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σ-Derivations on $\mathbb{C}[x]$

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras, $\mathcal{X}$ be a $\mathcal{B}$-bimodule and $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $d : \mathcal{A} \to \mathcal{X}$ is called a σ-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$. In this paper we characterize all σ-derivations of $\mathbb{C}[x]$ in terms of $d(x), \sigma(x)$ and $\sigma(x^2)$. We also characterize all linear mappings σ which possesses a nonzero σ-derivation.

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1. INTRODUCTION

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras, $\mathcal{X}$ be a $\mathcal{B}$-bimodule and $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $d : \mathcal{A} \to \mathcal{X}$ is called a σ-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$. These maps have been extensively investigated in pure algebra. Recently, they have been treated in the Banach algebra theory; see [4, 11, 12, 13] and references therein. Though this notion is a generalization of derivations, this is not the only one. Some other generalizations can be found in [3] and [7] which is studied in the realm of analysis and algebra. The study of theory of derivations in operator algebras is motivated by questions in quantum physics and statistical mechanics, cf. [1, 2]. There are some applications of σ-derivations to develop an approach to deformations of

Lie algebras which have many applications in models of quantum phenomena and in analysis of complex systems; cf. [5].

A wide range of examples are as follows:

(i) Every ordinary derivation of an algebra $\mathcal{A}$ into $\mathcal{A}$ is an $I_{\mathcal{A}}$-derivation, where $I_{\mathcal{A}}$ denotes the identity map on the algebra $\mathcal{A}$;

(ii) Every endomorphism α on an algebra $\mathcal{A}$ is a $\frac{\alpha}{2}$-derivation;

(iii) For a given homomorphism ρ on an algebra $\mathcal{A}$ and a fixed arbitrary element a_0 in $\mathcal{A}$, the linear mapping $d(a) = [a_0, \rho(a)] = a_0\rho(a) - \rho(a)a_0$ is a ρ-derivation of $\mathcal{A}$ which is said to be an inner ρ-derivation.

(iv) Every point derivation $d : \mathcal{A} \to \mathbb{C}$ at the character θ is a θ-derivation.

Among well-known problems, the problem of automatic continuity of σ-derivations is studied in [9], some problems concerning σ-dynamics are studied in [11] and σ-amenability is considered in [10]. See also [6] for an approach to continuity of generalized derivations without linearity.

If σ is the identity mapping on $\mathcal{A}$ then a σ-derivation is an ordinary derivation. As an example of a σ-derivation which is not ordinary, we can consider an automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ regarded as an $\frac{\alpha}{2}$-derivation. This shows that the theory of σ-derivations links the theory of derivations and automorphisms to each other and this is a reason that we are interested in the study of σ-derivations. Naturally, we are interested to generalize the classical results concerning derivations to σ-derivations.

Here, we are interested in σ-derivations on $\mathbb{C}[x]$, the algebra of all polynomials over $\mathbb{C}$. Note that if $d : \mathbb{C}[x] \to \mathbb{C}[x]$ is an ordinary derivation, then $d(f) = f'd(x)$ for each $f \in \mathbb{C}[x]$. Whence d is precisely determined by $d(x)$.

We assume that $\sigma : \mathbb{C}[x] \to \mathbb{C}[x]$ is a linear mapping. The importance of our approach is that σ is a linear mapping in general, not necessarily an algebra homomorphism. We show that each σ-derivation $d : \mathbb{C}[x] \to \mathbb{C}[x]$ is precisely determined by $d(x), \sigma(x)$ and $\sigma(x^2)$. This poses the problem of characterizing all linear mappings σ which possesses a nonzero σ-derivation d. These mappings are called derivable and we will characterize them in terms of $\sigma(x)$ and $\sigma(x^2)$. Furthermore, all derivable mappings that are homomorphisms, all σ-derivations that are homomorphisms and all σ-derivations that are ordinary derivations are characterized. Moreover, as an illustration of the discussion, a non-trivial example is given. A discussion about (σ, τ)-derivations on $\mathbb{C}[x]$ and $\mathbb{C}[x_1, \ldots, x_n]$ is also given. For a nice result concerning local derivations on $\mathbb{C}[x_1, \ldots, x_n]$ one can see [8].

2. The Results

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras, $\mathcal{X}$ be a $\mathcal{B}$-bimodule and $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $d : \mathcal{A} \to \mathcal{X}$ is called a σ-derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$. In this paper we consider derivations on $\mathcal{A} = \mathbb{C}[x]$, the algebra of all polynomials over $\mathbb{C}$. Thus we assume that $\mathcal{X} = \mathcal{B} = \mathcal{A} = \mathbb{C}[x]$ and $\sigma : \mathbb{C}[x] \to \mathbb{C}[x]$ is a linear mapping.

If $\sigma = I$, the identity mapping on $\mathbb{C}[x]$, then a σ-derivation is an ordinary derivation. As an example of a σ-derivation which is not ordinary, put $\sigma(f) = \frac{f}{2}$ and $d(f) = ff_0$ on $\mathbb{C}[x]$, where f_0 is an arbitrary nonzero element of $\mathbb{C}[x]$. Note that d in not an automorphism. Also note that $d(1) \neq 0$ and $\sigma(1) \neq 1$.

Let $d : \mathbb{C}[x] \to \mathbb{C}[x]$ be a σ-derivation. Then $d(1) = d(1^2) = 2d(1)\sigma(1)$. Thus $(2\sigma(1) - 1)d(1) = 0$. This implies that $2\sigma(1) - 1$ and $d(1)$ are polynomials of degree 0.

Proposition 2.1. *Let $d : \mathbb{C}[x] \to \mathbb{C}[x]$ be a σ-derivation with $d(1) = \lambda$, where λ is a nonzero complex number. Then 2σ is a homomorphism and $d = 2\sigma$.*

Proof. We have $(2\sigma(1) - 1)\lambda = (2\sigma(1) - 1)d(1) = 0$. Thus $2\sigma(1) = 1$ and so

$$2d(f) = 2d(f)\sigma(1) + 2\sigma(f)d(1) = d(f) + 2\sigma(f)$$

for all $f \in \mathbb{C}[x]$. This implies that $d = 2\sigma$. Now we have

$$2\sigma(fg) = d(fg) = d(f)\sigma(g) + \sigma(f)d(g) = (2\sigma(f))(2\sigma(g))$$

for all $f, g \in \mathbb{C}[x]$. Hence 2σ is a homomorphism. □

Under the above discussion, from now on we assume that $d(1) = 0$.

Lemma 2.2. *Let $d : \mathbb{C}[x] \to \mathbb{C}[x]$ be a σ-derivation. If $d_k = d(x^k)$ and $s_k = \sigma(x^k)$ for $k \in \mathbb{N}$, then*

(i) $d_{k+\ell} = d_k s_\ell + s_k d_\ell$;

(ii) $(s_k - s_1 s_{k-1})d_1 = d_{k-1}(s_2 - s_1^2)$;

(iii) $d_k = d_1 \sum_{i=0}^{k-1} s_1^{k-1-i} s_i$;

(iv) $s_k d_1 = [s_1 s_{k-1} + (s_2 - s_1^2) \sum_{i=0}^{k-2} s_1^{k-2-i} s_i] d_1$;

(v) $s_k d_1 = [2s_1 s_{k-1} + (s_2 - 2s_1^2) s_{k-2}] d_1$.

Proof. (i) We have

$$d_{k+\ell} = d(x^{k+\ell}) = d(x^k x^\ell) = d(x^k)\sigma(x^\ell) + \sigma(x^k)d(x^\ell) = d_k s_\ell + s_k d_\ell.$$

(ii) By (i) we can write

$$(d_{k-1}s_1 + s_{k-1}d_1)s_1 + s_k d_1 = d_{k+1} = d_{k-1}s_2 + s_{k-1}d_2 = d_{k-1}s_2 + 2s_{k-1}s_1 d_1,$$

which implies (ii).

(iii) We use induction on k. For $k = 1$, the result is obvious. Let (iii) be true for k. Thus

$$d_{k+1} = d_k s_1 + s_k d_1 = s_1 d_1 (\sum_{i=0}^{k-1} s_1^{k-1-i} s_i) + s_k d_1 = d_1 (\sum_{i=0}^{k-1} s_1^{k-i} s_i + s_k) = d_1 \sum_{i=0}^{k} s_1^{k-i} s_i.$$

(iv) This is obvious by (ii) and (iii).

(v) By (iv) we have $s_{k-1}d_1 = [s_1 s_{k-2} + (s_2 - s_1^2)\sum_{i=0}^{k-3} s_1^{k-3-i} s_i]d_1$ and so

$$\begin{aligned}
[s_1 s_{k-1} + (s_2 - 2s_1^2)s_{k-2}]d_1 &= [s_1^2 s_{k-2} + (s_2 - s_1^2)\sum_{i=0}^{k-3} s_1^{k-2-i} s_i \\
&\quad + (s_2 - s_1^2)s_{k-2} - s_1^2 s_{k-2}]d_1 \\
&= [(s_2 - s_1^2)\sum_{i=0}^{k-2} s_1^{k-2-i} s_i]d_1.
\end{aligned}$$

Now we can write

$$\begin{aligned}
s_k d_1 &= [s_1 s_{k-1} + (s_2 - s_1^2)\sum_{i=0}^{k-2} s_1^{k-2-i} s_i]d_1 \\
&= [s_1 s_{k-1} + s_1 s_{k-1} + (s_2 - 2s_1^2)s_{k-2}]d_1 \\
&= [2s_1 s_{k-1} + (s_2 - 2s_1^2)s_{k-2}]d_1.
\end{aligned}$$

□

In the following lemma we symbolically use a matrix representation to have a simple recursive formula for s_k.

Lemma 2.3. *Let $d : \mathbb{C}[x] \to \mathbb{C}[x]$ be a σ-derivation. If $d_k = d(x^k), s_k = \sigma(x^k)$ and*

$$S_k = \begin{bmatrix} s_k & s_{k-1} \\ s_{k-1} & s_{k-2} \end{bmatrix} \text{ and } T = \begin{bmatrix} 2s_1 & 1 \\ s_2 - 2s_1^2 & 0 \end{bmatrix},$$

then for each natural number $k \geq 3$ we have $d_1 S_k = d_1 S_3 T^{k-3}$.

Proof. We use induction on k. For $k = 3$ the assertion is obvious. If $d_1S_k = d_1S_3T^{k-3}$ then, by (v) of Lemma 2.2,

$$\begin{aligned} d_1S_{k+1} &= d_1\begin{bmatrix} s_{k+1} & s_k \\ s_k & s_{k-1} \end{bmatrix} \\ &= d_1\begin{bmatrix} 2s_1s_k + (s_2 - 2s_1^2)s_{k-1} & s_k \\ 2s_1s_{k-1} + (s_2 - 2s_1^2)s_{k-2} & s_{k-1} \end{bmatrix} \\ &= d_1\begin{bmatrix} s_k & s_{k-1} \\ s_{k-1} & s_{k-2} \end{bmatrix}\begin{bmatrix} 2s_1 & 1 \\ s_2 - 2s_1^2 & 0 \end{bmatrix} \\ &= d_1S_kT = d_1(S_3T^{k-3})T = d_1S_3T_{k-2}. \end{aligned}$$

□

We can summarize our results as follows:

Theorem 2.4. *Let $d : \mathbb{C}[x] \to \mathbb{C}[x]$ be a σ-derivation. Then for each $f = \sum_{k=0}^{m} \gamma_k x^k \in \mathbb{C}[x]$ we have $d(f) = \sum_{k=1}^{m} \gamma_k d_k$, where $d_k = d_1 \sum_{i=0}^{k-1} s_1^{k-1-i} s_i$ and s_k's are determined by the recursive relation*

$$d_1\begin{bmatrix} s_k & s_{k-1} \\ s_{k-1} & s_{k-2} \end{bmatrix} = d_1\begin{bmatrix} 3s_1s_2 - 2s_1^3 & s_2 \\ s_2 & s_1 \end{bmatrix}\begin{bmatrix} 2s_1 & 1 \\ s_2 - 2s_1^2 & 0 \end{bmatrix}^{k-3}.$$

Note that the definition of d is just related to s_1, s_2 and d_1. Indeed, we can say that each σ-derivation is determined by these three polynomials. For each three arbitrary polynomials s_1, s_2 and d_1, is the mapping d defined as above a σ-derivation? We affirmatively answer this question.

Suppose that $\sigma : \mathbb{C}[x] \to \mathbb{C}[x]$ is a linear mapping. Then σ is called derivable if there is a nonzero linear mapping $d : \mathbb{C}[x] \to \mathbb{C}[x]$ such that d is a σ-derivation. Our arguments show that if σ is derivable, then it is uniquely determined by $s_1 = \sigma(x)$ and $s_2 = \sigma(x^2)$. Furthermore, we have the following result.

Proposition 2.5. *A derivable mapping $\sigma : \mathbb{C}[x] \to \mathbb{C}[x]$ is a homomorphism if and only if $\sigma(1) = 1$ and $\sigma(x^2) = \sigma(x)^2$.*

Proof. Suppose that $s_1=\sigma(x), s_2=\sigma(x^2)$ and $s_2=s_1^2$. Then we have

$$\begin{aligned}
\begin{bmatrix} s_k & s_{k-1} \\ s_{k-1} & s_{k-2} \end{bmatrix} &= \begin{bmatrix} 3s_1s_2-2s_1^3 & s_2 \\ s_2 & s_1 \end{bmatrix}\begin{bmatrix} 2s_1 & 1 \\ s_2-2s_1^2 & 0 \end{bmatrix}^{k-3} \\
&= \begin{bmatrix} s_1^3 & s_1^2 \\ s_1^2 & s_1 \end{bmatrix}\begin{bmatrix} 2s_1 & 1 \\ -s_1^2 & 0 \end{bmatrix}^{k-3} \\
&= \begin{bmatrix} s_1^4 & s_1^3 \\ s_1^3 & s_1^2 \end{bmatrix}\begin{bmatrix} 2s_1 & 1 \\ -s_1^2 & 0 \end{bmatrix}^{k-4} \\
&= \dots \\
&= \begin{bmatrix} s_1^k & s_1^{k-1} \\ s_1^{k-1} & s_1^{k-2} \end{bmatrix}.
\end{aligned}$$

Hence if $f=\sum_{k=0}^m \gamma_k x^k \in \mathbb{C}[x]$ then $\sigma(f)=\sum_{k=0}^m \gamma_k s_1^k$. This shows that

$$\begin{aligned}
\sigma\left((\sum_{k=0}^m \gamma_k x^k)(\sum_{\ell=0}^n \lambda_\ell x^\ell)\right) &= \sigma\left(\sum_{i=0}^{n+m}\left[\sum_{k=0}^i \gamma_k\lambda_{i-k}\right]x^i\right) \\
&= \sum_{i=0}^{n+m}\left[\sum_{k=0}^i \gamma_k\lambda_{i-k}\right]s_1^i \\
&= (\sum_{k=0}^m \gamma_k s_1^k)(\sum_{\ell=0}^n \lambda_\ell s_1^\ell) \\
&= \sigma(\sum_{k=0}^m \gamma_k x^k)\sigma(\sum_{\ell=0}^n \lambda_\ell x^\ell),
\end{aligned}$$

where $s_1^0=\sigma(1)=1$. Thus σ is a homomorphism. The converse is trivial. □

Theorem 2.6. *Let $\sigma:\mathbb{C}[x]\to\mathbb{C}[x]$ be a linear mapping and $s_k=\sigma(x^k)$. Then σ is derivable if and only if $S_k=S_3T^{k-3}$ for each $k\geq 3$, where*

$$S_k=\begin{bmatrix} s_k & s_{k-1} \\ s_{k-1} & s_{k-2} \end{bmatrix} \text{ and } T=\begin{bmatrix} 2s_1 & 1 \\ s_2-2s_1^2 & 0 \end{bmatrix}.$$

Proof. Let $S_k=S_3T^{k-3}$ or equivalently $s_k=s_1s_{k-1}+(s_2-s_1^2)\sum_{i=0}^{k-2}s_1^{k-2-i}s_i$. For arbitrary function d_1 in $\mathbb{C}[x]$ with $d_1\neq 0$, define $d:\mathbb{C}[x]\to\mathbb{C}[x]$ by

$$d\left(\sum_{k=0}^m \gamma_k x^k\right)=\sum_{k=1}^m \gamma_k d_k,$$

where $d_k = d_1 \sum_{i=0}^{k-1} s_1^{k-1-i} s_i$. Then we can inductively prove that $d_{k+1} = d_k s_1 + s_k d_1$, and this inductively implies that $d_{k+\ell} = d_k s_\ell + s_k d_\ell$. Thus

$$\begin{aligned} d\left((\sum_{k=0}^{m} \gamma_k x^k)(\sum_{\ell=0}^{n} \lambda_\ell x^\ell) \right) &= d\left(\sum_{i=0}^{n+m} \left[\sum_{k=0}^{i} \gamma_k \lambda_{i-k} \right] x^i \right) \\ &= \sum_{i=1}^{n+m} \left[\sum_{k=0}^{i} \gamma_k \lambda_{i-k} \right] d_i \\ &= \sum_{i=1}^{n+m} \left[\sum_{k=0}^{i} \gamma_k \lambda_{i-k} (d_k s_{i-k} + s_k d_{i-k}) \right] \\ &= (\sum_{k=1}^{m} \gamma_k d_k)(\sum_{\ell=0}^{n} \lambda_\ell s_\ell) + (\sum_{k=0}^{m} \gamma_k s_k)(\sum_{\ell=1}^{n} \lambda_\ell d_\ell) \\ &= d\left(\sum_{k=0}^{m} \gamma_k x^k \right) \sigma \left(\sum_{\ell=0}^{n} \lambda_\ell x^\ell \right) \\ &\quad + \sigma\left(\sum_{k=0}^{m} \gamma_k x^k \right) d \left(\sum_{\ell=0}^{n} \lambda_\ell x^\ell \right). \end{aligned}$$

The converse is clear. □

Remark 2.7. The recursive relation $d_1 S_k = d_1 S_3 T^{k-3}$ has some other consequences. For example, taking determinant of both sides of the relation we have $d_1(s_k s_{k-2} - s_{k-1}^2) = d_1(3s_1^2 s_2 - 2s_1^4 - s_2^2)(2s_1^2 - s_2)^{k-3}$. This again implies that if $s_2 = s_1^2$ then $s_k s_{k-2} - s_{k-1}^2 = 0$ which yields $s_k = s_1^k$.

Corollary 2.8. *A σ-derivation $d : \mathbb{C}[x] \to \mathbb{C}[x]$ is a homomorphism if and only if $\sigma = \frac{d}{2}$.*

Proof. d is a homomorphism if and only if $d_k = d_1^k$. On the other hand, $d_1^2 = d_2 = 2d_1 s_1$. Hence $s_1 = \frac{d_1}{2}$. Thus $d_1^{k-1} s_1 + s_{k-1} d_1 = d_k = d_1^k$ and so

$$s_{k-1} d_1 = d_1^k - d_1^{k-1} s_1 = d_1^k - d_1^{k-1} \frac{d_1}{2} = \frac{d_1^k}{2}.$$

This implies that $s_k = \frac{d_1^k}{2} = \frac{d_k}{2}$. Thus $\sigma = \frac{d}{2}$.

Conversely, each homomorphism d is obviously a $\frac{d}{2}$-derivation. □

Corollary 2.9. *A nonzero σ-derivation $d : \mathbb{C}[x] \to \mathbb{C}[x]$ is an ordinary derivation if and only if $\sigma(x) = x$ and $\sigma(x^2) = x^2$.*

Proof. Let $d(fg) = d(f)\sigma(g) + \sigma(f)d(g)$ and $d(fg) = d(f)g + fd(g)$ for each $f, g \in \mathbb{C}[x]$. Then we must have

$$d(f)(\sigma(g) - g) + d(g)(\sigma(f) - f) = 0,$$

and so for $f = g = x$ we have $2d(x)(\sigma(x) - x) = 0$ or equivalently $\sigma(x) = x$. Also for $f = x$ and $g = x^2$ we have $d(x)(\sigma(x^2) - x^2) + d(x^2)(\sigma(x) - x) = 0$ or equivalently $d(x)(\sigma(x^2) - x^2) = 0$ which implies that $\sigma(x^2) = x^2$. The converse is obvious. □

Corollary 2.10. *A nonzero σ-derivation $d : \mathbb{C}[x] \to \mathbb{C}[x]$ is an ordinary derivation if and only if $d(x^2) = 2xd(x)$ and $d(x^3) = 3x^2d(x)$.*

Proof. We have $\sigma(x) = x$ and $\sigma(x^2) = x^2$ if and only if $d(x^2) = 2xd(x)$ and $d(x^3) = 3x^2d(x)$. □

As we noticed, we can arbitrarily choose s_1, s_2 and a nonzero d_1 to construct a derivable mapping σ and its relative σ-derivation d. To illustrate this, let us give an example.

Example 2.11. Suppose that $\sigma(1) = 1, s_1 = \frac{x}{2}$ and $s_2 = 2s_1^2$, so that σ is not a homomorphism. We have $T = \begin{bmatrix} x & 1 \\ 0 & 0 \end{bmatrix}$ and so $T^k = \begin{bmatrix} x^k & x^{k-1} \\ 0 & 0 \end{bmatrix}$. Thus we can write

$$\begin{aligned} S_k = S_3T^{k-3} &= \begin{bmatrix} 3s_1s_2 - 2s_1^3 & s_2 \\ s_2 & s_1 \end{bmatrix} \begin{bmatrix} x^{k-3} & x^{k-4} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{x^3}{2} & \frac{x^2}{2} \\ \frac{x^2}{2} & \frac{x}{2} \end{bmatrix} \begin{bmatrix} x^{k-3} & x^{k-4} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{x^k}{2} & \frac{x^{k-1}}{2} \\ \frac{x^{k-1}}{2} & \frac{x^{k-2}}{2} \end{bmatrix}. \end{aligned}$$

Hence $\sigma(x^k) = \frac{x^k}{2}$ and so $\sigma(f) = \frac{f+f(0)}{2}$, since $\sigma(1) = 1$. On the other hand,

$$d_k = d_{k-1}s_1 + s_{k-1}d_1 = d_{k-1}\frac{x}{2} + \frac{x^{k-1}}{2}d_1,$$

and we can inductively deduce that $d_k = x^k d_1$. Thus $d(f) = [f - f(0)]d_1$, since $d(1) = 0$. Note that

$$\begin{aligned} d(fg) &= [fg - f(0)g(0)]d_1 \\ &= [f - f(0)]d_1 \left(\frac{g + g(0)}{2}\right) + \left(\frac{f + f(0)}{2}\right)[g - g(0)]d_1 \\ &= d(f)\sigma(g) + \sigma(f)d(g). \end{aligned}$$

Remark 2.12. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras, $\mathcal{X}$ be a $\mathcal{B}$-bimodule and $\sigma, \tau : \mathcal{A} \to \mathcal{B}$ be two linear mappings. A linear mapping $d : \mathcal{A} \to \mathcal{X}$ is called a (σ, τ)-derivation if $d(ab) = d(a)\sigma(b) + \tau(a)d(b)$ for all $a, b \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are involutive algebras and σ, τ, d preserve $*$ then using the fact that $d(b^*a^*) = d(ab)^* = \sigma(b^*)d(a^*) + d(b^*)\tau(a^*)$ we can conclude that d is also a (τ, σ)-derivation and so it is $(\frac{\sigma+\tau}{2}, \frac{\sigma+\tau}{2})$-derivation. Hence in the case of $*$-linear mappings we can study σ-derivations instead of (σ, τ)-derivations. As $\mathbb{C}[x]$ can be regarded as an involutive algebra, our results hold for (σ, τ)-derivations on $\mathbb{C}[x]$, provided that σ, τ and d preserve $*$.

Remark 2.13. Let $\sigma : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ be a linear mapping and $d : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ be a σ-derivation. Since $d(x^k y^\ell) = d(x^k)\sigma(y^\ell) + \sigma(x^k)d(y^\ell)$, we can determine d by $d(x), d(y), \sigma(x), \sigma(y)$ and $\sigma(x^2), \sigma(y^2)$. This shows that if $\sigma : \mathbb{C}[x, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ is a linear mapping and $d : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ is a σ-derivation, then d can be determined by
$d(x_1), \ldots, d(x_n), \sigma(x_1), \ldots, \sigma(x_n)$ and $\sigma(x_1^2), \ldots, \sigma(x_n^2)$.

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