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Notes on $\mathbb{Z}^{2^n}$-Graded Algebras

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Abstract

We define a Lie-admissible algebra and its antisymmetrized algebra. We also show that those algebras are simple. Finding an auto-invariant set of a non-associative algebra in the paper, we find the automorphism group of the algebra.

Keywords: Lie-admissible algebra, Lie algebra, simple, automorphism

1 Preliminaries

Let $\mathbb{F}$ be the field of characteristic zero (not necessarily algebraically closed.) Throughout the paper, $\mathbb{N}$ and $\mathbb{Z}$ will denote the set of all the non-negative integers and the integers, respectively. For $m \leq n$, let us define the $\mathbb{F}$-algebra $\mathbb{F}[e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n]$ spanned by

$$\{e^{a_{1,\cdots,m}x_1\cdots x_m}e^{a_{1,\cdots,m-1}x_1\cdots x_{m-1}}\cdots e^{a_{2,\cdots,m}x_2\cdots x_m}e^{a_1x_1}\cdots e^{a_mx_m}x_1^{i_1}\cdots x_n^{i_n}| \\ a_{1,\cdots,m},\cdots,a_m \in \mathbb{Z}, i_1,\cdots,i_n \in \mathbb{N}\}$$

Thus we can define a Lie-admissible algebra $NW([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$ with the set $\{f\partial_u | f \in \mathbb{F}[e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n], 1 \leq u \leq n\}$ with the obvious addition and the multiplication $*$ is defined as follows:

$$f\partial_u * g\partial_v = f\partial_u(g)\partial_v$$

for $f\partial_u, g\partial_v \in NW([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$. Using the commutator $[,]$ of the algebra $NW([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$, we can define its antisymmetrized algebra $W([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$ naturally. We can consider $[x_1, \cdots, x_m]$ as the set $S_m = \{x_1\cdots x_m, x_1\cdots x_{m-1}, \cdots, x_2\cdots x_m, \cdots, x_1, \cdots, x_m\}$. Thus for any subset S of S_m, we can define the $\mathbb{F}$-subalgebra $\mathbb{F}[e^{\pm[S]}, x_1, \cdots, x_n]$ of $\mathbb{F}[e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n]$ spanned by

$$\{e^{a_1s_1}\cdots e^{a_rs_r}x_1^{i_1}\cdots x_n^{i_n}|s_1,\cdots,s_r \in S_m, a_1,\cdots,a_r \in \mathbb{Z}, i_1,\cdots,i_n \in \mathbb{N}\}$$

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In a similar manner, we can define a Lie-admissible algebra $NW([e^{\pm[S]}, x_1, \cdots, x_n])$ and its antisymmetrized algebra $W([e^{\pm[S]}, x_1, \cdots, x_n])$ as $NW([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$ and $W([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$. Note that the antisymmetrized algebra $W([e^{\pm[S]}, x_1, \cdots, x_n])$ is a Lie algebra, i.e., the Jacobi identity holds on the algebra. The set of all the right identities of the algebra $NW([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$ is $\{\sum_{1\le u\le n}(x_u + c_u)\partial_u | c_1, \cdots, c_u \in \mathbb{F}\}$. It is easy to know that the algebras $NW([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$ and $W([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$ are $\mathbb{Z}^{2^m}$-graded (see [5], [6]). Thus their subalgebras are appropriate graded algebras.

2 Simplicities

Theorem 1 *If S is a non-empty set, then the Lie-admissible algebra $NW([e^{\pm[S]}, x_1, \cdots, x_n])$ is simple.*

Proof. Since the algebra $NW([e^{\pm[S]}, x_1, \cdots, x_n])$ is graded, it is easy to prove that any ideal generated by a non-zero element of the algebra is the algebra $NW([e^{\pm[S]}, x_1, \cdots, x_n])$ itself. This implies that the algebra is simple. □

Theorem 2 *If S is a non-empty set, then the Lie (i.e., antisymmetrized) algebra $W([e^{\pm[S]}, x_1, \cdots, x_n])$ is simple.*

Proof. The proof of the theorem is similar to the proof of Theorem 1, so it is omitted. □

Corollary 1 *The Lie-admissible algebra $NW([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$ and the Lie algebra $W([e^{\pm[x_1,\cdots,x_m]}, x_1, \cdots, x_n])$ are simple.*

3 Automorphism group of the algebra $NW[e^{\pm xy}, x, y]$

Lemma 1 *For any automorphism θ in the automorphism group $Aut(NW[e^{\pm xy}, x, y])$ of the algebra $NW[e^{\pm xy}, x, y]$, either $\theta(\partial_1) = c_1\partial_1$ and $\theta(\partial_2) = c_2\partial_2$ hold, or $\theta(\partial_1) = c_3\partial_2$ and $\theta(\partial_2) = c_4\partial_1$ hold for $c_1, \cdots, c_4 \in \mathbb{F}^\bullet$.*

Proof. Let θ be the automorphism of the algebra $NW[e^{\pm xy}, x, y]$ in the lemma. Since $x\partial_1 + y\partial_2$ is a right identity of $NW[e^{\pm xy}, x, y]$, we have that $\theta(x\partial_1 + y\partial_2) = (x + d_1)\partial_1 + (y + d_2)\partial_2$ for $d_1, d_2 \in \mathbb{F}$. Since the right annihilator of $NW[e^{\pm xy}, x, y]$ is spanned by ∂_1 and ∂_2, we also have that $\theta(\partial_1) = d_3\partial_1 + d_4\partial_2$ and $\theta(\partial_2) = d_5\partial_1 + d_6\partial_2$ such that $d_3d_6 - d_4d_5 \neq 0$. Let us assume that d_3 and d_4 are not zeroes. Since $x\partial_1$ is a right identity of the element ∂_1, we are able to prove that $\theta(x\partial_1) = (x + c_7)\partial_1 + (y + c_8)\partial_2$ for $c_7, c_8 \in \mathbb{F}$. This contradicts the fact that $x\partial_1$ is not a right identity of the algebra. Thus either d_3 or d_4 is zero. Similarly we can prove that only one of d_5 and d_6 is zero. We have the following two cases, Case I: $\theta(\partial_1) = d_3\partial_1$ and Case II: $\theta(\partial_1) = d_4\partial_2$.

Case I. Let us assume that $\theta(\partial_1) = d_3\partial_1$ holds for $d_3 \in \mathbb{F}^\bullet$. Since $x\partial_1$ is a right identity of ∂_1, $x\partial_1$ is an idempotent, and ∂_2 is in the left annihilator of $x\partial_1$, we can prove that $\theta(x\partial_1) = (x + d_{10})\partial_1$ for $d_{10} \in \mathbb{F}$. Similarly we can also prove that $\theta(y\partial_2) = (y + d_{11})\partial_2$ holds for $d_{11} \in \mathbb{F}$ by switching appropriate coefficients in the lemma. These imply that $\theta(\partial_1) = c_1\partial_1$ and $\theta(\partial_2) = c_2\partial_2$ hold by switching appropriate coefficients in the lemma.

Case II. let us assume that $\theta(\partial_1) = d_4\partial_2$ holds for $d_4 \in \mathbb{F}^\bullet$. Because of Case I, similarly we can prove that $\theta(\partial_2) = d_5\partial_1$ by taking appropriate coefficients in the lemma.

Thus by Case I and Case II, the lemma follows immediately. □

Notes. For any basis element $e^{axy}x^iy^p\partial_u$, $1 \leq u \leq 2$, of the algebra $NW[e^{\pm xy}, x, y]$ (resp. the Lie algebra $W[e^{\pm xy}, x, y]$) and for $c_1, c_5 \in \mathbb{F}^\bullet$, if we define the $\mathbb{F}$-maps θ_{I_1,c_1,c_5}, θ_{I_2,c_1,c_5}, θ_{II_1,c_1,c_5}, θ_{II_2,c_1,c_5}, θ_{III_1,c_1,c_5}, θ_{III_2,c_1,c_5}, θ_{IV_1,c_1,c_5}, and θ_{IV_2,c_1,c_5} of the algebra $NW[e^{\pm xy}, x, y]$ (resp. the Lie algebra $W[e^{\pm xy}, x, y]$) as follows:

$$
\begin{aligned}
&\theta_{I_1,c_1,c_5}(e^{axy}x^iy^p\partial_1) = c_1^{1-a-i+p}c_5^a e^{axy}x^iy^p\partial_1\\
&\theta_{I_1,c_1,c_5}(e^{axy}x^iy^p\partial_2) = c_1^{-1-a-i+p}c_5^a e^{axy}x^iy^p\partial_2\\
&\theta_{I_2,c_1,c_5}(e^{axy}x^iy^p\partial_1) = c_1^{-1-a+i-p}c_5^a e^{axy}x^iy^p\partial_1\\
&\theta_{I_2,c_1,c_5}(e^{axy}x^iy^p\partial_2) = c_1^{1-a+i-p}c_5^a e^{axy}x^iy^p\partial_2\\
&\theta_{II_1,c_1,c_5}(e^{axy}x^iy^p\partial_1) = (-1)^{-p}c_1^{1-a-i+p}c_5^a e^{-axy}x^iy^p\partial_1\\
&\theta_{II_1,c_1,c_5}(e^{axy}x^iy^p\partial_2) = (-1)^{1-p}c_1^{-1-a-i+p}c_5^a e^{-axy}x^iy^p\partial_2\\
&\theta_{II_2,c_1,c_5}(e^{axy}x^iy^p\partial_1) = (-1)^{1-p}c_1^{-1-a+i-p}c_5^a e^{-axy}x^iy^p\partial_1\\
&\theta_{II_2,c_1,c_5}(e^{axy}x^iy^p\partial_2) = (-1)^{-p}c_1^{1-a+i-p}c_5^a e^{-axy}x^iy^p\partial_2\\
&\theta_{III_1,c_1,c_5}(e^{axy}x^iy^p\partial_1) = c_3^{1-a-i+p}c_9^a e^{axy}x^py^i\partial_2\\
&\theta_{III_1,c_1,c_5}(e^{axy}x^iy^p\partial_2) = c_3^{-1-a-i+p}c_9^a e^{axy}x^iy^p\partial_1\\
&\theta_{III_2,c_1,c_5}(e^{axy}x^iy^p\partial_1) = c_3^{-1-a+i-p}c_9^a e^{axy}x^py^i\partial_2\\
&\theta_{III_2,c_1,c_5}(e^{axy}x^iy^p\partial_2) = c_3^{1-a+i-p}c_9^a e^{axy}x^iy^p\partial_1\\
&\theta_{IV_1,c_1,c_5}(e^{axy}x^iy^p\partial_1) = (-1)^{p}c_3^{1-a-i+p}c_{10}^a e^{-axy}x^py^i\partial_2\\
&\theta_{IV_1,c_1,c_5}(e^{axy}x^iy^p\partial_2) = (-1)^{1-p}c_3^{-1-a-i+p}c_{10}^a e^{-axy}x^iy^p\partial_1\\
&\theta_{IV_2,c_1,c_5}(e^{axy}x^iy^p\partial_1) = (-1)^{1-p}c_3^{1-a+i-p}c_{10}^a e^{-axy}x^py^i\partial_2\\
&\theta_{IV_2,c_1,c_5}(e^{axy}x^iy^p\partial_2) = (-1)^{p}c_3^{-1-a+i-p}c_{10}^a e^{-axy}x^iy^p\partial_1
\end{aligned}
\tag{1}
$$

then they can be linearly extended to (resp. Lie) algebra automorphisms of the algebra $NW[e^{\pm xy}, x, y]$ (resp. the Lie algebra $W[e^{\pm xy}, x, y]$). □

Lemma 2 *For any automorphism θ in $Aut(NW[e^{\pm xy}, x, y])$, if $\theta(\partial_1) = c\partial_1$, then θ is one of the automorphisms θ_{I_1,c_1,c_5}, θ_{I_2,c_1,c_5}, θ_{II_1,c_1,c_5}, and θ_{II_2,c_1,c_5} as shown in the above Notes and if $\theta(\partial_1) = c'\partial_1$, then θ is one of the automorphisms θ_{III_1,c_1,c_5}, θ_{III_2,c_1,c_5}, θ_{IV_1,c_1,c_5}, and θ_{IV_2,c_1,c_5} as shown in Notes where $c, c' \in \mathbb{F}^\bullet$.*

Proof. Let θ be the automorphism of the algebra $NW[e^{\pm xy}, x, y]$ in the lemma. By Lemma 1, we have two cases, Case I: $\theta(\partial_1) = c_1\partial_1$ and $\theta(\partial_2) = c_2\partial_2$, Case II: $\theta(\partial_1) = c_3\partial_2$ and $\theta(\partial_2) = c_4\partial_1$ hold for $c_1, c_2, c_4, c_5 \in \mathbb{F}^\bullet$.

Case I. Let us assume that $\theta(\partial_1) = c_1\partial_1$ and $\theta(\partial_2) = c_2\partial_2$ hold. This implies that $\theta(x\partial_1) = (x + d_1)\partial_1$ and $\theta(y\partial_2) = (y + d_2)\partial_2$ hold for $d_1, d_2 \in \mathbb{F}$. By induction on i and p of $x^iy^p\partial_u$, $1 \le u \le 2$, we are able to prove that

$$
\begin{aligned}
&\theta(x^iy^p\partial_1) = c_1^{1-i}c_2^{-p}(x + d_1)^i(y + d_2)^p\partial_1\\
&\theta(x^iy^p\partial_2) = c_1^{-i}c_2^{1-p}(x + d_1)^i(y + d_2)^p\partial_2
\end{aligned}
\tag{2}
$$

By $\theta(e^{xy}\partial_1 * e^{-xy}\partial_1) = -(y+d_2)\partial_1$, we have that

$$\theta(e^{xy}\partial_1) * \theta(e^{-xy}\partial_1) = -(y+d_2)\partial_1 \tag{3}$$

Let us assume that $\theta(e^{xy}\partial_1)$ is in the subalgebra N_0 of $NW[e^{\pm xy}, x, y]$ spanned by $\{x^j y^q \partial_u | j, q \in \mathbb{N}, 1 \leq u \leq 2\}$. By (3), we have that $\theta(e^{xy}\partial_1)$ $\theta(e^{-xy}\partial_1)$ are in N_0. Since the algebra $NW[e^{\pm xy}, x, y]$ is generated by $\theta(e^{xy}\partial_u)$, $\theta(e^{-xy}\partial_u)$, $1 \leq u \leq 2$, and N_0, we have that $\theta(NW[e^{\pm xy}, x, y]) \subset N_0$. This contradiction shows that $\theta(e^{xy}\partial_1)$ is not in the subalgebra N_0. Since $x\partial_1$ is a right identity of $e^{xy}\partial_1$, $\theta(e^{xy}\partial_1)$ does not have a term which contains ∂_2. $\theta(e^{xy}\partial_1)$ can be written as follows:

$$\theta(e^{xy}\partial_1) = c_{b,k,r} e^{bxy} x^k y^r \partial_1 + \#_1 \tag{4}$$

where $\#_1$ is the sum of the remaining terms of $\theta(e^{xy}\partial_1)$. By (3) and (4), $\theta(e^{-xy}\partial_1) = c_{-b,k,r} e^{-bxy}\partial_1 + \#_2$ and $r = 0$. Since the algebra $NW[e^{\pm xy}, x, y]$ is generated by $\theta(e^{xy}\partial_u)$, $\theta(e^{-xy}\partial_u)$, $1 \leq u \leq 2$, and N_0, we have that $b = \pm 1$. Otherwise θ cannot be surjective. We have the following two subcases, Subcase I: $b = 1$ and Subcase II: $b = -1$.

Subcase I. Let us assume that $b = 1$ holds, i.e., $\theta(e^{xy}\partial_1) = c_5 e^{xy}\partial_1$ and $\theta(e^{-xy}\partial_1) = c_6 e^{-xy}\partial_1$ for $c_5, c_6 \in \mathbb{F}^\bullet$. By $\theta(\partial_2 * e^{xy}\partial_1) = \theta(e^{xy}x\partial_1)$, we have that $\theta(e^{xy}x\partial_1) = c_2 c_5 e^{xy} x\partial_1$. Similarly we can prove that $\theta(e^{xy}x^2\partial_1) = c_2^2 c_5 e^{xy} x^2 \partial_1$. By $\theta(x\partial_2 * e^{xy}\partial_1) = \theta(e^{xy}x^2\partial_1)$ and (2), we have that $d_1 = 0$ and $c_2 = c_1^{-1}$. Similarly we can also prove that $d_2 = 0$. This implies that by (2),

$$\begin{aligned} \theta(x^i y^p \partial_1) &= c_1^{1-i+p} x^i y^p \partial_1 \\ \theta(x^i y^p \partial_2) &= c_1^{-1-i+p} x^i y^p \partial_2 \end{aligned} \tag{5}$$

$\theta(e^{xy}x\partial_1) = c_1^{-1} c_5 e^{xy} x\partial_1$, and $\theta(e^{xy}x^2\partial_1) = c_1^{-2} c_5 e^{xy} x^2 \partial_1$. Since $\theta(e^{xy}\partial_1 * x\partial_2) = \theta(e^{xy}\partial_2)$, we have that $\theta(e^{xy}\partial_2) = c_1^{-2} c_5 e^{xy}\partial_2$. We can prove that c_6 is c_5^{-1}. By induction on a, i, and p of $e^{axy}x^i y^p \partial_1$, and by (5), we can prove that $\theta(e^{axy}x^i y^p \partial_1) = c_1^{1-a-i+p} c_5^a e^{axy} x^i y^p \partial_1$ holds where $a \in \mathbb{Z}$, $i, p \in \mathbb{N}$. Similarly we can alsoprove that $\theta(e^{axy}x^i y^p \partial_2) = c_1^{-1-a-i+p} c_5^a e^{axy} x^i y^p \partial_2$ holds. This implies that θ is the automorphism θ_{I_1,c_1,c_5} as shown in Notes. Symmetrically we can prove that θ can also be the automorphism θ_{I_2,c_1,c_5} as shown in Notes. □

Subcase II. Let us assume that $b = -1$ holds, i.e., $\theta(e^{xy}\partial_1) = c_7 e^{-xy}\partial_1$ and $\theta(e^{-xy}\partial_1) = c_8 e^{xy}\partial_1$ for $c_7, c_8 \in \mathbb{F}^\bullet$. By $\theta(\partial_2 * e^{xy}\partial_1) = \theta(e^{xy}x\partial_1)$, we have that $\theta(e^{xy}x\partial_1) = -c_2 c_7 e^{-xy} x\partial_1$. Similarly we are able to prove that $\theta(e^{xy}x^2\partial_1) = c_2^2 c_7 e^{-xy} x^2 \partial_1$. We can also prove that $c_2 = -c_1^{-1}$ and $d_1 = d_2 = 0$

in (2). Similarly by (2), we can prove that

$$\begin{aligned} \theta(x^i y^p \partial_1) &= (-1)^{-p} c_1^{1-i+p} x^i y^p \partial_1 \\ \theta(x^i y^p \partial_2) &= (-1)^{1-p} c_1^{-1-i+p} x^i y^p \partial_2 \end{aligned} \tag{6}$$

$\theta(e^{xy} x\partial_1) = c_1^{-1} c_7 e^{-xy} \partial_1$, and $\theta(e^{xy} x^2 \partial_1) = c_1^{-2} c_7 e^{-xy} x^2 \partial_1$. By $\theta(e^{xy}\partial_1 * x\partial_2) = \theta(e^{xy}\partial_2)$, we have that $\theta(e^{xy}\partial_2) = -c_1^{-2} c_7 e^{-xy}\partial_2$. By $\theta(e^{xy}\partial_2 * e^{xy}\partial_1) = \theta(e^{2xy} x\partial_1)$, we have that $\theta(e^{2xy} x\partial_1) = c_1^{-2} c_7^2 e^{-2xy} x\partial_1$. We can prove that $c_8 = c_7^{-1}$, i.e., $\theta(e^{-xy}\partial_1) = c_7^{-1} e^{xy}\partial_1$. By induction on a, i, and p of $e^{axy} x^i y^p \partial_1$ and by (6), we can prove that $\theta(e^{axy} x^i y^p \partial_1) = (-1)^{-p} c_1^{1-a-i+p} c_7^a e^{-axy} x^i y^p \partial_1$ where $a \in \mathbb{Z}$, $i, p \in \mathbb{N}$. Similarly we can also prove that $\theta(e^{axy} x^i y^p \partial_2) =$ $(-1)^{1-p} c_1^{-1-a-i+p} c_5^a e^{-axy} x^i y^p \partial_2$. This implies that θ is the automorphism θ_{II_1,c_1,c_5} as shown in Notes. Symmetrically we can prove that θ can also be the automorphism θ_{II_2,c_1,c_5} as shown in Notes. □

Case II. Let us assume that $\theta(\partial_1) = c_3\partial_2$ and $\theta(\partial_2) = c_4\partial_1$ hold. Since $x\partial_1$ is not a right identity of the algebra, we can prove that $\theta(x\partial_1) = (y + d_3)\partial_2$ and $\theta(y\partial_2) = (x + d_4)\partial_1$ for $d_3, d_4 \in \mathbb{F}$. By induction on i and p of $x^i y^p \partial_u$, $1 \le u \le 2$, we can also prove that

$$\begin{aligned} \theta(x^i y^p \partial_1) &= c_3^{1-i} c_4^{-p} (x + d_4)^p (y + d_3)^i \partial_2 \\ \theta(x^i y^p \partial_2) &= c_3^{-i} c_4^{1-p} (x + d_4)^p (y + d_3)^i \partial_1 \end{aligned} \tag{7}$$

Similarly to Case I, we are able to prove that there are two subcases, Subcase III: $\theta(e^{xy}\partial_1) = c_9 e^{xy}\partial_2$ and Subcase IV: $\theta(e^{xy}\partial_1) = c_{10} e^{-xy}\partial_2$ for $c_9, c_{10} \in \mathbb{F}^\bullet$.

Subcase III. Let us assume that $b = 1$ holds, i.e., $\theta(e^{xy}\partial_1) = c_9 e^{xy}\partial_2$ and $\theta(e^{-xy}\partial_1) = c_{11} e^{-xy}\partial_2$ for $c_9, c_{11} \in \mathbb{F}^\bullet$. By $\theta(\partial_2 * e^{xy}\partial_1) = \theta(e^{xy} x\partial_1)$, we have that $\theta(e^{xy} x\partial_1) = c_4 c_9 e^{xy} y\partial_2$. Similarly we can prove that $\theta(e^{xy} x^2\partial_1) = c_4^2 c_9 e^{xy} y^2 \partial_2$. By $\theta(x\partial_1 * e^{xy}\partial_1) = \theta(e^{xy} x^2 \partial_1)$ and (2), we have that $d_3 = 0$ and $c_4 = c_3^{-1}$. Similarly we can prove that $d_4 = 0$. This implies that by (7), we can also prove

$$\begin{aligned} \theta(x^i y^p \partial_1) &= c_3^{1-i+p} x^p y^i \partial_2 \\ \theta(x^i y^p \partial_2) &= c_3^{-1-i+p} x^p y^i \partial_1 \end{aligned} \tag{8}$$

$\theta(e^{xy} x\partial_1) = c_3^{-1} c_9 e^{xy} y\partial_2$, and $\theta(e^{xy} x^2 \partial_1) = c_3^{-2} c_9 e^{xy} y^2 \partial_2$. We can also prove that $c_{11} = c_3^2 c_9^{-1}$, i.e., $\theta(e^{-xy}\partial_1) = c_3^2 c_9^{-1} e^{-xy}\partial_2$. By induction on a, i, and p of $e^{axy} x^i y^p \partial_1$, and by (5), we can prove that $\theta(e^{axy} x^i y^p \partial_1) = c_3^{1-a-i+p} c_9^a e^{axy} x^p y^i \partial_2$ where $a \in \mathbb{Z}$, $i, p \in \mathbb{N}$. Similarly we can prove that $\theta(e^{axy} x^i y^p \partial_2) =$ $c_3^{-1-a-i+p} c_9^a e^{axy} x^i y^p \partial_1$. This implies that θ is the automorphism θ_{III_1,c_1,c_5} as shown in Notes by switching appropriate coefficients. Similarly we can prove that θ can be the automorphism θ_{III_2,c_1,c_5} as shown in Notes

Subcase IV. Let us assume that $b = -1$ holds, i.e., $\theta(e^{xy}\partial_1) = c_{10}e^{-xy}\partial_2$ and $\theta(e^{-xy}\partial_1) = c_{12}e^{xy}\partial_2$ for $c_{10}, c_{12} \in \mathbb{F}^\bullet$. By $\theta(\partial_2 * e^{xy}\partial_1) = \theta(e^{xy}x\partial_1)$, we have that $\theta(e^{xy}x\partial_1) = -c_4c_{10}e^{-xy}y\partial_1$. Similarly we can also prove that $\theta(e^{xy}x^2\partial_1) = c_4^2c_{10}e^{-xy}y^2\partial_1$. By $\theta(x\partial_2 * e^{xy}\partial_1) = \theta(e^{xy}x^2\partial_1)$ and (2), we have that $d_3 = 0$ and $c_4 = -c_3^{-1}$. Similarly we can prove that $d_4 = 0$. This implies that by (7), we can also prove

$$\begin{aligned}\theta(x^iy^p\partial_1) &= (-1)^p c_3^{1-i+p}x^py^i\partial_2 \\ \theta(x^iy^p\partial_2) &= (-1)^{1-p} c_3^{-1-i+p}x^py^i\partial_1\end{aligned} \tag{9}$$

$\theta(e^{xy}x\partial_1) = c_3^{-1}c_{10}e^{xy}y\partial_2$, and $\theta(e^{xy}x^2\partial_1) = c_3^{-2}c_{10}e^{xy}y^2\partial_2$. We can also prove that $c_{11} = c_3^2c_{10}^{-1}$, i.e., $\theta(e^{-xy}\partial_1) = c_3^2c_9^{-1}e^{-xy}\partial_2$. By induction on a, i, and p of $e^{axy}x^iy^p\partial_1$, and by (5), we can prove that $\theta(e^{axy}x^iy^p\partial_1) =$
$(-1)^pc_3^{1-a-i+p}c_{10}^ae^{-axy}x^py^i\partial_2$ where $a \in \mathbb{Z}, i, p \in \mathbb{N}$. Similarly we can prove that $\theta(e^{axy}x^iy^p\partial_2) = (-1)^{1-p}c_3^{-1-a-i+p}c_{10}^ae^{-axy}x^iy^p\partial_1$. This implies that θ is the automorphism θ_{IV_1,c_1,c_5} as shown in Notes by switching appropriate coefficients. Similarly we can prove that θ can be the automorphism θ_{IV_2,c_1,c_5} as shown in Notes

Therefore by Case I and Case II, we have proven the lemma. □

Theorem 3 *The automorphism group $Aut(NW[e^{\pm xy}, x, y])$ of the algebra $NW[e^{\pm xy}, x, y]$ is generated by the automorphisms θ_{I_1,c_1,c_5}, θ_{I_2,c_1,c_5}, θ_{II_1,c_1,c_5}, θ_{II_2,c_1,c_5}, θ_{III_1,c_1,c_5}, θ_{III_2,c_1,c_5}, θ_{IV_1,c_1,c_5}, and θ_{IV_2,c_1,c_5} as shown in Notes.*

Proof. The proof of the theorem is straightforward by Lemma 1 and Lemma 2. □

Remark 1. Note an algebra automorphism of the algebra $NW[e^{\pm xy}, x, y]$ is a Lie automorphism of the Lie algebra $W[e^{\pm xy}, x, y]$. This implies that the automorphism group $Aut(NW[e^{\pm xy}, x, y])$ of the algebra $NW[e^{\pm xy}, x, y]$ is a subgroup of the Lie automorphism group $Aut_{Lie}(W[e^{\pm xy}, x, y])$ of the Lie algebra $W[e^{\pm xy}, x, y]$. Because of Theorem 3, it will be an interesting problem to check whether the groups $Aut(NW[e^{\pm xy}, x, y])$ and $Aut_{Lie}(W[e^{\pm xy}, x, y])$ are isomorphic or not. □

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