Note on Bi-Ideals in Γ-Semigroups

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Abstract

The motivation mainly comes from the conditions of bi-ideals to be (0-)minimal or maximal that are of importance and interest in semigroups. It is well known that any semigroup can be reduced to a Γ-semigroup. The aim of this paper is to study the concept of (0-)minimal and maximal bi-ideals in Γ-semigroups, and give some characterizations of (0-)minimal and maximal bi-ideals in Γ-semigroups analogous to the characterizations of (0-)minimal and maximal bi-ideals in semigroups considered by Iampan [3].

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1 Introduction and Prerequisites

Let $S$ be a semigroup. A subsemigroup $B$ of $S$ is called a bi-ideal of $S$ if $BSB \subseteq B$. The notion of a bi-ideal was first introduced by Good and Hughes [2] as early as 1952, and it has been widely studied. In 2008, Iampan [3] characterized the (0-)minimal and maximal bi-ideals in semigroups, and gave some characterizations of (0-)minimal and maximal bi-ideals in semigroups.

The concept of a bi-ideal is a very interesting and important thing in semigroups. Now we also characterize the (0-)minimal and maximal bi-ideals in Γ-semigroups, and give some characterizations of (0-)minimal and maximal bi-ideals in Γ-semigroups.

To present the main results we first recall some definitions which is important here.

Let $M$ and $\Gamma$ be any two nonempty sets. $M$ is called a Γ-semigroup [6] if there exists a mappings $M \times \Gamma \times M \rightarrow M$, written as $(a, \gamma, b) \mapsto a\gamma b$, satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and
\(\alpha, \beta \in \Gamma\). A nonempty subset \(K\) of \(M\) is called a sub-\(\Gamma\)-semigroup of \(M\) if \(a \gamma b \in K\) for all \(a, b \in K\) and \(\gamma \in \Gamma\). For nonempty subsets \(A, B\) of \(M\), let 
\[
A \Gamma B := \{ a \gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma \}.
\]
We also write \(a \Gamma B, A \Gamma b\) and \(\{a\} \Gamma \{b\}\), respectively.

Examples of \(\Gamma\)-semigroups can be seen in [4, 5] and [7] respectively.

The following example comes from Dixit and Dewan [1].

**Example 1.1** Let \(M = \{-i, 0, i\}\) and \(\Gamma = M\). Then \(M\) is a \(\Gamma\)-semigroup under the multiplication over complex number while \(M\) is not a semigroup under complex number multiplication.

A nonempty subset \(I\) of a \(\Gamma\)-semigroup \(M\) is called an ideal of \(M\) if \(M \Gamma I \subseteq I\) and \(I \Gamma M \subseteq I\). A sub-\(\Gamma\)-semigroup \(Q\) of a \(\Gamma\)-semigroup \(M\) is called a quasi-ideal of \(M\) if \(M \Gamma Q \cap Q \Gamma M \subseteq Q\). A sub-\(\Gamma\)-semigroup \(B\) of a \(\Gamma\)-semigroup \(M\) is called a bi-ideal of \(M\) if \(B \Gamma M \Gamma B \subseteq B\). Then the notion of a quasi-ideal is a generalization of the notion of an ideal, and the notion of a bi-ideal is a generalization of the notion of a quasi-ideal. The intersection of all bi-ideals of a sub-\(\Gamma\)-semigroup \(K\) of a \(\Gamma\)-semigroup \(M\) containing a nonempty subset \(A\) of \(K\) is called the bi-ideal of \(K\) generated by \(A\). For \(A = \{a\}\), let \(B_K(a)\) denote the bi-ideal of \(K\) generated by \(\{a\}\). If \(K = M\), then we also write \(B_M(a)\) as \(B(a)\). An element \(a\) of a \(\Gamma\)-semigroup \(M\) with at least two elements is called a zero element of \(M\) if \(x \gamma a = a \gamma x = a\) for all \(x \in M\) and \(\gamma \in \Gamma\), and denote it by 0. If \(M\) is a \(\Gamma\)-semigroup with zero, then every bi-ideal of \(M\) containing a zero element. A \(\Gamma\)-semigroup \(M\) without zero is called \(B\)-simple if it has no proper bi-ideals. A \(\Gamma\)-semigroup \(M\) with zero is called \(0\)-\(B\)-simple if it has no nonzero proper bi-ideals and \(M \Gamma M \neq \{0\}\). A bi-ideal \(B\) of a \(\Gamma\)-semigroup \(M\) without zero is called a minimal bi-ideal of \(M\) if there is no bi-ideal \(A\) of \(M\) such that \(A \subseteq B\). Equivalently, if for any bi-ideal \(A\) of \(M\) such that \(A \subseteq B\), we have \(A = B\). A nonzero bi-ideal \(B\) of a \(\Gamma\)-semigroup \(M\) with zero is called a 0-minimal bi-ideal of \(M\) if there is no nonzero bi-ideal \(A\) of \(M\) such that \(A \subseteq B\). Equivalently, if for any nonzero bi-ideal \(A\) of \(M\) such that \(A \subseteq B\), we have \(A = B\). A proper bi-ideal \(B\) of a \(\Gamma\)-semigroup \(M\) is called a maximal bi-ideal of \(M\) if for any bi-ideal \(A\) of \(M\) such that \(B \subseteq A\), we have \(A = M\). Equivalently, if for any proper bi-ideal \(A\) of \(M\) such that \(B \subseteq A\), we have \(A = B\).

Our purpose in this paper is fourfold.

1. To introduce the concept of a \(B\)-simple and \(0\)-\(B\)-simple \(\Gamma\)-semigroup.

2. To characterize the properties of bi-ideals in \(\Gamma\)-semigroups.
3. To characterize the relationship between (0-)minimal bi-ideals and (0-)\(B\)-simple \(\Gamma\)-semigroups.

4. To characterize the relationship between maximal bi-ideals and (0-)\(B\)-simple \(\Gamma\)-semigroups.

We shall assume throughout this paper that \(M\) stands for a \(\Gamma\)-semigroup. Before the characterizations of bi-ideals for the main theorems, we give some auxiliary results which are necessary in what follows. The following three lemmas are also necessary for our considerations, and easy to verify.

**Lemma 1.2**  For any \(a \in M\),
\[
B(a) = a\Gamma M \Gamma a \cup a \Gamma a \cup \{a\}.
\]

**Lemma 1.3**  The set \(a\Gamma M \Gamma a\) is a bi-ideal of \(M\) for all \(a \in M\).

**Lemma 1.4**  Let \(\{B_\gamma \mid \gamma \in \Lambda\}\) be a collection of bi-ideals of \(M\). Then
\[
\bigcap_{\gamma \in \Lambda} B_\gamma \text{ is a bi-ideal of } M \text{ if } \bigcap_{\gamma \in \Lambda} B_\gamma \neq \emptyset.
\]

**Lemma 1.5**  If \(M\) has no zero element, then the following statements are equivalent:

(i) \(M\) is \(B\)-simple.

(ii) \(a\Gamma M \Gamma a = M\) for all \(a \in M\).

(iii) \(B(a) = M\) for all \(a \in M\).

**Proof.** Since \(M\) is \(B\)-simple, it follows from Lemma 1.3 that \(a\Gamma M \Gamma a = M\) for all \(a \in M\). Therefore (i) implies (ii). By Lemma 1.2, \(B(a) = a\Gamma M \Gamma a \cup a \Gamma a \cup \{a\} = M \cup a \Gamma a \cup \{a\} = M\) for all \(a \in M\). Thus (ii) implies (iii). Let \(B\) be a bi-ideal of \(M\), and let \(a \in B\). Then \(M = B(a) \subseteq B \subseteq M\), so \(B = M\). Hence \(M\) is \(B\)-simple, we have that (iii) implies (i). \(\square\)

**Lemma 1.6**  If \(M\) has a zero element, then the following statements hold:

(i) If \(M\) is 0-\(B\)-simple, then \(B(a) = M\) for all \(a \in M \setminus \{0\}\).

(ii) If \(B(a) = M\) for all \(a \in M \setminus \{0\}\), then either \(MTM = \{0\}\) or \(M\) is 0-\(B\)-simple.

**Proof.** (i) Assume that \(M\) is 0-\(B\)-simple. Then \(B(a)\) is a nonzero bi-ideal of \(M\) for all \(a \in M \setminus \{0\}\). Hence \(B(a) = M\) for all \(a \in M \setminus \{0\}\).

(ii) Assume that \(B(a) = M\) for all \(a \in M \setminus \{0\}\) and \(MTM \neq \{0\}\). Let \(B\) be a nonzero bi-ideal of \(M\), and let \(a \in B \setminus \{0\}\). Then \(M = B(a) \subseteq B \subseteq M\), so \(B = M\). Therefore \(M\) is 0-\(B\)-simple. \(\square\)
Lemma 1.7 If \( B \) is a bi-ideal of \( M \), and \( K \) is a sub-\( \Gamma \)-semigroup of \( M \), then the following statements hold:

(i) If \( K \) is \( B \)-simple such that \( K \cap B \neq \emptyset \), then \( K \subseteq B \).

(ii) If \( K \) is \( 0 \)-\( B \)-simple such that \( K \setminus \{0\} \cap B \neq \emptyset \), then \( K \subseteq B \).

Proof. (i) Assume that \( K \) is \( B \)-simple such that \( K \cap B \neq \emptyset \). Then, let \( a \in K \cap B \). By Lemma 1.3, \( a\Gamma K \Gamma a \) is a bi-ideal of \( K \). It follows that \( a\Gamma K \Gamma a = a\Gamma K \Gamma K \subseteq B \), so \( K \subseteq B \).

(ii) Assume that \( K \) is \( 0 \)-\( B \)-simple such that \( K \setminus \{0\} \cap B \neq \emptyset \). Then, let \( a \in K \setminus \{0\} \cap B \). By Lemma 1.2 and 1.6 (i), \( K = B_{K}(a) = a\Gamma K \Gamma a \cup a\Gamma a \cup \{a\} \subseteq a\Gamma M \Gamma a \cup a\Gamma a \cup \{a\} = B(a) \subseteq B \). Hence \( K \subseteq B \).

Hence the proof is completed. \( \square \)

We now give the main theorem of this paper as follow.

2 (0-)Minimal Bi-ideals

The aim of this section is to characterize the relationship between minimal bi-ideals and \( B \)-simple \( \Gamma \)-semigroups, and 0-minimal bi-ideals and \( 0 \)-\( B \)-simple \( \Gamma \)-semigroups.

Theorem 2.1 If \( M \) has no zero element, and \( B \) is a bi-ideal of \( M \), then the following statements hold:

(i) \( B \) is a minimal bi-ideal without zero of \( M \) if and only if \( B \) is \( B \)-simple.

(ii) If \( B \) is a minimal bi-ideal with zero of \( M \), then either \( B \Gamma B = \{0\} \) or \( B \) is \( 0 \)-\( B \)-simple.

Proof. (i) Assume that \( B \) is a minimal bi-ideal without zero of \( M \). Then \( B \) is a sub-\( \Gamma \)-semigroup of \( M \). Now, let \( A \) be a bi-ideal of \( B \). Then \( A \Gamma B \Gamma A \subseteq A \). Define \( H := \{h \in A \mid h = a_{\gamma_{1}}b_{\gamma_{2}}a_{2} \text{ for some } a_{1}, a_{2} \in A, b \in B \) and \( \gamma_{1}, \gamma_{2} \in \Gamma \} \). Then \( \emptyset \neq H \subseteq A \subseteq B \). To show that \( H \) is a bi-ideal of \( M \), let \( h_{1}, h_{2} \in H \), \( x \in M \) and \( \gamma_{1}, \gamma_{2} \in \Gamma \). Then \( h_{1} = a_{1}\alpha_{1}b_{1}\alpha_{1}'a_{1}' \) and \( h_{2} = a_{2}\alpha_{2}b_{2}\alpha_{2}'a_{2}' \) for some \( a_{1}, a_{1}', a_{2}, a_{2}' \in A, b_{1}, b_{2} \in B \) and \( \alpha_{1}, \alpha_{1}', \alpha_{2}, \alpha_{2}' \in \Gamma \), so \( h_{1} = a_{1}\alpha_{1}b_{1}\alpha_{1}'a_{1}' \gamma_{1}a_{2}\alpha_{2}b_{2}\alpha_{2}'a_{2}' \) and \( h_{1} \gamma_{1}x \gamma_{2}h_{2} = a_{1}\alpha_{1}b_{1}\alpha_{1}'a_{1}' \gamma_{1}x \gamma_{2}a_{2}\alpha_{2}b_{2}\alpha_{2}'a_{2}' \). Since \( B \Gamma M \Gamma B \subseteq B \), \( b_{1}\alpha_{1}'a_{1}' \gamma_{1}a_{2}\alpha_{2}b_{2} \subseteq B \) and \( b_{1}\alpha_{1}'a_{1}' \gamma_{1}x \gamma_{2}a_{2}\alpha_{2}b_{2} \subseteq B \). Since \( h_{1} \gamma_{1}h_{2} \in H \cap H \subseteq A \Gamma A \subseteq A \), we get \( h_{1} \gamma_{1}h_{2} \in H \). Thus \( H \) is a sub-\( \Gamma \)-semigroup of \( M \).

Since \( A \Gamma B \Gamma A \subseteq A \), we get \( h_{1} \gamma_{1}x \gamma_{2}h_{2} = a_{1}\alpha_{1}b_{1}\alpha_{1}'a_{1}' \gamma_{1}x \gamma_{2}a_{2}\alpha_{2}b_{2}\alpha_{2}'a_{2}' \subseteq A \). Hence \( h_{1} \gamma_{1}x \gamma_{2}h_{2} \in H \), so \( H \Gamma \Gamma H \subseteq H \). Therefore \( H \) is a bi-ideal of \( M \).

Since \( B \) is a minimal bi-ideal of \( M \), we get \( H = B \). Hence \( A = B \), so \( B \) is \( B \)-simple.
Conversely, assume that $B$ is $B$-simple. Let $A$ be a bi-ideal of $M$ such that $A \subseteq B$. Then $A \cap B \neq \emptyset$, it follows from Lemma 1.7 (i) that $B \subseteq A$. Hence $A = B$, so $B$ is a minimal bi-ideal of $M$.

(ii) Similar to the proof of necessary condition of statement (i).

Therefore we complete the proof of the theorem. □

Using the same proof of Theorem 2.1 (i) and Lemma 1.7 (ii), we have Theorem 2.2.

**Theorem 2.2** If $M$ has a zero element, and $B$ is a nonzero bi-ideal of $M$, then the following statements hold:

(i) If $B$ is a 0-minimal bi-ideal of $M$, then either $A \Gamma B \Gamma A = \{0\}$ for some nonzero bi-ideal $A$ of $B$ or $B$ is 0-$B$-simple.

(ii) If $B$ is 0-$B$-simple, then $B$ is a 0-minimal bi-ideal of $M$.

**Theorem 2.3** If $M$ has no zero element but it has a proper bi-ideal, then every proper bi-ideal of $M$ is minimal if and only if the intersection of any two distinct proper bi-ideals is empty.

**Proof.** Assume $B_1$ and $B_2$ are two distinct proper bi-ideals of $M$. Then $B_1$ and $B_2$ are minimal. If $B_1 \cap B_2 \neq \emptyset$, then $B_1 \cap B_2$ is a bi-ideal of $M$ by Lemma 1.4. Since $B_1$ and $B_2$ are minimal, $B_1 = B_2$. It is a contradiction. Hence $B_1 \cap B_2 = \emptyset$.

The converse is obvious. □

Using the same proof of Theorem 2.3, we have Theorem 2.4.

**Theorem 2.4** If $M$ has a zero element and a nonzero proper bi-ideal, then every nonzero proper bi-ideal of $M$ is 0-minimal if and only if the intersection of any two distinct nonzero proper bi-ideals is $\{0\}$.

## 3 Maximal Bi-ideals

The aim of this section is to characterize the relationship between maximal bi-ideals and the set $\mathcal{U}$ in $\Gamma$-semigroups.

**Theorem 3.1** Let $B$ be a bi-ideal of $M$. If either $M \setminus B = \{a\}$ for some $a \in M$ or $M \setminus B \subseteq b \Gamma M \Gamma b$ for all $b \in M \setminus B$, then $B$ is a maximal bi-ideal of $M$. 
Proof. Let $A$ be a bi-ideal of $M$ such that $B \subseteq A$. Then we consider the following two cases:

Case 1: $M \setminus B = \{a\}$ for some $a \in M$.

Then $M = B \cup \{a\}$. Since $B \subset A$, $\emptyset \neq A \setminus B \subseteq M \setminus B = \{a\}$. Thus $A \setminus B = \{a\}$ and $A = B \cup \{a\} = M$.

Case 2: $M \setminus B \subseteq b\Gamma M\Gamma b$ for all $b \in M \setminus B$.

If $b \in A \setminus B \subseteq M \setminus B$, then $M \setminus B \subseteq b\Gamma M\Gamma b \subseteq A\Gamma M\Gamma A \subseteq A$. Hence $M = B \cup M \setminus B \subseteq B \cup A = A \subseteq M$, so $A = M$.

Therefore $B$ is a maximal bi-ideal of $M$.

\[ \square \]

Theorem 3.2 If $B$ is a maximal bi-ideal of $M$, and $B \cup B(a)$ is a bi-ideal of $M$ for all $a \in M \setminus B$, then either

(i) $M \setminus B \subseteq a\Gamma a \cup \{a\}$ and $a\Gamma a \Gamma a \subseteq B$ for some $a \in M \setminus B$, and $b\Gamma M\Gamma b \subseteq B$ for all $b \in M \setminus B$ or

(ii) $M \setminus B \subseteq B(a)$ for all $a \in M \setminus B$.

Proof. Assume that $B$ is a maximal bi-ideal of $M$ and $B \cup B(a)$ is a bi-ideal of $M$ for all $a \in M \setminus B$. Then we have the following two cases:

Case 1: $a\Gamma M\Gamma a \subseteq B$ for some $a \in M \setminus B$.

Then $a\Gamma a \Gamma a \subseteq a\Gamma M\Gamma a \subseteq B$, so $a\Gamma a \Gamma a \subseteq B$. Since $B \cup a\Gamma a \cup \{a\} = (B \cup a\Gamma M\Gamma a) \cup a\Gamma a \cup \{a\} = B \cup (a\Gamma M\Gamma a \cup a\Gamma a \cup \{a\}) = B \cup B(a)$, so $B \cup a\Gamma a \cup \{a\}$ is a bi-ideal of $M$. Since $a \in M \setminus B$, we have $B \subseteq B \cup a\Gamma a \cup \{a\}$. Thus $B \cup a\Gamma a \cup \{a\} = M$ because $B$ is a maximal bi-ideal of $M$, so $M \setminus B \subseteq a\Gamma a \cup \{a\}$. If $b \in M \setminus B$, then $b \in a\Gamma a \cup \{a\}$. If $b = a\gamma a$ for some $\gamma \in \Gamma$, then $b\Gamma M\Gamma b = a\gamma a\Gamma M\Gamma a\gamma a \subseteq a\Gamma M\Gamma a \subseteq B$. Hence $b\Gamma M\Gamma b \subseteq B$ for all $b \in M \setminus B$.

Case 2: $a\Gamma M\Gamma a \not\subseteq B$ for all $a \in M \setminus B$.

If $a \in M \setminus B$, then $B \subseteq B \cup a\Gamma M\Gamma a \subseteq B \cup B(a)$ by Lemma 1.2. Since $B \cup B(a)$ is a bi-ideal of $M$, and $B$ is a maximal bi-ideal of $M$, we have $B \cup B(a) = M$. Hence $M \setminus B \subseteq B(a)$ for all $a \in M \setminus B$.

Hence the proof is completed. \[ \square \]

For a $\Gamma$-semigroup $M$, let $\mathcal{U}$ denote the union of all nonzero proper bi-ideals of $M$ if $M$ has a zero element, and let $\mathcal{U}$ denote the union of all proper bi-ideals of $M$ if $M$ has no zero element. Then it is easy to verify Lemma 3.3.

Lemma 3.3 $M = \mathcal{U}$ if and only if $B(a) \neq M$ for all $a \in M$.

As a consequence of Theorem 3.2 and Lemma 3.3, we obtain Theorem 3.4.
Theorem 3.4 If \( M \) has no zero element, then one of the following four conditions is satisfied:

(i) \( \mathcal{U} \) is not bi-ideal of \( M \).

(ii) \( B(a) \neq M \) for all \( a \in M \).

(iii) There exists \( a \in M \) such that \( B(a) = M, a \Gamma a \cup \{a\} \not\subseteq a \Gamma M \Gamma a \) and \( a \Gamma a \Gamma a \subseteq \mathcal{U}, M \) is not \( B \)-simple, \( M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\} \) and \( \mathcal{U} \) is the unique maximal bi-ideal of \( M \).

(iv) \( M \setminus \mathcal{U} \subseteq B(a) \) for all \( a \in M \setminus \mathcal{U}, M \) is not \( B \)-simple, \( M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\} \) and \( \mathcal{U} \) is the unique maximal bi-ideal of \( M \).

Proof. Assume that \( \mathcal{U} \) is a bi-ideal of \( M \). Then \( \mathcal{U} \neq \emptyset \). Now, we consider the following two cases:

Case 1: \( \mathcal{U} = M \).

By Lemma 3.3, \( B(a) \neq M \) for all \( a \in M \). In this case, the condition (ii) is satisfied.

Case 2: \( \mathcal{U} \neq M \).

Then \( M \) is not \( B \)-simple. To show that \( \mathcal{U} \) is the unique maximal bi-ideal of \( M \), let \( A \) be a bi-ideal of \( M \) such that \( \mathcal{U} \subset A \). If \( A \neq M \), then \( A \) is a proper bi-ideal of \( M \). Thus \( A \subset \mathcal{U} \), so it is a contradiction. Hence \( \mathcal{U} \) is a maximal bi-ideal of \( M \). Next, assume that \( B \) is a maximal bi-ideal of \( M \). Then \( B \subset \mathcal{U} \subset M \) because \( B \) is a proper bi-ideal of \( M \). Since \( B \) is a maximal bi-ideal of \( M \), we have \( B = \mathcal{U} \). Hence \( \mathcal{U} \) is the unique maximal bi-ideal of \( M \). Since \( \mathcal{U} \neq M \), it follows from Lemma 3.3 that \( B(a) = M \) for some \( a \in M \).

Clearly, \( B(a) = M \) for all \( a \in M \setminus \mathcal{U} \). Thus \( M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\} \), so \( \mathcal{U} \cup B(a) = M \) is a bi-ideal of \( M \) for all \( a \in M \setminus \mathcal{U} \). By Theorem 3.2, we have the following two cases:

(i) \( M \setminus \mathcal{U} \subseteq a \Gamma a \cup \{a\} \) and \( a \Gamma a \Gamma a \subseteq \mathcal{U} \) for some \( a \in M \setminus \mathcal{U} \), and \( b \Gamma M \Gamma b \subseteq \mathcal{U} \) for all \( b \in M \setminus \mathcal{U} \) or

(ii) \( M \setminus \mathcal{U} \subseteq B(a) \) for all \( a \in M \setminus \mathcal{U} \).

Assume \( M \setminus \mathcal{U} \subseteq a \Gamma a \cup \{a\} \) and \( a \Gamma a \Gamma a \subseteq \mathcal{U} \) for some \( a \in M \setminus \mathcal{U} \), and \( b \Gamma M \Gamma b \subseteq \mathcal{U} \) for all \( b \in M \setminus \mathcal{U} \). If \( a \Gamma a \cup \{a\} \subseteq a \Gamma M \Gamma a \), then \( M = B(a) = a \Gamma M \Gamma a \cup a \Gamma a \cup \{a\} = a \Gamma M \Gamma a \) by Lemma 1.2. By hypothesis, \( M = a \Gamma M \Gamma a \subseteq \mathcal{U} \) and so \( \mathcal{U} = M \). This is a contradiction. Hence \( a \Gamma a \cup \{a\} \not\subseteq a \Gamma M \Gamma a \). In this case, the condition (iii) is satisfied. Now, assume \( M \setminus \mathcal{U} \subseteq B(a) \) for all \( a \in M \setminus \mathcal{U} \). In this case, the condition (iv) is satisfied.

Hence the theorem is now completed.

Using the same proof of Theorem 3.4, we have Theorem 3.5.
Theorem 3.5 If $M$ has a zero element and $M\Gamma M \neq \{0\}$, then one of the following five conditions is satisfied:

(i) $U$ is not bi-ideal of $M$.

(ii) $B(a) \neq M$ for all $a \in M$.

(iii) $U = \{0\}, M \setminus U = \{x \in M \mid B(x) = M\}$ and $U$ is the unique maximal bi-ideal of $M$.

(iv) There exists $a \in M$ such that $B(a) = M, a\Gamma a \cup \{a\} \not\subseteq a\Gamma M\Gamma a$ and $a\Gamma a \subseteq U$, $M$ is not $0$-$B$-simple, $M \setminus U = \{x \in M \mid B(x) = M\}$ and $U$ is the unique maximal bi-ideal of $M$.

(v) $M \setminus U \subseteq B(a)$ for all $a \in M \setminus U$, $M$ is not $0$-$B$-simple, $M \setminus U = \{x \in M \mid B(x) = M\}$ and $U$ is the unique maximal bi-ideal of $M$.

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