International Journal of Algebra, Vol. 2, 2008, no. 10, 493 - 499

On Primitive Boolean Algebras

Lahcen Taoufiq

Université Moulay Ismaïl, Faculté des Sciences et Techniques
Département de Mathématiques, Groupe d'Algèbre et Applications
B. P. 509 Boutalamine, Errachidia; Morocco
lahcentaoufiq@gmail.com

Abstract

In this work we are particularly interesting by the characterization of primitive Boolean algebras. In return for the theorem of Mostowski and Tarski this problem is equivalent to find a sufficient and necessary condition on a chain L such that the intervals algebra over L, Int(L), either primitive. The main result acquired in this sense is that a superatomic boolean algebra is primitive if and only if, it is isomorphic to $Int(\omega^{\alpha})$ where α is a countable ordinal, that is equivalent to say that a scattered boolean space is primitive if and only if it is homeomorphic to $\omega^{\alpha}+1$ where α is a countable ordinal. In order to generalize this result to non-metrizable case, we qualify weakly primitive a boolean space X such that $Pi(X)$ disjointly generates $Clop(X)$, if furthermore X is pseudo-indecomposable, we said that X is quasi primitive. Seen the complexity of the problem one amounts to scattered interval spaces and prove that: For every integer n and every ordinals $\mu_0 ...., \mu_n, \nu_0 .... \nu_n$, the topological space $\sum_{1 \leq i \leq n} L(\mu_i, \nu_i)$ is weakly primitive. To prove the Reciprocal result, it seems that the general form of scattered chains is essential.

Mathematics Subject Classification: 03G05, 04A10, 54G12

Keywords: Boolean algebras, scattered boolean spaces, ordinals

1 Universal primitive boolean algebra

Definition 1.1 *Let A be a boolean algebra.*
1. *A is said to be pseudo-indecomposable (p.i.) if A is not isomorphic to any product of boolean algebras wish all factors are not isomorphic to A.*
2. *$a \in A$ is said to be pseudo-indecomposable if $A|a = \{b \in A/b \leq a\}$ is p.i.,*

the set of p.i. elements of A is denoted by $Pi(A)$.
3. *A boolean algebra A is said to be generated by a subset E if each element of A is the supremum of a finite disjoint family of elements of E.*
4. *A boolean algebra A is said to be primitive if it is countable, p.i. and disjointly generated by $Pi(A)$.*

Remarque that A is p.i. if and only if, for each $a \in A$, $A \simeq A|a$ or $A \simeq A|a^c$.

Exemple. The superatomic intervals algebra $Int(\omega)$ is primitive.
Indeed, since $Int(\omega)$ is generated by $\{[n, +\infty[, n \in \omega\}$ then $Int(\omega)$ is countable. On the other hand, for every $a \in Int(\omega)$, a or a^c content a final section isomorphic to ω, then $Int(\omega)$ is p.i.. It remain to prove that $Int(\omega)$ is generated by his p.i. elements, that is immediate since

$$Pi(Int(\omega)) = \{a \in Int(\omega) / \parallel a \parallel = 1 \text{ or } \parallel a \parallel = \aleph_0\}.$$

Remark that $Int(\omega)$ is a superatomic boolean algebra.

Proposition 1.2 *Let η be the order type of $\mathbb{Q}$ endowed with the usual order. The intervals algebra over η, $Int(\eta)$, is primitive.*
Proof. Int(η) is countable because η it is. Let $a \in Int(\eta) \setminus \{\emptyset\}$, $a \neq \emptyset$ then there is $x < y \in \eta$, such that $]x, y[\subseteq a$, it arise that $A|a$ is countable and atomless, then by the Vaught's theorem (cf[3]) A is p.i. and $Pi(A) = A \setminus \{\emptyset\}$ generates disjointly A. ∎

Theorem 1.3 *A boolean algebra A is isomorphic to an intervals algebra if and only if A is generated by a chain C. If moreover $A \simeq Int(L)$, then we can choose C isomorphic to L.*

Proof. If $A = Int(L)$ then $C = \{[0_L, x[, x \in L\}$ is a chain isomorphic to L generating A. Reciprocally, suppose that A is Generated by a chain C, without loss of generality , we can suppose that $0_A \in C$ and $1_A \in C$, then there is a morphism f from A to $Int(C)$ such that $f(c) = [0_A, c[\forall c \in C$. But $f(C)$ generates $Int(C)$ then f is surjective. Consequently $A \simeq < \{[0_A, c[, c \in C\} >= Int(C)$. ∎

In [4], Mostowski and Tarski have proved that every countable boolean algebra is generated by a chain.

Corollary 1.4 *Every countable boolean algebra is an intervals algebra.*

Proposition 1.5 ([6], Theorem2) *Every countable chain is embedding in η.*

Corollary 1.6 *$Int(\eta)$ is the universal primitive boolean algebra in such away that every primitive boolean algebra is embedding in $Int(\eta)$.*

Definition 1.7 *Let X be a topological space.*
1. *A Clopen (closed and open) U of X is said to be pseudo-indecomposable (p.i.) if U is not homeomorphic to a disjoint sum of two clopens of X wish are not homeomorphic to U. The set of p.i. clopens of X is denoted by $Pi(X)$.*
2. *The boolean space X is primitive if his dual algebra $ClopX$ it is. In exchange for the Stone duality, a boolean space X is primitive if and only if X is metrizable, p.i. and $Pi(X)$ disjointly generates $clop(X)$, i.e., every Clopen of X is the union of a finite disjoint family of elements of $Pi(X)$.*

2 Characterization of primitive superatomic boolean algebras

Definition 2.1 *A boolean algebra A is superatomic if every homeomorphic image of A is atomic.*

Definition 2.2 1. *A boolean space X is scattered if every nonempty closed subset of X has an isolated point.*
2. *A topological space X is of witness if there is $w \in X$ such that every open of X that content w is homeomorphic to X.*

In [2] it is shown that a non-trivial metrizable boolean space X is scattered if and only if there is a countable ordinal α and an integer n such that $X \cong \omega^\alpha n + 1$, moreover α and n are two topological invariants. Accordingly, the characterization of primitive scattered boolean spaces amounts to the one of primitive boolean spaces of the form $\omega^\alpha n + 1$ where α is a countable ordinal and n an integer.

Lemma 2.3 ([2], Lemma 7.6. page 98) *Let X be a boolean space. If $B \subseteq ClopX$ is a topological base of X, stable by the finite intersection then $B = ClopX$.*

Corollary 2.4 *For $X = \omega^\alpha n + 1$ we have:*

$$ClopX = \{\bigcup [a_i, b_i[/ a_i \le b_i \in \omega^\alpha n{+}1, n \in \omega, a_i \text{ and } b_i \text{ non-limit ordinals}\} \bigcup \{X\}$$
$$= \{\bigcup [a_i, b_i] / a_i \le b_i \in \omega^\alpha n + 1, n \in \omega, a_i \text{ non-limit ordinals}\} \bigcup \{\emptyset, X\}$$

Theorem 2.5 (Cantor's normal forme, [5] Theorem 3.46, page 61) *Every ordinal α have a decomposition of the form $\omega^{\alpha_1} n_1 + \omega^{\alpha_2} n_2 + \dots\dots\dots\dots\dots + \omega^{\alpha_k} n_k$ where $\alpha > \alpha_1 > \dots\dots > \alpha_k$ are ordinals and $k, n_1, \dots\dots, n_k$ are integers. Moreover, this decomposition is unique.*

Theorem 2.6 *A scattered boolean space X non reduced to only one point is primitive if and only if there is a ordinal α such that X is homeomorphic to $\omega^\alpha + 1$.*

Proof. Let X be a scattered boolean space. According to the precedent theorem we can suppose that X is of the shape $\omega^\alpha n + 1$ where α is a countable ordinal and $n \in \omega$. If $\alpha = 0$ then X is p.i. if and only if $\mid X \mid = 1$, in this case X is primitive. If $\alpha \neq 0$ and $n > 1$ then $X = \omega^\alpha n + 1 \cong \omega^\alpha(n-1) + 1 \bigcup \omega^\alpha + 1$, but X is not homeomorphic neither to $\omega^\alpha(n-1) + 1$ nor to $\omega^\alpha + 1$, then X is not p.i., thereafter not primitive. Consequently we can suppose that $n = 1$ i.e. $X = \omega^\alpha + 1$. Let $U \in Clop(X)$ such that $U = \bigcup [a_i, b_i[$ where $a_i < b_i < a_j < b_j \forall 0 < i < j$, a_i, b_i non-limit for every $i \leq n$. $\forall x \in U$, $\exists i_x \leq n$ such that $x \in [a_{i_x}, b_{i_x}[$. The function f from U to X defined by: $f(x) = \sum\limits_{1 \leq i \leq i_x - 1} (b_i - a_i) + x - a_{i_x}$ for every $x \in U$ is an homeomorphism, then $X$is a witness space, consequently X is p.i.. It remains to prove that $Pi(X)$ generates disjointly $Clop(X)$, what result from the Cantor theorem since $Pi(X) = \{\{x\}/x \text{ non-limit in } \omega^\alpha + 1\} \cup \{U \in Clop(X)/\exists \beta \leq \alpha; U \cong \omega^\beta + 1\}$. In all that follows we designates by $Ic(\omega^\alpha)$ the set of the initial sections of ω^α provided with the order topology associated to inclusion. ∎

Proposition 2.7 *For every ordinal α, $Ic(\omega^\alpha)$ is homeomorphic to $\omega^\alpha + 1$ endowed with his order topology.*

Proof. The application f from $Ic(\omega^\alpha)$ to $\omega^\alpha + 1$ defined by:

$$F(C) = Min((\omega^\alpha + 1) \setminus C), \text{ for every } C \in Ic(\omega^\alpha)$$

is an homeomorphism. ∎

Corollary 2.8 *A non-trivial superatomic boolean algebra A is primitive if and only if there is a countable ordinal α such that $A \simeq Int(\omega^\alpha)$.*

According to [3] one knows that a countable boolean algebra is either superatomic, or uniform, or mixed i.e. product of superatomic and uniform algebras.The mixed, not being p.i. cannot be primitive.The perfect countable boolean algebras are all isomorphic to $Int(\eta)$ that one showed primitive. After several tentative, without success, we abandon the unperfect uniforms.

3 Quasi-primitive boolean algebras

Definition 3.1 *A boolean space (not necessary metrizable) X is weakly primitive if $Pi(X)$ generate disjointly $ClopX$, if furthermore X is p.i. it is said to be quasi-primitive.*

Proposition 3.2 *Let ω_1 be the first uncountable ordinal, the boolean space $X = \omega_1 + 1 + \omega_1^*$ is quasi-primitive.*

Proof. Since X is of witness, ($X_w = \{\omega_1\}$), then X is p.i.. From lemma 2.3, $Pi(X) = \{\{\alpha\}/\alpha \in \omega_1$ and α non-limit $\} \cup \{U \in ClopX/\exists\, \alpha < \omega_1; U \cong \omega^\alpha + 1\}$ $\cup \{U \in ClopX/U \cong X\}$. It remain to prove that $Pi(X)$ generates disjointly $ClopX$, but every clopen of X is of the form $U = \bigcup [a_i, b_i]$, therefore it suffices to show that $Pi(X)$ generates disjointly the elementary clopens of X. Let $[a, b]$ be a such clopen. If $[a, b]$ is finite then $[a, b] = \{x_i, i \in n\} = \cup_{i \in n}\{x_i\}$, no x_i can be limit because otherwise $[a, b]$ will be infinite. If $\|[a, b]\| = \aleph_0$, without loss of generality, we can suppose that $[a, b] \subseteq \omega_1 + 1$. Let $b - a = \omega^{\alpha_1} n_1 + ... + \omega^{\alpha_m} n_m$ the Cantor normal form of $b - a$, we have $[a, b] = [a, a + \omega^{\alpha_1} n_1] \cup .... \cup [a + \omega^{\alpha_1} n_1 + ... + \omega^{\alpha_{m-1}} n_{m-1} + 1, b] = [a, a + \omega^{\alpha_1}] \cup ...... \cup [a + \omega^{\alpha_1}(n_1 - 1) + 1, a + \omega^{\alpha_1} n_1] \cup [a + \omega^{\alpha_1} n_1 + 1, a + \omega^{\alpha_1} n_1 + \omega^{\alpha_2}] \cup ...... \cup [a + \omega^{\alpha_1} n_1 + \omega^{\alpha_2}(n_2 - 1) + 1, a + \omega^{\alpha_1} n_1 + \omega^{\alpha_2} n_2] \cup .... \cup [a + \omega^{\alpha_1} n_1 + ... + \omega^{\alpha_{m-1}} n_{m-1} + 1, b]$, thereby $[a, b]$ is a finite union of disjoint elements of $Pi(X)$. If $\|a, b\| = \aleph_1$ then $\omega_1 \in [a, b]$ so that $[a, b] \cong X$ is p.i.. ∎

Definition 3.3 *Let α and β two nonzero limit ordinals, the β-sequence $(\alpha_\xi)_{\xi<\beta}$ is cofinal in α if $\lim\limits_{\xi \to \beta} \alpha_\xi = \alpha$. Likewise a subset A of α is said to be cofinal in α if $cf(A) = \alpha$. The cofinality of α denoted $cf(\alpha)$ is the smallest limit ordinal β such that there is an increasing β-sequence cofinal in α.*

Remark. In the previous paragraph one himself interesse not to boolean spaces of the forme $\alpha + 1 + \beta^*$, since in countable case $\alpha + 1 + \beta^* \cong Sup(\alpha, \beta) + 1$ but this is false beyond of the countable as testifies the following result:

Proposition 3.4 *Let α an ordinal. if $cf(\omega^\alpha) \geq \omega_1$, then the boolean spaces $\omega^\alpha + 1$ and $\omega^\alpha + 1 + (\omega^\alpha)^*$ are not homeomorphics.*

Proof. Let $X = \omega^\alpha + 1 = U + \{a\}$ and $Y = \omega^\alpha + 1 + (\omega^\alpha)^* = V + \{b\} + W$. Suppose that there is a continues function f from X to Y. Since V and W are two disjoint opens of Y, then $f^{-1}(V)$ and $f^{-1}(W)$ are two disjoint opens of X. Considered as chains, $f^{-1}(V)$ and $f^{-1}(W)$ are cofinals in U, indeed, suppose for example, that $f^{-1}(V)$ is bounded in U, then $F = \overline{f^{-1}(V)}$ is a compact subspace of U such that $rg(F) = \beta < \alpha$, where $rg(F)$ is the Cantor Bendixson rang of F (cf[2]), but $f(F)$ is a compact subspace of Y such that $V \subseteq f(F)$, then $b \in f(F)$, consequently $rg(f(F)) = \alpha$ what is absurd, one deducts that $f^{-1}(V)$ is cofinal in U. Likewise we prove that $f^{-1}(W)$ is cofinal in U. Then, if $(x_n)_{n\in\omega}$ is a strictly increasing sequence of elements of U such that $x_{2n} \in f^{-1}(V)$ and $x_{2n+1} \in f^{-1}(W)$ for every $n \in \omega$. For $x = \lim\limits_{n \to \omega} x_n$ we have $f(x) = \lim\limits_{n \to \omega} x_{2n}$, but $cf(\omega^\alpha) \geq \omega_1$ then $f(x) \in V$, likewise $f(x) = \lim\limits_{n \to \omega} x_{2n+1} \in W$ what is absurd, then there is no continues function from $\omega^\alpha + 1$ to $\omega^\alpha + 1 + \omega^{\alpha *}$. ∎

This result justify the following notation:

Notation. Let α and β are two limit ordinals, the boolean space $\alpha+1+\beta^*$ is denoted by $L(\alpha,\beta)$. Remark that $L(\alpha,\beta)$ is a scattered intervals space and $L(\omega^\alpha,\omega^\beta)$ is p.i. since it is of witness for every ordinals α and β.

Lemma 3.5 *Let α be an ordinal, The space α is boolean if and only if there is an ordinal β and a integer n such that $\alpha \cong \omega^\beta n+1$.*

Proof. This condition is evidently sufficient. Conversely, let α be a boolean ordinal and $\sum_{i\in m} \omega^i n_i$ his Cantor's normal form where $\alpha<\alpha_0<........<\alpha_{m-1}$ and $n_i\in\omega\forall i\in m$. Since α is compact, then $\alpha_{m-1}=0$ and $\alpha=\omega^{\alpha_0}n_0+......+\omega^{\alpha_{m-2}n_{m-2}}+n_{m-1}$. For every $i<m-1$, we have $\alpha_i<\alpha_0$ and thus $\alpha\cong(n_{m-1}-1)+\omega^{\alpha_{m-2}}n_{m-2}.....+\omega^{\alpha_1}n_1+\omega^{\alpha_0}n_0+1\cong\omega^{\alpha_0}n_0+1.$ ∎

In all that follows $\mathcal{O}n$ will designate the class of all ordinals.

Lemma 3.6 *For every limit ordinals α and β, there is μ , $\nu\in\mathcal{O}n$ and $m,n\in\omega$ such that $L(\alpha,\beta)\cong L(\omega^\mu m,\omega^\nu n)$.*

Proof. Let α and β two limit ordinals; there is $\mu,\nu\in\mathcal{O}n$ and $m,n\in\omega$ such that $\alpha\cong\omega^\mu m$ and $\beta^*\cong\omega^\nu n$. Put $L(\alpha,\beta)=\alpha\cup\{\Omega\}\cup\beta^*$ and $L(\omega^\mu m,\omega^\nu n)=\omega^\mu m\cup\{\Omega'\}\cup(\omega^\nu n)^*$. If f from α to $\omega^\mu m$ and g from β to $\omega^\nu n$ are two homeomorphisms, then the application h from $L(\alpha,\beta)$ to $L(\omega^\mu m,\omega^\nu n)$ defined by: $h(\xi)=f(\xi)$ if $\xi\in\alpha$, $h(\xi)=g(\xi)$ if $\xi\in\beta^*$ and $h(\Omega)=\Omega'$ is an homeomorphism. ∎

Proposition 3.7 *For every limit ordinals μ and ν, the boolean space $L(\mu,\nu)$ is weakly primitive.*

Proof. Let μ and ν are two limit ordinals and $X=L(\mu,\nu)$. By the previous lemma we can suppose that $X=L(\omega^\alpha m,\omega^\beta n)=\omega^\alpha m\cup\{\infty\}\cup(\omega^\beta n)^*$. Let $[a,b]$ be an elementary clopen of X. If $b<\omega^\alpha m$ (resp. $a>\omega^\beta n$) then $[a,b]\in clop(\omega^\alpha m+1)$ (resp. $\text{clop}(1+(\omega^\beta n)^*)$). But $\omega^\alpha m+1$ (Resp $1+(\omega^\beta n)^*$) is waekly primitive then $[a,b]$ is the union of a finite disjoint family of p.i. clopens. If $\infty\in[a,b]$ then $[a,b]=U_1\bigcup U_2\bigcup U_3$ where $U_1=[a,b]\bigcap[0,\omega^\alpha(m-1)]$, $U_2=[a,b]\bigcap[\omega^\alpha(m-1)+1,(\omega^\beta(n-1)+1)^*]$, $U_3=[a,b]\bigcap(\omega^\beta(m-1))^*$. On the one hand, since $\infty\in[a,b]$ then $U_2=\omega^\alpha+1+(\omega^\beta)^*$ is p.i., On the other hand, U_1 and U_3 are evidently unions of finite disjoint familys of p.i. clopens. ∎

Corollary 3.8 *For every ordinals α and β, the boolean space $L(\omega^\alpha,\omega^\beta)$ is quasi-primitive.*

Corollary 3.9 *For every integer n and every limit ordinals $\mu_1,......,\mu_n,\nu_1,......,\nu_n$ the boolean space $\sum_{1\le i\le n} L(\mu_i,\nu_i)$ is weakly primitive.*

References

[1] G. Hansoul, Algébres de Boole primitives, *Discrete Mathematics* **53** (1985), 103-116, North Holland.

[2] S. Koppelberg, *Hand book of boolean algebras,* **vol. 1**, North Holland (1989).

[3] R. S. Pierce, Countable boolean algebras, chapter 21, Monk (Ed), *Hand book of boolean algebras,* **Vol. 3**, North Holland (1989), 775-876.

[4] A. Mostowski and A. Tarski, Boolesche Ringe mit geordneter basis, *Fund. Math.,* **Vol. 32** (1939), 69-86.

[5] J. G. Rosenstein, Linear Orderings, *Academic press* (1982).

[6] W. Sierpinski, *Cardinal and ordinal numbers,* Tom 34, Ponst Wome Wydawinctwo, Warszawa (1958).

Received: December 5, 2007