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On $\mathcal{A}$-Quasi-Projective Modules and $\mathcal{A}$-Semiperfect Modules

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Abstract

If M is $\mathcal{A}$-quasi-projective with (C_6) for A, then A is $\mathcal{A}$-semi perfect. K. Oshiro defined the notion dual to *the extending property of modules for* $\mathcal{A}$ in [3]. This dualization leads us to the dualizations of continuous modules and quasi-continuous modules. These are the semiperfect modules and quasi-semiperfect modules. We revisit his paper [3] and give some different proofs from Oshiro which are more simpler and easier than [3].

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1 Introduction and Preliminaries

Throughout this note, R is an associative ring with unit, and M is an unitary right R-module. Let $\mathcal{A}$ be a subfamily of the family $\mathcal{L}(M)$ of all submodules of M satisfying some condition [3], [4]. M is said to have the extending property of modules for $\mathcal{A}$ provided that, for any $A \in \mathcal{A}$, there exists a summand A^* of M which contains A as an essential submodule, i.e., $\forall A \in \mathcal{A}$, $\exists\ A^* <^{\oplus} M \ni A \trianglelefteq A^*$. For the dual of the extending property, Oshiro defined the concepts *co-essential extension* and *co-closed submodule* in M which correspond to *essential* *extension* and *closed submodule* in M, respectively. A submodule A of M is essential if and only if the image of the inclusion map $i : A \to M$ is large in M. Thus the following definition is quite natural!

Definition 1.1. *[3] Let $N_1 \leq N_2 \leq M$. We say N_1 is a co-essential submodule of N_2 in M($N_1 \trianglelefteq_c N_2$ in M) if the kernel of the canonical map $M/N_1 \to M/N_2 \to 0$ is small in M/N_1.*

$N_1 \trianglelefteq_c N_2$ in M if and only if $N_2 + X = M$ implies $N_1 + X = M$. N is a *co-closed* submodule in M if $N' \trianglelefteq_c N$ in M implies $N = N'$. Any summand is co-closed. If $N_0 \leq N_1 \leq N_2 \leq M$ then $N_0 \trianglelefteq_c N_2$ if and only if $N_0 \trianglelefteq_c N_1$ and $N_1 \trianglelefteq_c N_2$. If $f : P \to M$ is a homomorphism and $B \trianglelefteq_c A$ in P then $f(B) \trianglelefteq_c f(A)$ in M. In fact, $\bar{f} : P/B \to M/f(B)$ is a homomorphism and $A/B \ll P/B$, so $f(A)/f(B) \ll M/f(B)$ therefore $f(B) \trianglelefteq_c f(A)$ in M. Moreover, if f is an epimorphism and $B \trianglelefteq_c A$ in M then $f^{-1}(B) \trianglelefteq_c f^{-1}(A)$ in P. Indeed, if $f^{-1}(A) + X = P$ then $f(f^{-1}(A) + X) = A + f(X) = M$, so $B + f(X) = M$ thus $f^{-1}(B + f(X)) = f^{-1}(B) + X = P$.

Lemma 1.2. *[3] Let $M = N^* \bigoplus N^{**}$ with $N^* \leq N \leq M$. The followings are equivalent:*

(1) $N^* \trianglelefteq_c N$ in M.

(2) $N \cap N^{**} \ll M$(i.e., $N \cap N^{**}$ is small in M).

(3) $N \cap N^{**} \ll N^{**}$.

Proof. (1)$\Longleftrightarrow$(2) see [3].
(2)$\Longleftrightarrow$(3) It is immediate from the fact $N^{**} <^{\oplus} M$. □

Definition 1.3. *Let N and N' be submodules of M. N' is said to be a supplement of N in M if $M = N + N'$ but $M \neq N + X$ for every $X \leq N'$.*

Note that N' is a supplement of N in M if and only if $M = N + N'$ and $N \cap N' \ll N'$. Thus the previous lemma says that $N^* \trianglelefteq_c N$ in M if and only if N^{**} is a supplement of N in M. If N is a supplement for some submodule in M then M is co-closed. Conversely, if N has a supplement N' and N is co-closed, then N is a supplement of N'(Proposition 1.2 in [3]). First, we reprove Theorem 1.3 of [3].

2 The main results

Proposition 2.1. *Let $N \leq M$. If M and M/N have projective covers then there exists a co-closed submodule N^* of M with $N^* \trianglelefteq_c N$ in M.*

Proof. Let $f : P \to M$, $Q \to M/N$ be the projective covers and let $\eta \circ f : P \to M \to M/N$, where η is the canonical epimorphism. Thus we have: $P = P_1 \bigoplus P_2$, $P_1 \cong Q$, $P_2 \subset f^{-1}(N)$ and $P_1 \cap f^{-1}(N) \ll P_1$(see [1]). So by the Lemma 1.4 of [?], $P_2 \trianglelefteq_c f^{-1}(N)$ in P. Hence $f(P_2) \trianglelefteq_c N$ by the previous remarks. We claim that $f(P_2)$ is co-closed. Indeed, suppose $K \trianglelefteq_c f(P_2)$ in M, then by the minimality of the projective cover, $f^{-1}(K) \cap P_2 \trianglelefteq_c P_2$, so $P_2 \subset f^{-1}(K)$, that is, $K = f(P_2)$. □

Corollary 2.2. *If M and M/N have projective covers, then there exists a decomposition of M as co-closed submodules which are supplement of each other : $M = M_1 + M_2$, $M_1 \cap M_2 \ll M_i$, $i = 1, 2$ and $M_2 \trianglelefteq_c N$.*

Proof. Under the same notations as Proposition 2.1, we put $M_1 = f(P_1)$, $M_2 = f(P_2)$. The Proposition says that $M_2 \trianglelefteq_c N$ and M_2 is co-closed. It suffices to show that $f(P_1)$ is a supplement of N, and hence $f(P_1)$ is a supplement of $f(P_2)$. If $f(I) + N = M$, $I \leq P_1$, then $f^{-1}(f(I) + N) = I + f^{-1}(N) = P$. Since $P_2 \trianglelefteq_c f^{-1}(N)$, $I \bigoplus P_2 = P$. Hence $I = P_1$, so $f(I) = f(P_1)$. □

Which modules have such a decomposition as above? : $\forall A \leq M$, $\exists\, M_1, M_2$ submodules of $M \ni M = M_1 + M_2$, $M_2 \trianglelefteq_c A$ in M and $M_1 \cap M_2 \ll M_i$, i=1, 2. Semiperfect modules have such a decomposition[3]. Moreover, if every submodule of M has a co-closed submodule which is co-essential in M and if M is H-supplemented(i.e., $\forall A \leq M$, $\exists A' <^{\oplus} M \ni A + X = M$ iff $A' + X = M$) then M has above decomposition : Let $A \leq M$. Then $\exists\, A' \leq M \ni M = A' \bigoplus B$ for some $B \leq M$ and $A + X = M \Leftrightarrow A' + X = M$. So $M = A + B$ and B is a supplement of A, i.e. $A \cap B \ll B$. By hyperthesis A has co-closed submodule C which is co-essential in M. Then $M = C + B$, $C \trianglelefteq_c A$ and $C \cap B \ll C$ as required. Now when is the composition direct? Semiperfect modules satisfy this. Oshiro defined *the lifting property of modules for $\mathcal{A}$* as the dual concept of the extending property as follows; M has the lifting property of modules for $\mathcal{A}$ provided that, for any $A \in \mathcal{A}$, there exists a direct summand $A^* <^{\oplus} M$ such that $A^* \trianglelefteq_c A$ in M. The following lemma plays an important role in this paper.

Lemma 2.3. *Suppose $M = M_1 + M_2$, $M_1 \cap M_2 \ll M_i$, i=1, 2. The followings are equivalent :*

(1) $M = M_1 \bigoplus M_2$.

(2) $\exists\, g : M \to M_1 \ni Im(1_M - g) \subset M_2$.

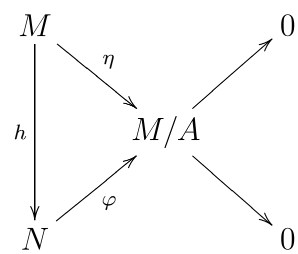
Proof. Since $M_1 \cap M_2 \ll M_1$, g is epimorphism, and since $0 \trianglelefteq_c M_1 \cap M_2$ in M_1, $g^{-1}(0) \trianglelefteq_c g^{-1}(M_1 \cap M_2) = M_2$ in M. Now M_2 is co-closed in M, so $g^{-1}(0)$ must be equal to M_2. Finally, $M_2 = g^{-1}(0) = g^{-1}(M_1 \cap M_2)$, therefore $g(M_2) = 0 = M_1 \cap M_2$. □

In [3], Oshiro introduced the concepts of $\mathcal{A}$-semiperfect modules, $\mathcal{A}$-quasi-semiperfect modules and $\mathcal{A}$-quasi-projective modules as notions dual to those of $\mathcal{A}$-continuous modules, $\mathcal{A}$-quasi-continuous modules and $\mathcal{A}$-quasi-injective modules, respectively.

Definition 2.4. *[3] M is $\mathcal{A}$-semiperfect (respectively, $\mathcal{A}$-quasi-semiperfect) if the conditions (C_1) and (C_2)(respectively, (C_1) and (C_3)) below are satisfied;*
(C_1) M has the lifting property for $\mathcal{A}$.
(C_2) For any $A \in \mathcal{A}$ such that $A <^{\oplus} M$, any sequence $M \to M/A \to 0$ splits.
(C_3) Let $A \in \mathcal{A}$ and $N \in \mathcal{L}(M)$ which are summands of M. If $X = A \cap N$ is small in M and $(A/X \bigoplus N/X) <^{\oplus} M/X$, then $X = 0$.

In particular, we simply say that M is semi-perfect(respectively, quasi-semiperfect) when it is $\mathcal{L}(M)$-semiperfect(respectively, $\mathcal{L}(M)$-quasi-semiperfect).

Definition 2.5. *[3] M is $\mathcal{A}$-quasi-projective if it satisfies the condition :*
(C_4) For any $A \in \mathcal{A}$, $N \in \mathcal{L}(M)$ and any sequence $N \to M/A \to 0$, there exists a homomorphism $h : M \to N$ which makes the diagram commute



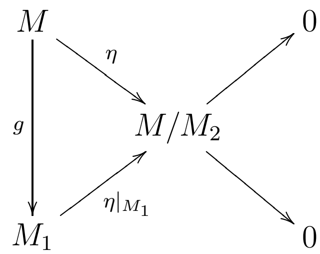
where η is the canonical map.

We have "projective $\Rightarrow$ quasi-projective $\not\Rightarrow$ semipect $\Rightarrow$ quasi-semiperfect $\Rightarrow$ (C_1) "(see [2]).

(C_1) implies (C_6) : $\forall A \in \mathcal{A}$, $\exists\ N <^{\oplus} M \ni M = N + A$ and $N \cap A \ll M$. Clearly (C_4) implies (C_2). Now we prove Theorem 2.1 of [3] again. It is exactly different and obvious method. Oshiro showed this by a slight modification of the proof of Wu-Jans [5]. That is too long.

Proposition 2.6. *Assume that M and M/A have projective covers for all $A \in \mathcal{A}$. Then if M is $\mathcal{A}$-quasi-projective, then it is $\mathcal{A}$-semiperfect.*

Proof. Claim (C_1) : Let $A \in \mathcal{A}$. Then by the Corollary of the Proposition 2.1, M has a decomposition $M = M_1 + M_2 \ni M_1 \cap M_2 \ll M_i$, i=1,2 and $M_2 \trianglelefteq_c A$ in M. Since $M_2 \trianglelefteq_c A$, $M_2 \in \mathcal{A}$ by the condition (β) of $\mathcal{A}$. So the quasi-projectivity implies the existence of a homomorphism $g : M \to M_1$ such that



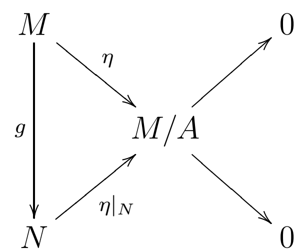
where η is the canonical map.

Then $m - g(m) \in M_2$ for every $m \in M$, that is, $(1_M - g)M \subset M_2$. Hence by applying the Lemma 2.3 we have $M = M_1 \bigoplus M_2$ and $M_2 \trianglelefteq_c A$ in M. $\square$

Theorem 2.12 of [3] is trivial. Theorem 3.18 of [3] says that if M is a quasi-projective R-module satisfying (C_6) for $\mathcal{L}(M)$ then it is semiperfect. We generalize this theorem for $\mathcal{A}$ and it's proof is simple.

Theorem 2.7. *If M is $\mathcal{A}$-quasi-projective with (C_6) for $\mathcal{A}$ then M is $\mathcal{A}$-semiperfect.*

Proof. Let A be in $\mathcal{A}$ then by (C_6), $\exists\, N <^{\oplus} M \ni M = N + A$ and $N \cap A \ll M$. "say" $M = N \bigoplus B$. By (C_4) there exists a homomorphism $g : M \to N$ such that



where η is the canonical map. so $(1_M - g)M \subset A$. Applying the proof of Lemma 2.3, we have $M = N + g^{-1}(0)$ and $g^{-1}(0) \trianglelefteq_c A$ in M. $M/g^{-1}(0) \cong N \cong M/B$, hence by (α) and (β) for $\mathcal{A}$, $B \in \mathcal{A}$. If $g^{-1}(0) <^{\oplus} M$ then $g^{-1}(0)$ is co-closed in M, and hence $M = N + g^{-1}(0)$, $N \cap g^{-1}(0) \ll N$ and $N \cap g^{-1}(0) \ll g^{-1}(0)$. Again, applying Lemma 2.3, we obtain : $M = N \bigoplus g^{-1}(0)$ and $g^{-1}(0) \trianglelefteq_c A$. For the completeness of the proof we must claim that $g^{-1}(0) <^{\oplus} M$. Consider $h \circ \eta : M \to N$ where $\eta : M \to M/g^{-1}(0)$ is the canonical epimorphism and $h : M/g^{-1}(0) \to N$ is an isomorphism. (C_4) implies (C_2), therefore the map $q \circ h \circ \eta : M \to M/B \to 0$ splits by, say, $f : M/B \to M$, where $q : N \to M/B$ is an isomorphism. Hence $M = \mathrm{Im}(f) \bigoplus \ker(q \circ h \circ \eta) = \mathrm{Im}(f) \bigoplus g^{-1}(0)$ as required $\square$

In particular, any quasi-projective module with (C_6) for $\mathcal{L}(M)$ is written as a direct sum of hollow modules[3]. A module M is called *supplemented* if for any submodules A and B with $A + B = M$, B contains a supplement of

A. Using Lemma 2.3, we can also show that if M is $\mathcal{A}$-quasi-projective and supplemented then it is $\mathcal{A}$-semiperfect.

References

[1] Frank W. Anderson and Kent R. Fuller, *Rings and Categories of Modules,* Springer-Verlag, 1974.

[2] Saad H. Mohamed and Bruno J. Müller, *Continuous and discrete modules,* London Mathematical Society Lecture Note Series 147.

[3] K. Oshiro, Semiperfect modules and quasi-semiperfect modules, *Osaka J. Math.,* **20**(1983), 337 - 372.

[4] K. Oshiro, Continuous modules and quasi-continuous modules, *Osaka J. Math.,* **20**(1983), 681 - 694.

[5] L.E.T. Wu and J.P. Jans, On quasi-projectives, *Illinois J. Math.,* 11(1967), 439-448

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