On $\mathcal{A}$-Quasi-Projective Modules and $\mathcal{A}$-Semiperfect Modules

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Abstract

If $M$ is $\mathcal{A}$-quasi-projective with $(C_6)$ for $\mathcal{A}$, then $A$ is $\mathcal{A}$-semi perfect. K. Oshiro defined the notion dual to the extending property of modules for $\mathcal{A}$ in [3]. This dualization leads us to the dualizations of continuous modules and quasi-continuous modules. These are the semiperfect modules and quasi-semiperfect modules. We revisit his paper [3] and give some different proofs from Oshiro which are more simpler and easier than [3].

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1 Introduction and Preliminaries

Throughout this note, $R$ is an associative ring with unit, and $M$ is an unitary right $R$-module. Let $\mathcal{A}$ be a subfamily of the family $\mathcal{L}(M)$ of all submodules of $M$ satisfying some condition [3], [4]. $M$ is said to have the extending property of modules for $\mathcal{A}$ provided that, for any $A \in \mathcal{A}$, there exists a summand $A^*$ of $M$ which contains $A$ as an essential submodule, i.e., $\forall A \in \mathcal{A}, \exists A^* \prec M \ni A \leq A^*$. For the dual of the extending property, Oshiro defined the concepts co-essential extension and co-closed submodule in $M$ which correspond to essential...
extension and closed submodule in $M$, respectively. A submodule $A$ of $M$ is essential if and only if the image of the inclusion map $i : A \to M$ is large in $M$. Thus the following definition is quite natural!

**Definition 1.1.** [3] Let $N_1 \leq N_2 \leq M$. We say $N_1$ is a co-essential submodule of $N_2$ in $M$ ($N_1 \trianglelefteq c N_2$ in $M$) if the kernel of the canonical map $M/N_1 \to M/N_2 \to 0$ is small in $M/N_1$.

$N_1 \trianglelefteq c N_2$ in $M$ if and only if $N_2 + X = M$ implies $N_1 + X = M$. $N$ is a co-closed submodule in $M$ if $N' \trianglelefteq c N$ in $M$ implies $N = N'$. Any summand is co-closed. If $N_0 \leq N_1 \leq N_2 \leq M$ then $N_0 \trianglelefteq c N_2$ if and only if $N_0 \trianglelefteq c N_1$ and $N_1 \trianglelefteq c N_2$. If $f : P \to M$ is a homomorphism and $B \trianglelefteq c A$ in $P$ then $f(B) \subseteq c f(A)$ in $M$. In fact, $\tilde{f} : P/B \to M/f(B)$ is a homomorphism and $A/B \ll P/B$, so $f(A)/f(B) \ll M/f(B)$ therefore $f(B) \subseteq c f(A)$ in $M$. Moreover, if $f$ is an epimorphism and $B \trianglelefteq c A$ in $M$ then $f^{-1}(B) \subseteq c f^{-1}(A)$ in $P$. Indeed, if $f^{-1}(A) + X = P$ then $f(f^{-1}(A) + X) = A + f(X) = M$, so $B + f(X) = M$ thus $f^{-1}(B + f(X)) = f^{-1}(B) + X = P$.

**Lemma 1.2.** [3] Let $M = N^* \bigoplus N^{**}$ with $N^* \leq N \leq M$. The followings are equivalent:

(1) $N^* \trianglelefteq c N$ in $M$.

(2) $N \cap N^{**} \ll M$ (i.e., $N \cap N^{**}$ is small in $M$).

(3) $N \cap N^{**} \ll N^{**}$.

**Proof.** (1)$\iff$(2) see [3].

(2)$\iff$(3) It is immediate from the fact $N^{**} <^{\oplus} M$. $\square$

**Definition 1.3.** Let $N$ and $N'$ be submodules of $M$. $N'$ is said to be a supplement of $N$ in $M$ if $M = N + N'$ but $M \neq N + X$ for every $X \leq N'$.

Note that $N'$ is a supplement of $N$ in $M$ if and only if $M = N + N'$ and $N \cap N' \ll N'$. Thus the previous lemma says that $N^* \trianglelefteq c N$ in $M$ if and only if $N^{**}$ is a supplement of $N$ in $M$. If $N$ is a supplement for some submodule in $M$ then $M$ is co-closed. Conversely, if $N$ has a supplement $N'$ and $N$ is co-closed, then $N$ is a supplement of $N'$ (Proposition 1.2 in [3]). First, we reprove Theorem 1.3 of [3].

## 2 The main results

**Proposition 2.1.** Let $N \leq M$. If $M$ and $M/N$ have projective covers then there exists a co-closed submodule $N^*$ of $M$ with $N^* \trianglelefteq c N$ in $M$. 
Proof. Let \( f : P \to M, Q \to M/N \) be the projective covers and let \( \eta \circ f : P \to M \to M/N \), where \( \eta \) is the canonical epimorphism. Thus we have: 
\[ P = P_1 \oplus P_2, P_1 \cong Q, P_2 \subset f^{-1}(N) \text{ and } P_1 \cap f^{-1}(N) \ll P_1 \] (see [1]).
So by the Lemma 1.4 of \([?]\), \( P_2 \leq_c f^{-1}(N) \) in \( P \). Hence \( f(P_2) \leq_c N \) by the previous remarks. We claim that \( f(P_2) \) is co-closed. Indeed, suppose \( K \leq_c f(P_2) \) in \( M \), then by the minimality of the projective cover, \( f^{-1}(K) \cap P_2 \leq_c P_2 \), so \( P_2 \subset f^{-1}(K) \), that is, \( K = f(P_2) \).

\[ \square \]

**Corollary 2.2.** If \( M \) and \( M/N \) have projective covers, then there exists a decomposition of \( M \) as co-closed submodules which are supplement of each other: \( M = M_1 + M_2, M_1 \cap M_2 \ll M_i, i = 1, 2 \) and \( M_2 \leq c N \).

Proof. Under the same notations as Proposition 2.1, we put \( M_1 = f(P_1), M_2 = f(P_2) \). The Proposition says that \( M_2 \leq c N \) and \( M_2 \) is co-closed. It suffices to show that \( f(P_1) \) is a supplement of \( N \), and hence \( f(P_1) \) is a supplement of \( f(P_2) \). If \( f(I) + N = M, I \leq P_1 \), then \( f^{-1}(f(I) + N) = I + f^{-1}(N) = P \). Since \( P_2 \leq_c f^{-1}(N) \), \( I \oplus P_2 = P \). Hence \( I = P_1 \), so \( f(I) = f(P_1) \).

Which modules have such a decomposition as above? : \( \forall A \leq M, \exists M_1, M_2 \) submodules of \( M \ni M = M_1 + M_2, M_2 \leq c A \) in \( M \) and \( M_1 \cap M_2 \ll M_i, i = 1, 2 \). Semiperfect modules have such a decomposition[3]. Moreover, if every submodule of \( M \) has a co-closed submodule which is co-essential in \( M \) and if \( M \) is \( H \)-supplemented (i.e., \( \forall A \leq M, \exists A' \leq c M \ni A + X = M \text{ iff } A' + X = M \)) then \( M \) has above decomposition : Let \( A \leq M \). Then \( \exists A' \leq c M \ni A = A' \oplus B \) for some \( B \leq M \) and \( A + X = M \Leftrightarrow A' + X = M \). So \( M = A + B \) and \( B \) is a supplement of \( A \), i.e. \( A \cap B \ll B \). By hypothesis \( A \) has co-closed submodule \( C \) which is co-essential in \( M \). Then \( M = C + B, C \leq c A \) and \( C \cap B \ll C \) as required. Now when is the composition direct? Semiperfect modules satisfy this. Oshiro defined the lifting property of modules for \( A \) as the dual concept of the extending property as follows; \( M \) has the lifting property of modules for \( \mathcal{A} \) provided that, for any \( A \in \mathcal{A} \), there exists a direct summand \( A^* \leq c M \) such that \( A^* \leq c A \) in \( M \). The following lemma plays an important role in this paper.

**Lemma 2.3.** Suppose \( M = M_1 + M_2, M_1 \cap M_2 \ll M_i, i = 1, 2 \). The followings are equivalent :

1. \( M = M_1 \oplus M_2 \).
2. \( \exists g : M \to M_1 \ni \text{Im}(1_M - g) \subset M_2 \).

Proof. Since \( M_1 \cap M_2 \ll M_1, g \) is epimorphism, and since \( 0 \leq c M_1 \cap M_2 \) in \( M_1, g^{-1}(0) \leq c g^{-1}(M_1 \cap M_2) = M_2 \) in \( M \). Now \( M_2 \) is co-closed in \( M \), so \( g^{-1}(0) \) must be equal to \( M_2 \). Finally, \( M_2 = g^{-1}(0) = g^{-1}(M_1 \cap M_2) \), therefore \( g(M_2) = 0 = M_1 \cap M_2 \).

\[ \square \]
In [3], Oshiro introduced the concepts of $A$-semiperfect modules, $A$-quasi-semiperfect modules and $A$-quasi-projective modules as notions dual to those of $A$-continuous modules, $A$-quasi-continuous modules and $A$-quasi-injective modules, respectively.

**Definition 2.4.** [3] $M$ is $A$-semiperfect (respectively, $A$-quasi-semiperfect) if the conditions $(C_1)$ and $(C_2)$ (respectively, $(C_1)$ and $(C_3)$) below are satisfied;

$(C_1)$ $M$ has the lifting property for $A$.

$(C_2)$ For any $A \in A$ such that $A \vartriangleleft M$, any sequence $M \rightarrow M/A \rightarrow 0$ splits.

$(C_3)$ Let $A \in A$ and $N \in L(M)$ which are summands of $M$. If $X = A \cap N$ is small in $M$ and $(A/X \bigoplus N/X) \vartriangleleft M/X$, then $X = 0$.

In particular, we simply say that $M$ is semi-perfect (respectively, quasi-semiperfect) when it is $L(M)$-semiperfect (respectively, $L(M)$-quasi-semiperfect).

**Definition 2.5.** [3] $M$ is $A$-quasi-projective if it satisfies the condition:

$(C_4)$ For any $A \in A$, $N \in L(M)$ and any sequence $N \rightarrow M/A \rightarrow 0$, there exists a homomorphism $h : M \rightarrow N$ which makes the diagram commute

\[
\begin{align*}
M & \xrightarrow{\eta} M/A \xrightarrow{h} N \xrightarrow{\varphi} 0 \\
M/A & \xrightarrow{\text{can}} N \xrightarrow{0} 0
\end{align*}
\]

where $\eta$ is the canonical map.

We have “projective $\Rightarrow$ quasi-projective $\not\Rightarrow$ semipect $\Rightarrow$ quasi-semiperfect $\Rightarrow (C_1)$ ” (see [2]).

$(C_1)$ implies $(C_6)$ : $\forall A \in A, \exists N \vartriangleleft M \ni M = N + A$ and $N \cap A \ll M$.

Clearly $(C_4)$ implies $(C_2)$. Now we prove Theorem 2.1 of [3] again. It is exactly different and obvious method. Oshiro showed this by a slight modification of the proof of Wu-Jans [5]. That is too long.

**Proposition 2.6.** Assume that $M$ and $M/A$ have projective covers for all $A \in A$. Then if $M$ is $A$-quasi-projective, then it is $A$-semiperfect.

**Proof.** Claim $(C_1)$ : Let $A \in A$. Then by the Corollary of the Proposition 2.1, $M$ has a decomposition $M = M_1 + M_2 \ni M_1 \cap M_2 \ll M_i, i=1,2$ and $M_2 \ll A$ in $M$. Since $M_2 \ll A$, $M_2 \in A$ by the condition $(\beta)$ of $A$. So the quasi-projectivity implies the existence of a homomorphism $g : M \rightarrow M_1$ such
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that

\[
\begin{array}{ccc}
M & \xrightarrow{\eta} & M/M_2 \\
\downarrow g & & \downarrow 0 \\
M_1 & \xrightarrow{\eta|_{M_1}} & 0
\end{array}
\]

where $\eta$ is the canonical map.

Then $m - g(m) \in M_2$ for every $m \in M$, that is, $(1_M - g)M \subset M_2$. Hence by applying the Lemma 2.3 we have $M = M_1 \bigoplus M_2$ and $M_2 \leq_c A$ in $M$. \hfill \Box

Theorem 2.12 of [3] is trivial. Theorem 3.18 of [3] says that if $M$ is a quasi-projective $R$-module satisfying $(C_6)$ for $\mathcal{L}(M)$ then it is semiperfect. We generalize this theorem for $A$ and it’s proof is simple.

**Theorem 2.7.** If $M$ is $A$-quasi-projective with $(C_6)$ for $A$ then $M$ is $A$-semiperfect.

**Proof.** Let $A$ be in $A$ then by $(C_6)$, $\exists\ N <^\oplus M \ni M = N + A$ and $N \cap A \ll M$. "say" $M = N \bigoplus B$. By $(C_4)$ there exists a homomorphism $g : M \rightarrow N$ such that

\[
\begin{array}{ccc}
M & \xrightarrow{\eta} & M/A \\
\downarrow g & & \downarrow 0 \\
N & \xrightarrow{\eta|_N} & 0
\end{array}
\]

where $\eta$ is the canonical map. so $(1_M - g)M \subset A$. Applying the proof of Lemma 2.3, we have $M = N + g^{-1}(0)$ and $g^{-1}(0) \leq_c A$ in $M$. $M/g^{-1}(0) \cong N \cong M/B$, hence by $(\alpha)$ and $(\beta)$ for $A$, $B \in A$. If $g^{-1}(0) <^\oplus M$ then $g^{-1}(0)$ is co-closed in $M$, and hence $M = N + g^{-1}(0)$, $N \cap g^{-1}(0) \ll N$ and $N \cap g^{-1}(0) \ll g^{-1}(0)$. Again, applying Lemma 2.3, we obtain : $M = N \bigoplus g^{-1}(0)$ and $g^{-1}(0) \leq_c A$. For the completeness of the proof we must claim that $g^{-1}(0) <^\oplus M$. Consider $h \circ \eta : M \rightarrow N$ where $\eta : M \rightarrow M/g^{-1}(0)$ is the canonical epimorphism and $h : M/g^{-1}(0) \rightarrow N$ is an isomorphism. $(C_4)$ implies $(C_2)$, therefore the map $q \circ h \circ \eta : M \rightarrow M/B \rightarrow 0$ splits by, say, $f : M/B \rightarrow M$, where $q : N \rightarrow M/B$ is an isomorphism. Hence $M = \text{Im}(f) \bigoplus \ker(q \circ h \circ \eta) = \text{Im}(f) \bigoplus g^{-1}(0)$ as required \hfill \Box

In particular, any quasi-projective module with $(C_6)$ for $\mathcal{L}(M)$ is written as a direct sum of hollow modules[3]. A module $M$ is called supplemented if for any submodules $A$ and $B$ with $A + B = M$, $B$ contains a supplement of
A. Using Lemma 2.3, we can also show that if $M$ is $\mathcal{A}$-quasi-projective and supplemented then it is $\mathcal{A}$-semiperfect.

References


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