Additionally, based on the above definition, it leads to $g(x \vee y) = \uparrow (x \vee y)$. Moreover,

$$\begin{aligned} g(x) \vee_R g(y) &= sup\{X \in \mathcal{T}_{\mathcal{C}_R} | g(x), g(y) \subseteq_c X\} = sup\{X \in \mathcal{T}_{\mathcal{C}_R} | X \subseteq g(x), g(y)\} \\ &= sup\{X \in \mathcal{T}_{\mathcal{C}_R} | X \subseteq g(x) \cap g(y) = \uparrow x \cap \uparrow y = \uparrow (x \vee y)\} \\ &= g(x \vee y). \end{aligned}$$

Consequently, $L \cong (\mathcal{T}_{\mathcal{C}_R}, \subseteq_c)$ is correct.

In light of Theorem 1, $(\mathcal{T}_{\mathcal{C}_R}, \subseteq_c) \cong Gal(R)$ holds. So, $L \cong Gal(R)$ will be true.

We see that our proof method here is different from [4,5] after recalling back [4, p.593, Theorem 1&5,p.20,Theorem 3] and the result that there exists a context R satisfying $L \cong Gal(R)$ for any complete lattice L (cf.[4,5] or any of books about concept lattices).

Actually, [4,5] are the main essential theory about concept lattices. Therefore, we can say that our method here provides a new way to discuss the properties of concept lattices.

We need to add up a sentence to the value of Theorem 2. Theorem 2 is not one of main purposes in this paper, but its idea and the way of proof provide us an approach to Theorem 3. Certainly, it also gives us a method to the future research on concept lattices.

4 Example

By an example, in this section, we will show the truth and some uses of most of consequences in Section 3 especially Theorem 1. The context in Example 1 is come from [9]. Much more examples of concept lattices could be found in [10]. These examples also demonstrate that concept lattice theory is a good tool in the scope of data handling for databases.

Example 1 Let Figure 1 be the diagram of lattice L. $R = (O, P, I)$ is constituted by $O = \{\text{girl}, \text{woman}, \text{boy}, \text{man}\}$, $P = \{\text{female}, \text{juvenile}, \text{adult}, \text{male}\}$ and the incidence I. I is seen as Table 1.

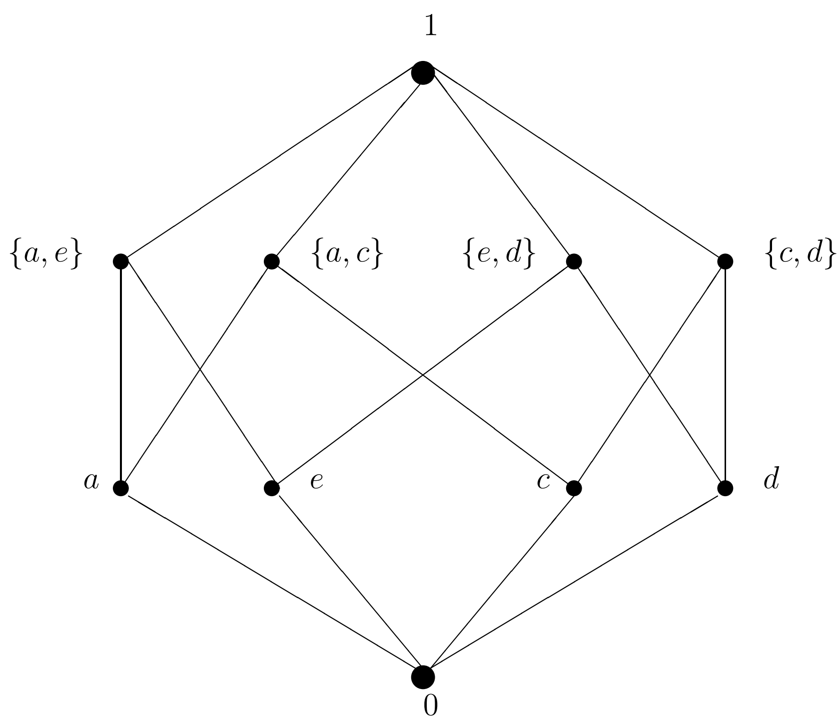

Figure 1

	female	juvenile	adult	male
girl	×	×		
woman	×		×	
boy		×		×
man			×	×

Table 1

Let fe = female, ju = juvenile, ad = adult, ma = male, b = boy, m = man, w = woman and g = girl. It is easy to obtain that the basis $\mathcal{C}$ determined by R is $\{\uparrow o | o \in O\} = \{C_j : j = 1, 2, 3, 4\}$ where $C_1 = \uparrow g = \{y \in P | gIy\} = \{fe, ju\}, C_2 = \uparrow w = \{fe, ad\}, C_3 = \uparrow b = \{ju, ma\}$ and $C_4 = \uparrow m = \{ad, ma\}$.

Because $\uparrow g \cap \uparrow w = \{fe\}, \uparrow g \cap \uparrow b = \{ju\}, \uparrow w \cap \uparrow m = \{ad\}, \uparrow b \cap \uparrow m = \{ma\}$, and $\uparrow g \cap \uparrow m = \uparrow w \cap \uparrow b = \uparrow g \cap \uparrow w \cap \uparrow b \cap \uparrow m = \emptyset$. So,
$\mathcal{T}_{\mathcal{C}} = \{C_j, (j = 1, 2, 3, 4); \uparrow g \cap \uparrow w, \uparrow g \cap \uparrow b, \uparrow b \cap \uparrow m, \uparrow w \cap \uparrow m, \uparrow g \cap \uparrow b \cap \uparrow w \cap \uparrow m\}$.

The diagram of $(\mathcal{T}_{\mathcal{C}}, \subseteq_c)$ is Figure 2.

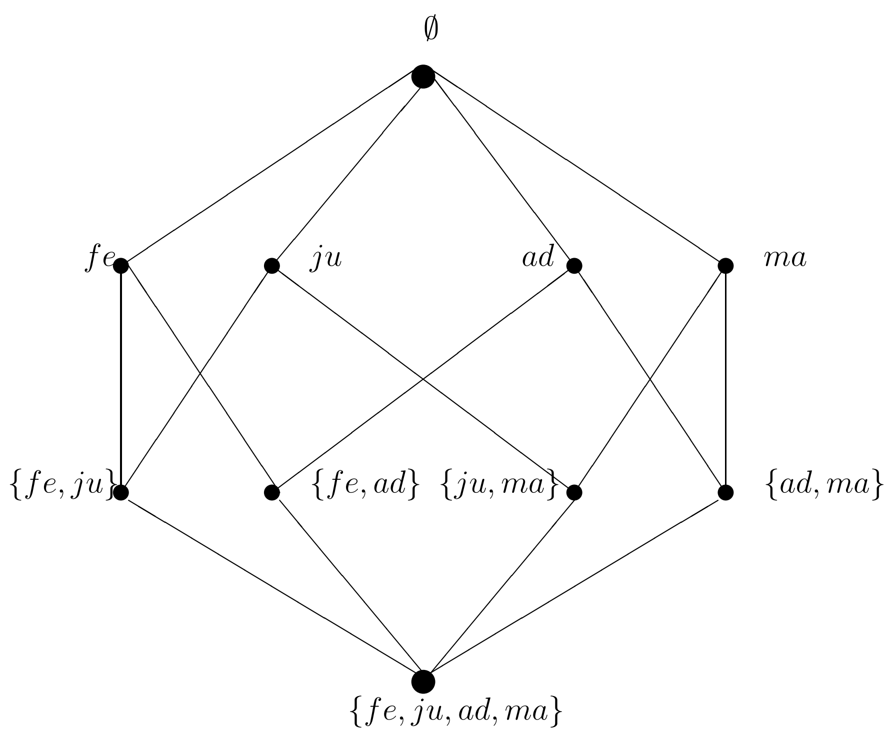

Figure 2

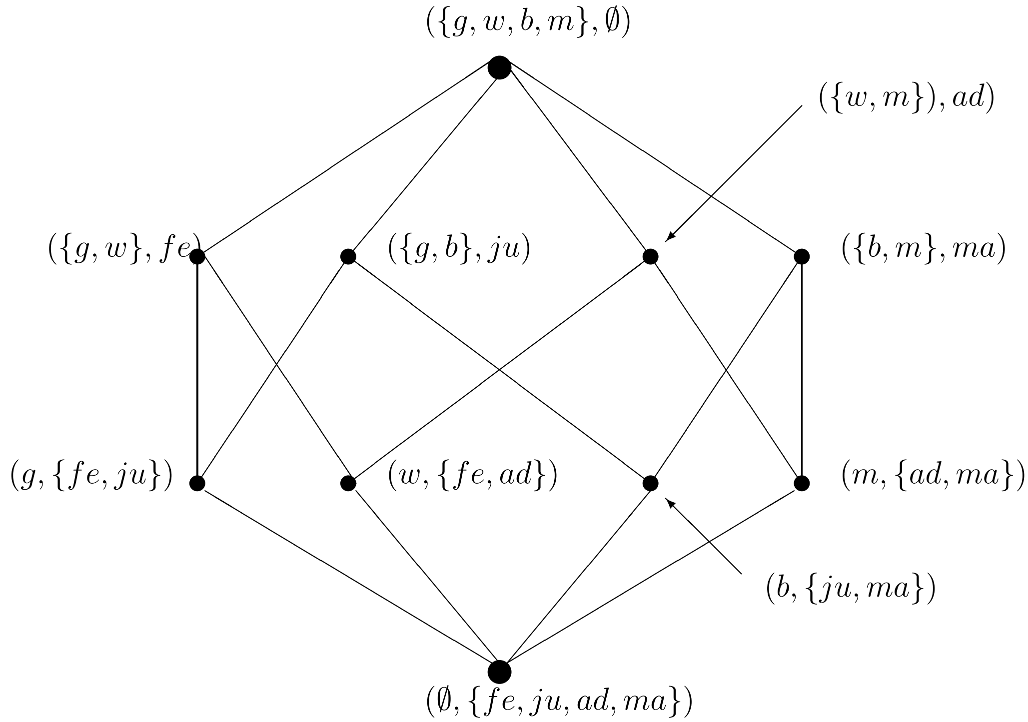

Figure 3

Let $B_5 =\uparrow g\cap \uparrow w, B_6 =\uparrow g\cap \uparrow b, B_7 =\uparrow w\cap \uparrow m, B_8 =\uparrow b\cap \uparrow m, B_9 =\uparrow g\cap \uparrow w\cap \uparrow b\cap \uparrow m$. Actually, by the consequences in Section 3, we obtain

$(\uparrow g)' = \{g\}, (\uparrow w)' = \{w\}, (\uparrow b)' = \{b\}, (\uparrow m)' = \{m\}$.

So $(A_1, B_1) = (g, \{fe, ju\}), (A_2, B_2) = (w, \{fe, ad\})$,

$(A_3, B_3) = (b, \{ju, ma\}), (A_4, B_4) = (m, \{ad, ma\})$;

$A_5 = B_5' = \{g, w\}, A_6 = B_6' = \{g, b\}, A_7 = B_7' = \{w, m\}$,

$A_8 = B_8' = \{b, m\}, A_9 = \{g, w, b, m\}$.

Hence, $Gal(R) = \{(\emptyset, \{fe, ju, ad, ma\}), (A_i, B_i), i = 1, \ldots, 9\}$.

The diagram of $Gal(R)$ is Figure 3.

In addition, it is evident $L \cong Gal(R) \cong (\mathcal{T}_{\mathcal{C}}, \subseteq_c)$.

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