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On Pseudo-Projective and Pseudo-Small-Projective Modules

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Abstract

Pseudo-projectivity and small-pseudo-projectivity is a generalization of projectivity. In this paper, we study the properties of pseudo-projective modules, and discuss the conditions that are equivalent to pseudo-projctivity and small-pseudo-projectvity.

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1 Introduction

Throughout this paper all rings are associative rings with identity, and all modules are unitary right modules.

Pseudo-injective and essential-pseudo-injective modules have been studied by many authors [2], [3], but here we study the properties of pseudo and small-pseudo-projective modules. Suppose that M is an R-module. A submodule A of M is said to be a *small submodule* of M (denoted by $A \ll M$) if for any $B \subseteq M$, $A + B = M$ implies $B = M$. Given two R-modules N and M, N is called *M-projective* if for every submodule A of M, any homomorphism $\alpha : N \to M/A$ can be lifted to a homomorphism $\beta : N \to M$. A module N is called *projective* if it is M-projective for every R-module M. On the other hand, N is called *quasi-projective* if N is N-projective.

It is well-known that an R-module N is projective if and only if satisfies any of the following equivalent conditions:
i) For any R-module M and $B \subseteq M$ every R-homomorphism $N \to M/B$ can be lifted to an R-homomorphism $N \to M$.
ii) For any R-module M, every R-epimorphism $M \to N$ splits.

Moreover, we will denote *projective cover* of M by $(P(M), \alpha_M)$ if, there is an epimorphism $\alpha_M : P(M) \to M$ with $P(M)$ projective and $ker(\alpha_M) \ll P(M)$.

Consider the following conditions for a module M which have been studied in [4],[5] :

D_1: For every submodule N of M there exist submodules K, L of M such that $M = K \oplus L$ and $K \leq N$ and $N \cap L \ll L$.

D_2: If N is a submodule of M such that M/N is isomorphic to a direct summand of M, then N is a direct summand of M.

D_3: For all direct summands K, L of M with $M = K + L$, $K \cap L$ is a direct summand of M.

If the module M satisfies D_1 and D_2 then it is called a *discrete* module.
If the module M satisfies D_1 and D_3 then it is called a *quasi discrete* module.
Let M be a module, and N and L be submodules of M. N is called a *supplement* of L if it is minimal with the property that $M = N + L$, equivalently $M = N + L$ and $N \cap L \ll N$.

2 Preliminary Notes

Recall that an epimorphism $f : M \to N$ is said to split if there exists a homomorphism $g : N \to M$ with $fog = I_N$.

Definition 2.1 *A module N is called M-pseudo-projective (or pseudo-projective relative to M) if for every submodule A of M, any epimorphism $\alpha : N \to M/A$ can be lifted to a homomorphism $\beta : N \to M$. Moreover N is called pseudo-projective if N is N-pseudo-projective. Two modules N and M are called mutually(pseudo-)projective if N is M-(pseudo-)projective and M is N-(pseudo-)projective .*

Definition 2.2 *A module N is called small-M-pseudo-projective if for every submodule A of M any epimorphism $\alpha : N \to M/A$ with $ker\alpha \ll N$ can be lifted to a homomorphism $\beta : N \to M$*

3 Main Results

Theorem 3.1 *If N is M-pseudo-projective then any epimorphism $f : M \to N$ splits.*

proof. Let $f : M \to N$ be an epimorphism. Then $N \cong M/ker(f)$ with an isomorphism g such that $g : N \to M/ker(f)$. As N is M-pseudo-projective then g can be lifted to homomorphism $f' : N \to M$. It is straight forward to show that fof' is the identity map of N.

Theorem 3.2 *N is projective if and only if N is M-pseudo-projective for all M.*

proof. Clear from definition.

Theorem 3.3 *Every direct summand of an M-pseudo-projective module is also M-pseudo-projective module.*

proof. Assume that N is M-pseudo-projective and $N = A \oplus B$. Let X be a submodule of M and $f : A \to M/X$ be an epimorphism. Define the epimorphism $g : A \oplus B \to M/X$ with $g = f \circ \pi_A$, where π_A is natural projection of N onto A. Then $g(a,b) = f(a)$, $a \in A$, $b \in B$. As N is M-pseudo-projective there is $g^* : A \oplus B \to M$ lifting g. Then $f^* = g^* \mid_A$ is homomorphism which lifts f. Therefore A is M-pseudo-projective.

Theorem 3.4 *Let N be an M-pseudo-projective module, $P(N)$ be a projective cover of N and $P(M)$ be a projective cover of M with epimorphism α_N and α_M respectively. If $\alpha : P(N) \to P(M)$ is an epimorphism then $\alpha(ker(\alpha_N)) \subseteq ker(\alpha_M)$.*

proof. We have $\alpha_N : P(N) \to N$ and $\alpha_M : P(M) \longrightarrow M$, then $P(N)/ker(\alpha_N) \cong N$ and $P(M)/ker(\alpha_M) \cong M$. Define $\alpha'_N : N \to P(N)/\ker(\alpha_N)$ be an isomorphism. Because of epimorphism $\alpha : P(N) \to P(M)$ there is another epimorphism $\alpha' : P(N)/\ker(\alpha_N) \to P(M)/\alpha(\ker(\alpha_M))$. Because of epimorphism α_M there is an epimorphism $\alpha'_M : P(M)/\alpha(\ker(\alpha_M)) \to M/\alpha_M(\alpha(\ker(\alpha_N)))$. Therefore there is epimorphism $\theta : N \to M/\alpha_M(\alpha(\ker(\alpha_N))$ such that $\theta = \alpha'_M \circ \alpha' \circ \alpha'_N$. As N is M-pseudo-projective θ can be lifted to a homomorphism $\beta : N \to M$. Therefore we have $(\beta \circ \alpha_N - \alpha_M \circ \alpha)(P(N)) = \alpha_M \circ \alpha(ker(\alpha_N)) \ll M$. On the other hand $(\beta \circ \alpha_N(P(N)) + (\beta \circ \alpha_N - \alpha_M \circ \alpha)(P(N)) = M$. Then $\beta \circ \alpha_N(P(M)) = M$. It means $\beta \circ \alpha_N$ is an epimorphism. Therefore $M/ker(\beta \circ \alpha_N) \cong M/ker(\alpha_M \circ \alpha)$. Then $ker(\beta \circ \alpha_N) \cong ker(\alpha_M \circ \alpha)$. Consequently $\alpha_M \circ \alpha(ker(\alpha_N)) = 0$ and $\alpha(ker(\alpha_N)) \subseteq ker(\alpha_M)$.

Theorem 3.5 *If M is pseudo-projective, then $\alpha(ker(\alpha_M)) \subseteq ker(\alpha_M)$, for every epimorphism α in $End(P(M))$.*

proof. It is clear.

Theorem 3.6 *Let A and B be mutually pseudo-projective. If $P(A) \cong P(B)$, then every isomorphism $P(A) \to P(B)$ reduces an isomorphism $ker(\alpha_A) \to ker(\alpha_B)$. See Proposition(2.4)* *proof.* Let $P(A) \to P(B)$ be an isomorphism. By proposition(2.4) $f(ker(\alpha_A)) \subseteq ker(\alpha_B)$. Similarly, $f \circ f^{-1}(ker(\alpha_B)) \subseteq ker(\alpha_A)$. Thus $ker(\alpha_B) = (f \circ f^{-1})(ker(\alpha_B)) \subseteq f(ker(\alpha_A))$, that means $f(ker(\alpha_A)) = ker(\alpha_B)$. Therefore $f|_{ker(\alpha_A)} : ker(\alpha_A) \to ker(\alpha_B)$ is an isomorphism.

Theorem 3.7 *Let A, B be mutually pseudo-projective. If $P(A) \cong P(B)$ then $A \cong B$. Consequently, A and B are pseudo-projective.*

proof. By Proposition (2.6) $ker(\alpha_A) \cong ker(\alpha_B)$ and $A \cong P(A)/ker(\alpha_A)$, $B \cong P(B)/ker(\alpha_B)$. Then we have $A \cong B$. As A is B-pseudo-projective and $A \cong B$, we have A is A-pseudo-projective, i.e. A and B are pseudo-projective.

Theorem 3.8 *Let M and N be two modules and $X = N \oplus M$. The following conditions are equivalent:*
(i) *N is pseudo-M-projective;*
(ii) *For any submodule A of X with $A + M = X$ and $N + A = X$, there exists a submodule T of A with $T \oplus M = X$.*

proof. $(i) \Longrightarrow (ii)$ Assume that holds (i) and A satisfies the assumptions of (ii) and also let $f : N \to M/(A \cap M)$. If $n \in N$ then there exist $m \in M$, $a \in A$ such that $n = a + m$. Define $f(n) = m + A \cap M$. If $n_1 = n_2$ in N then $n_1 = a_1 + m_1$ and $n_2 = a_2 + m_2$ for some $a_1, a_2 \in A$ and $m_1, m_2 \in$M. Then $a_1 - a_2 = m_2 - m_1$ is in $A \cap M$. Thus $f(n_1) = f(n_2)$. Clearly f is an epimorphism. As N is M-pseudo-projective, f can be lifted to a homomorphism $f' : N \to M$ such that $\pi \circ f' = f$ with $\pi : M \to M/M \cap A$. Define $N' = \{n - f'(n) | n \in N\}$. Let $x \in N'$, then $x = n - f'(n)$ for some $n \in N$. We have $\pi \circ f'(n) = f(n)$ and $n = a + m$ for some a in A and m in M. Therefore $f'(n) + A \cap M = m + A \cap M$. Then $f'(n) - m$ is in $A \cap M$ and $f'(n) - n + a$ is in $A \cap M$. Thus $x \in A$. Then $N' \subseteq A$. It is clear that $N' + M = X$. Also $N' \cap M = 0$. Consequently $N' \oplus M = X$.
$(ii) \Longrightarrow (i)$ Let $f : N \to M/B$ be an epimorphism and $\pi : M \to M/B$ is natural projection. Define $A = \{n + m | f(n) = -\pi(m)\}$. It is clear that $X = N + A$ and $X = A + M$, then there exists a submodule $N' \subseteq A$ with $X = N' \oplus M$. Define $\alpha : N' \oplus M \to M$ with $\alpha(n' + m) = m$. Then $\alpha|_N : N \to M$ lifts f. Therefore N is M- pseudo-projective.

Theorem 3.9 *If N be a M-pseudo projective module and B be a direct summand of M, then N is B-pseudo projective module.*

proof. Let $X' = N \oplus B$ and $X = N \oplus M$. Assume that $A \subseteq X'$ such that $A + B = X'$ and $N + A = X'$. Then we have $A + B = X$ and $N + A = X$. By Proposition (2.8) there exists a submodule $N' \subseteq A$ such that $N' \oplus M = X$. Then $N' \oplus B = X'$. Again by proposition(2.8) N is B-pseudo projective.

Theorem 3.10 *Any pseudo-projective module satisfies D_2.*

proof. Let A be a direct summand of M and B a submodule of M with $M/B \cong A$ such that isomorphism $f : M/B \to A$. Define $f^* : M \to A$ by $f^* = f \circ \pi_B$ where $\pi_B : M \to M/B$ is natural projection of M to M/B. It is clear that f^* is an epimorphism and $ker(f^*) = B$. As M is M-pseudo-projective, A is M-pseudo-projective by Proposition (2.3). Again by Proposition(2.1) f^* splits. It means $M = ker(f^*) \oplus C, C \leq M$. Therefore B is a direct summand of M.

Theorem 3.11 *A pseudo-projective D_1 module is discrete.*

proof. It is clear.

Consequently, we have the extended implications:

projective $\Longrightarrow$ pseudo projective $\Longrightarrow D_2$

pseudo-projective $+D_2 \Longrightarrow$ discrete

The following Theorem is a characterization of small-pseudo-projective

Theorem 3.12 *Let M and N be two modules and $X = M \oplus N$. The following conditions are equivalent:*
(i) N is small-M-pseudo-projective;
(ii) for any submodule A of X such that N is a supplement of A and $A + M = X$, there exists a submodule $N' \subseteq A$ such that $N' \oplus M = X$.

proof. $(i) \Longrightarrow (ii)$ Assume that holds $i)$ and A satisfies the assumptions of (ii) and also let $f : N \to M/(A \cap M)$. If $n \in N$ then there exists $m \in M$, $a \in A$ such that $n = a + m$. Define $f(n) = m + A \cap M$. If $n_1 = n_2$ in N then $n_1 = a_1 + m_1$ and $n_2 = a_2 + m_2$ for some a_1, a_2 in A and m_1, m_2 in M. Then $a_1 - a_2 = m_2 - m_1$ is in $A \cap M$. Thus $f(n_1) = f(n_2)$. Clearly f is an epimorphism. $ker(f) = A \cap N \ll N$. As N is small M-pseudo-projective, f can be lifted to a homomorphism $f' : N \to M$ such that $\pi \circ f' = f$, with $\pi : M \to M/(M \cap A)$. Define $N' = \{n - f'(n) | n \in N\}$ and let $x \in N'$, then $x = n - f'(n)$ for some $n \in N$. We have $\pi \circ f'(n) = f(n)$ and $n = a + m$ for some a in A and m in M. Therefore $f'(n) + A \cap M = m + A \cap M$. Then $f'(n) - m$ is in $A \cap M$ and $f'(n) - n + a$ is in $A \cap M$. Thus $x \in A$. Then $N' \subseteq A$. It is clear that $N' + M = X$. Also $N' \cap M = 0$. Consequently $N' \oplus M = X$.
$(ii) \Longrightarrow (i)$ Let $f : N \to M/B$ be an epimorphism with $ker(f) \ll N$ and $\pi : M \to M/B$ is natural projection. Define $A = \{n + m | f(n) = -\pi(m)\}$. It is clear that $X = N + A$ and $A \cap N = ker(f) \ll N$ and $X = A + M$, then there exists a submodule $N' \subseteq A$ with $X = N' \oplus M$. Define $\alpha : N' \oplus M \to M$ with $\alpha(n' + m) = m$ Then $\alpha|_N : N \to M$ is lifting homomorphism of f. Therefore N is M- pseudo-projective.

Theorem 3.13 *If N be a small-M-pseudo projective module and B be a direct summand of M, then N is small-B-pseudo projective module.*

proof. Let $X' = N \oplus B$ and $X = N \oplus M$. Assume that $A \subseteq X'$ such that N is supplement of A inX' and $X = N \oplus B$. Then we have N is supplement of A in X and$A + B = X$. By Proposition (3.1) there exists a submodule $N' \subseteq A$ such that $N' \oplus M = X$. Then $N' \oplus B = X'$. Again by Proposition(3.1) N is small-B-pseudo projective.

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