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On Relation $\mathcal{B}_\gamma$ in le-Γ-Semigroups

Jani Dine

Department of Mathematics, Faculty of Natural Sciences
University of Gjirokastra, Gjirokastra, Albania
jani_dine@yahoo.com

Abstract

In this paper we introduce and study the relation $\mathcal{B}_\gamma$ in $le-\Gamma$-*semigroups* that mimics the relation $\mathcal{B}$ in le-semigroups [3]. This relation in general turns out to have better properties than the relation $\mathcal{H}_\gamma$ studied in [2]. We give several properties that hold in every $\mathcal{B}_\gamma$-class of an $le-\Gamma$-semigroup and especially in every $\mathcal{B}_\gamma$-class satisfying the Green's condition. In particular, the γ-regularity and γ-intra-regularity of an $\mathcal{B}_\gamma$-class is studied. We also provide various conditions that ensure that an $\mathcal{B}_\gamma$-class of a $le-\Gamma$-semigroup M forms a subsemigroup of $M_\gamma = (M, \circ)$.

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1 Introduction and preliminaries

In [2] it is studied the relation $\mathcal{H}_\gamma$ investigating several properties that hold in every $\mathcal{H}_\gamma$-classes of an le-Γ-semigroup satisfying the so-called Green's condition and a necessary and sufficient condition when an $\mathcal{H}_\gamma$-class H of an $le-\Gamma$-semigroup M is a subgroup of $M_\gamma = (M, \circ)$ is provided. In [2] there are also provided several conditions that ensure that an $\mathcal{H}_\gamma$-class forms a subsemigroup of M_γ extending and generalizing those for le-semigroups studied in [4].

The aim of the present paper is to introduce and study the relation $\mathcal{B}_\gamma$ which turns out to be finer that $\mathcal{H}_\gamma$. This means that each $\mathcal{H}_\gamma$-class can be partitioned into $\mathcal{B}_\gamma$-classes. We investigate several properties that hold in every $\mathcal{B}_\gamma$-classes and prove also several results which shows that the relation

$\mathcal{B}_\gamma$ may be a better candidate than $\mathcal{H}_\gamma$ for developing the structure theory for $le-\Gamma$-semigroups. We show that the Green's condition is sufficient for a $\mathcal{B}_\gamma$-class to be γ-regular and γ-intra-regular. In the last section we concentrate on finding conditions ensuring that an $\mathcal{B}_\gamma$-class of an $le-\Gamma$-semigroup M forms a subsemigroup in $M_\gamma = (M, \circ)$. A necessary and sufficient condition is provided (Theorem 3.5); its corollaries as well as Theorem 3.10 give various sufficient conditions.

We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

In 1986 Sen and Saha [5] defined Γ-semigroup as a generalization of semigroup as follows:

Definition 1.1 *Let $M = \{a, b, c, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two nonempty sets. Then M is called a Γ-semigroup if there exist mapping $M \times \Gamma \times M \to M$, written as $(a, \gamma, b) \longmapsto a\gamma b$ satisfying the following identity*

$$(a\alpha b)\beta c = a\alpha(b\beta c) \text{ for all } a, b, c \in M \text{ and for all } \alpha, \beta \in \Gamma.$$

An element a of a Γ-semigroup M is called an *γ-idempotent* if exists $\gamma \in \Gamma$, $a\gamma a = a$.

Let M be a Γ-semigroup and γ be a fixed element of Γ. In [5] is defined $a \circ b$ in M by $a \circ b = a\gamma b, \forall a, b \in M$ and is shown that $(M, \circ)$ is a semigroup and this semigroup is denoted by M_γ. Also, it is shown that if M_γ is a group for some $\gamma \in \Gamma$, then M_γ is a group for all $\gamma \in \Gamma$. A Γ-semigroup M is called a Γ-*group* if M_γ is a group for some (hence for all) $\gamma \in \Gamma$ [5].

Definition 1.2 *A po-Γ-semigroup is an ordered set M at the same time Γ-semigroup such that for all $c \in M$ and for all $\gamma \in \Gamma$*

$$a \leq b \Rightarrow a\gamma c \leq b\gamma c, c\gamma a \leq c\gamma b$$

A *poe*-Γ-semigroup is a $po-\Gamma$-semigroup M with a greatest element "e" (i.e., $\forall a \in M, e \geq a$).

In a $po-\Gamma$-semigroup M, for any $\gamma \in \Gamma$, the element a is called an *γ-right* (resp. *γ-left*) *ideal element* if for all $b \in M$, $a\gamma b \leq a$ (resp. $b\gamma a \leq a$). And a is called an *γ-ideal element* if it is both an γ- right and γ-left ideal element. In a $poe-\Gamma$-semigroup M, for any $\gamma \in \Gamma$, a is called an *γ-right* (resp. *γ-left*) *ideal element* if $a\gamma e \leq a$ (resp. $e\gamma a \leq a$).

For $A \subseteq M$ we denote

$$(A] = \{t \in M | t \leq a, \text{ for some } a \in A\}$$

An element a of a $poe-\Gamma$-semigroup is called an γ-*quasi-ideal element* if $e\gamma a \wedge a\gamma e$ exists and $a\gamma e \wedge e\gamma a \leq a$. The γ-*zero* of a $poe-\Gamma$-semigroup M is an element of M denoted by 0_γ such that for every $a \in M$, $e \neq 0_\gamma \leq a$ and $0_\gamma \gamma a = a\gamma 0_\gamma = 0_\gamma$. Let M be a $poe-\Gamma$-semigroup with 0_γ. An γ-quasi-ideal element a of M is called *minimal* if $a \neq 0_\gamma$ and there exists no γ-quasi-ideal element t of M such that $0_\gamma < t < a$. We say that $a \in M$ is an γ-*bi-ideal element* of M if and only if $a\gamma e\gamma a \leq a$.

Definition 1.3 *Let M be a semilattice under $\vee$ with a greatest element e and at the same time a $po-\Gamma$-semigroup such that for all $a, b, c \in M$ and for all $\gamma \in \Gamma$*

$$a\gamma(b \vee c) = a\gamma b \vee a\gamma c$$

and

$$(a \vee b)\gamma c = a\gamma c \vee b\gamma c.$$

Then M is called a $\vee e-\Gamma$-semigroup.

A $\vee e-\Gamma$-semigroup which is also a lattice is called an le-Γ-*semigroup*.

Throught this paper M will stand for an $le-\Gamma$-semigroup. The usual order relation $\leq$ on M is defined in the following way

$$a \leq b \Leftrightarrow a \vee b = b.$$

Then we can show that for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Example 1.4 [1] *Let $(X, \leq)$ and $(Y, \leq)$ be two finite chains. Let M be the set of all isotone mappings from X into Y and Γ be the set of all isotone mappings from Y into X. Let , $f, g \in M$ and $\alpha \in \Gamma$. We define $f\alpha g$ to denote the usual mapping composition of f, α and g. Then M is a Γ-semigroup. For $f, g \in M$, the mappings $f \vee g$ and $f \wedge g$ are defined by letting, for each $a \in X$*

$$(f \vee g)(a) = max\{f(a), g(a)\},\ (f \wedge g)(a) = min\{f(a), g(a)\}$$

(the maximum and minimum are considered with respect to the order $\leq$ in X and Y). The greatest element e is the mapping that sends every $a \in X$ to the greatest element of finite chains $(Y, \leq)$. Then M is an $le-\Gamma$-semigroup.

Example 1.5 [1] *Let M be a $po-\Gamma$-semigroup. Let M_1 be the set of all ideals of M. Then $(M_1, \subseteq, \cap, \cup)$ is an $le-\Gamma$-semigroup.*

Example 1.6 [1] *Let M be a $po-\Gamma$-semigroup. Let $M_1 = P(M)$ be the set of all subsets of M and $\Gamma_1 = P(\Gamma)$ the set of all subsets of Γ. Then M_1 is a $po-\Gamma_1$-semigroup if*

$$A\Lambda B = \begin{cases} (A](\Lambda](B] = (A\Lambda B] & \text{if } A, B \in M_1 \backslash \{\emptyset\}, \Lambda \in \Gamma_1 \backslash \{\emptyset\} \\ \emptyset & \text{if } A = \emptyset \text{ or } B = \emptyset \end{cases}$$

Then $(M_1, \subseteq, \cap, \cup)$ is an $le-\Gamma_1$-semigroup.

In [2] for any $\gamma \in \Gamma$ two mappings r_γ and l_γ are defined by for any $x \in M$ as follows:

$$r_\gamma : M \to M, r_\gamma(x) = x\gamma e \vee x,$$
$$l_\gamma : M \to M, l_\gamma(x) = e\gamma x \vee x$$

In an arbitrary $le-\Gamma$-semigroup M, the Green's relations are defined in [2] as follows:

$\mathcal{L}_\gamma = \{(x, y) \in M^2 | e\gamma x \vee x = e\gamma y \vee y\}$,
or
$\mathcal{L}_\gamma = \{(x, y) \in M^2 | l_\gamma(x) = l_\gamma(y)\}$,
$\mathcal{R}_\gamma = \{(x, y) \in M^2 | x\gamma e \vee x = y\gamma e \vee y\}$,
or
$\mathcal{R}_\gamma = \{(x, y) \in M | r_\gamma(x) = r_\gamma(y)\}$,
$\mathcal{H}_\gamma = \mathcal{L}_\gamma \cap \mathcal{R}_\gamma$.

It is clear that an element $a \in M$ is an γ-left [resp. γ-right] ideal element if $l_\gamma(x) = x$ [resp. $r_\gamma(x) = x$].

We define now in a $\vee e-\Gamma$-semigroup M for all $a \in M$ and for any $\gamma \in \Gamma$ the mappings q_γ and b_γ as follows:

$$b_\gamma : M \to M, b_\gamma(x) = x \vee x\gamma e\gamma x$$
$$q_\gamma : M \to M, q_\gamma(x) = x \vee (e\gamma x \wedge x\gamma e)$$

One can easily verify that for every $x \in M$, the elements $l_\gamma(x)$, $r_\gamma(x)$, $q_\gamma(x)$, $b_\gamma(x)$, are respectively the least γ-left, γ-right, γ-quasi and γ-bi-ideal elements above the x.

A $\mathcal{H}_\gamma$-class H of Γ-semigroup M satisfy Green's condition if there exist elements x and y of H such that $x\gamma y \in H$[2].

We define now the following equivalence relation $\mathcal{B}_\gamma$ in $le-\Gamma$-semigroup M:

$\mathcal{B}_\gamma = \{(x, y) \in M^2 | x \vee x\gamma e\gamma x = y \vee y\gamma e\gamma y\}$,
or
$\mathcal{B}_\gamma = \{(x, y) \in M^2 | b_\gamma(x) = b_\gamma(y)\}$,

Lemma 1.7 *Let M be an $le-\Gamma$-semigroup. Then $\mathcal{B}_\gamma \subseteq \mathcal{H}_\gamma$.*

Proof. Let $(x,y) \in \mathcal{B}_\gamma$. Then from the definition of $\mathcal{B}_\gamma$, since $b_\gamma(x) = b_\gamma(y)$,

$$x \vee x\gamma e\gamma x = y \vee y\gamma e\gamma y,$$

Multiplying this equality through on the right by e, we have that $x\gamma e \vee x\gamma e\gamma x\gamma e = y\gamma e \vee y\gamma e\gamma y\gamma e$ and then combining both the equalities, we get

$$x \vee x\gamma e\gamma x \vee x\gamma e \vee x\gamma e\gamma x\gamma e = y \vee y\gamma e\gamma y \vee y\gamma e \vee y\gamma e\gamma y\gamma e \qquad (1)$$

Since e is the greatest element of M, we have $e \geq e\gamma x$ and $e \geq e\gamma x\gamma e$. Thus we obtain $x\gamma e \geq x\gamma e\gamma x$ and $x\gamma e \geq x\gamma e\gamma x\gamma e$. Therefore the left-hand side part of (1) reduces to $x \vee x\gamma e$; similarly, the right hand side part of (1) reduces to $y \vee y\gamma e$. Thus, (1) is equivalent to $x \vee x\gamma e = y \vee y\gamma e$. In a symmetric way one can prove that $x \vee e\gamma x = y \vee e\gamma y$. This complete the proof.

The following example shows that in general the inclusion of Lemma 1.7 is strict.

Example 1.8 *In Example 1.6, if we take as $M = Z_8$, $\Gamma = \{\overline{1}\}$ where for all $a, b \in M$ and $\gamma \in \Gamma$, $\overline{a}\ \overline{\gamma}\ \overline{b} = \overline{a\gamma b}$, then $M_1 = P(M)$ is an $le-\Gamma$-semigroup with the biggest element $e = Z_8$. We observe that:*

$$\begin{aligned}\{\overline{2}\} \vee \{\overline{2}\}\overline{1}e\overline{1}\{\overline{2}\} &= \{\overline{2}\} \cup \{\overline{2}\}\overline{1}Z_8\overline{1}\{\overline{2}\} = \{\overline{0}, \overline{2}, \overline{4}\},\\ \{\overline{6}\} \vee \{\overline{6}\}\overline{1}e\overline{1}\{\overline{6}\} &= \{\overline{6}\} \cup \{\overline{6}\}\overline{1}Z_8\overline{1}\{\overline{6}\} = \{\overline{0}, \overline{4}, \overline{6}\}\end{aligned}$$

which shows that $\{\overline{2}\}$ is not $\mathcal{B}_{\overline{1}}$-related to $\{\overline{6}\}$. On the other side, we have

$$\begin{aligned}\{\overline{2}\} \vee \{\overline{2}\}\overline{1}e &= \{\overline{2}\} \cup \{\overline{2}\}\overline{1}Z_8 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\},\\ \{\overline{6}\} \vee \{\overline{6}\}\overline{1}e &= \{\overline{6}\} \cup \{\overline{6}\}\overline{1}Z_8 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}\end{aligned}$$

which shows that $\{\overline{2}\}\mathcal{R}_{\overline{1}}\{\overline{6}\}$. Since $M_1 = P(Z_8)$ is commutative, the relation $\mathcal{R}_{\overline{1}}$ and $\mathcal{H}_{\overline{1}}$ coincide, whence $\{\overline{2}\}\mathcal{H}_{\overline{1}}\{\overline{6}\}$ which implies that $\mathcal{B}_{\overline{1}} \neq \mathcal{H}_{\overline{1}}$.

Remark 1. Concerning the above lemma we notice that : *In distributive γ-regular $le-\Gamma$-semigroups, $\mathcal{B}_\gamma = \mathcal{H}_\gamma$.* In fact, let M be a distributive γ-regular $le-\Gamma$-semigroup and $(x,y) \in \mathcal{H}_\gamma$. Since $(x,y) \in \mathcal{R}_\gamma$ and $(x,y) \in \mathcal{L}_\gamma$, we get $x\gamma e \vee x = y\gamma e \vee y$ and $e\gamma x \vee x = e\gamma y \vee y$. Then,

$$(x\gamma e \vee x) \wedge (e\gamma x \vee x) = (y\gamma e \vee y) \wedge (e\gamma y \vee y).$$

Since M is distributive, we have

$$x \vee (x\gamma e \wedge e\gamma x) = y \vee (y\gamma e \wedge e\gamma y).$$

Thus,

$$x \leq y \vee (y\gamma e \wedge e\gamma y) \text{ and } y \leq x \vee (x\gamma e \wedge e\gamma x).$$

Since M is γ-regular and since $y\gamma e$ is r_γ-closed and $e\gamma b$ is l_γ-closed, we have [1, Theorem 2.2 (1)$\Leftrightarrow$ (2)], $y\gamma e \wedge e\gamma y = y\gamma e\gamma e\gamma y$ and $x\gamma e \wedge e\gamma x = x\gamma e\gamma e\gamma x$. Then,

$$\begin{aligned} x \vee x\gamma e\gamma x &\leq (y \vee y\gamma e\gamma y) \vee (y \vee y\gamma e\gamma y)\gamma e\gamma(y \vee y\gamma e\gamma y) \\ &= y \vee y\gamma e\gamma y \vee y\gamma e\gamma y \vee y\gamma e\gamma y\gamma e\gamma y \vee y\gamma e\gamma y\gamma e\gamma y \vee y\gamma e\gamma y\gamma e\gamma y\gamma e\gamma y \\ &\leq y \vee y\gamma e\gamma y. \end{aligned}$$

Similarly, we have $y \vee y\gamma e\gamma y \leq x \vee x\gamma e\gamma x$. Thus, we have

$$x \vee x\gamma e\gamma x = y \vee y\gamma e\gamma y$$

i.e, $(x, y) \in \mathcal{B}_\gamma$.

Lemma 1.9 *Let M be an $le - \Gamma$-semigroup. For each $x \in M$ and $\gamma \in \Gamma$, we have $b_\gamma(b_\gamma(x)) = b_\gamma(x)$.*

Proof. In fact,

$$\begin{aligned} b_\gamma(b_\gamma(x)) &= b_\gamma(x \vee x\gamma e\gamma x) = (x \vee x\gamma e\gamma x) \vee (x \vee x\gamma e\gamma x)\gamma e\gamma(x \vee x\gamma e\gamma x) \\ &= x \vee x\gamma e\gamma x \vee x\gamma e\gamma x\gamma e\gamma x \vee x\gamma e\gamma x\gamma e\gamma x\gamma e\gamma x = x \vee x\gamma e\gamma x = b_\gamma(x) \end{aligned}$$

as both $x\gamma e\gamma x\gamma e\gamma x$ and $x\gamma e\gamma x\gamma e\gamma x\gamma e\gamma x$ are not greater than $x\gamma e\gamma x$.

Lemma 1.10 *Let M be an $le - \Gamma$-semigroup and $x, y \in M$. If $(x, y) \in \mathcal{H}_\gamma$, then $x\gamma e = y\gamma e, e\gamma x = e\gamma y$ and $x\gamma e\gamma x = y\gamma e\gamma y$.*

Proof. Let $x, y \in M$ such that $(x, y) \in \mathcal{H}_\gamma$. Then we have $(x, y) \in \mathcal{R}_\gamma$ and $(x, y) \in \mathcal{L}_\gamma$. By Lemma 1.5[2] we have $x\gamma e = y\gamma e$ and $e\gamma x = e\gamma y$. From the first equality we get $x\gamma e\gamma y = y\gamma e\gamma y$. Replacing $e\gamma y$ by $e\gamma x$ we obtain the result.

Lemma 1.11 *Let M be an $le - \Gamma$-semigroup. Each $\mathcal{B}_\gamma$-class B of M contains a unique γ-bi-ideal element which is the greatest element of the class.*

Proof. For every element $x \in B$, by Lemma 1.9 and the definition of relation $\mathcal{B}_\gamma$, we have $b_\gamma(x) \in B$. If z is an γ-bi-ideal element belonging to B, then $b_\gamma(x) = b_\gamma(z) = z$, which shows that $b_\gamma(x)$ is the only γ-bi-ideal element of the class. Since $x \leq b_\gamma(x)$, we see that $b_\gamma(x)$ is the greatest element of B.

Lemma 1.11 implies that for each $a \in M$, the γ-bi-ideal element $b_\gamma(x)$ depends on the $\mathcal{B}_\gamma$-class B of x rather than on x itself. We call the γ-bi-ideal element $b_\gamma(x)$ the *representative γ-bi-ideal element* of the $\mathcal{B}_\gamma$-class B and denote it by b_B.

2 γ-Regularity and γ-intra-regularity of $\mathcal{B}_\gamma$-classes

In this section we give some necessary and sufficient conditions for a $\mathcal{B}_\gamma$-class to be γ-regular or γ-intra-regular. We have also included here some auxiliary results which may be found in [2]. This is for convenience of the reader and since they are quite similar to the results obtained in the present paper, so it will be easier for everyone to see the analogy between the relations $\mathcal{H}_\gamma$ and $\mathcal{B}_\gamma$ and how the results improve for the $\mathcal{B}_\gamma$-case.

An element x of an $le-\Gamma$-semigroup M is called γ-*regular*[2] if and only if $x \leq x\gamma l_\gamma(x)$ or equivalently $x \leq x\gamma e\gamma x$. An $le-\Gamma$-semigroup M is called γ-*regular*[2] if and only if every element of M is γ-regular. An element x of an $le-\Gamma$-semigroup M is called γ-*intra-regular* if and only if $x \leq e\gamma x\gamma x\gamma e$. An $le-\Gamma$-semigroup M is called γ-*intra-regular* if and only if every element of M is γ-intra-regular.

Proposition 2.1 *Let M be an $le-\Gamma$-semigroup. A $\mathcal{B}_\gamma$-class B of M is γ-regular if and only if B contains an γ-regular element.*

Proof. Let $x \in B$ be an γ-regular element and $z \in B$. Then from Lemma 1.10 we have $z \vee z\gamma e\gamma z = b_\gamma(z) = b_\gamma(x) = x \vee x\gamma e\gamma x = x\gamma e\gamma x = z\gamma e\gamma z$, which implies that z is γ-regular. The converse is obvious.

The following proposition is an immediately corollary of the above proposition.

Corollary 2.2 *Let M be an $le-\Gamma$-semigroup. M is γ-regular if and only if each $\mathcal{B}_\gamma$-class of M contains an γ-regular element. In particular, M is γ-regular if and only if every γ-bi-ideal element of M is γ-regular.*

Proposition 2.3 *Let M be an $le-\Gamma$-semigroup. A $\mathcal{B}_\gamma$-class B of M is γ-intra-regular if and only if B contains an γ-intra-regular element.*

Proof. For every $x \in B$ we have the following:

$$\begin{aligned} e\gamma b_B\gamma b_B\gamma e &= e\gamma(x \vee x\gamma e\gamma x)\gamma(x \vee x\gamma e\gamma x)\gamma e = \\ &= e\gamma(x\gamma x \vee x\gamma x\gamma e\gamma x \vee x\gamma e\gamma x\gamma x \vee x\gamma e\gamma x\gamma x\gamma e\gamma x)\gamma e = e\gamma x\gamma x\gamma e \end{aligned}$$

If x is γ-intra-regular, then $x \leq e\gamma x\gamma x\gamma e = e\gamma b_B\gamma b_B\gamma e$. This implies that

$$x\gamma e\gamma x \leq e\gamma b_B\gamma b_B\gamma e\gamma e\gamma e\gamma b_B\gamma b_B\gamma e \leq e\gamma b_B\gamma b_B\gamma e$$

and consequently $b_\gamma(x) = x \vee x\gamma e\gamma x \leq e\gamma b_B\gamma b_B\gamma e$. Therefore for every $y \in B$ we have $y \leq b_\gamma(y) = b_\gamma(x) \leq e\gamma b_B\gamma b_B\gamma e = e\gamma y\gamma y\gamma e$, which shows that y is γ-intra-regular. The converse is obvious.

Corollary 2.4 *Let M be an $le-\Gamma$-semigroup. M is γ-intra-regular if and only if each $\mathcal{B}_\gamma$-class of M contains an γ-intra-regular element. In particular, M is γ-intra-regular if and only if every γ-bi-ideal element of M is γ-intra-regular.*

Proposition 2.5 *Let M be an $le-\Gamma$-semigroup. If B_x and B_y are two γ-regular $\mathcal{B}_\gamma$-classes contained in the same $\mathcal{H}_\gamma$-class of M, then they coincide.*

Proof. From the γ-regularity of both x and y, we have $b_\gamma(x) = x\gamma e\gamma x$ and $b_\gamma(y) = y\gamma e\gamma y$. Since x and y are in the same $\mathcal{H}_\gamma$-class, Lemma 1.10 yields $x\gamma e\gamma x = y\gamma e\gamma y$. Hence we have $b_\gamma(x) = b_\gamma(y)$ and consequently $(x, y) \in \mathcal{B}_\gamma$.

In [6], Theorem 2 shows a nice situation in Γ-semigroups concerning the transmission of regularity from elements to subsets, that is, if an element is regular then the whole $\mathcal{D}_\gamma$-class containing it is γ-regular too. In contrast with the Γ-semigroup case, the Proposition 2.5 shows that in $le-\Gamma$-semigroups the γ-regularity of a $\mathcal{H}_\gamma$-class H is "localized" in a unique $\mathcal{B}_\gamma$-class B contained in H, that is, an element x of M is γ-regular together with its own $\mathcal{B}_\gamma$-class B_x and none of the other $\mathcal{B}_\gamma$-classes included in H_x (if there is any) is γ-regular. The following problem arises:

Problem 1 *Does γ-regularity of an element x imply γ-regularity of H_x, or equivalently, does it imply $B_x = H_x$?*

An approach to find a non-γ-regular $\mathcal{H}_\gamma$-class containing an γ-regular element would be to construct an $le-\Gamma$-semigroup with a non-γ-regular $\mathcal{H}_\gamma$-class satisfying the Green's condition.

Problem 2 *Is there an $le-\Gamma$-semigroup containing an $\mathcal{H}_\gamma$-class that satisfies the Green's condition but is not γ-regular?*

We say that a $\mathcal{B}_\gamma$-class B of an $le-\Gamma$-semigroup satisfies the Green's condition if there exist elements $x, y \in B$ such that $x\gamma y \in B$.

In [2] it is proved the following theorem:

Theorem 2.6 [*2, Theorem*2.1] *Let M be an $le-\Gamma$-semigroup. If H is an $\mathcal{H}_\gamma$-class of M satisfying the Green's condition and $q = q_H$, then:*

1. *$q\gamma q \in H$ and $q = q\gamma e \wedge e\gamma q$;*
2. *q is the only γ-quasi-ideal element of H;*
3. *if $x, y \in H$, then $y \leq x\gamma e$ and $x \leq e\gamma y$;*

4. *$q\gamma q = q\gamma e\gamma q = (q\gamma)^{n-1}q$ for all integers $n \geq 2$; in particular, $q\gamma q$ is γ-idempotent;*

5. *every element of H is γ-intra-regular;*

6. *$q = q\gamma q$ if and only if q is γ-regular in which case every element of H is γ-regular.*

The following result comparing with the above result shows that imposing the Green's condition on a $\mathcal{B}_\gamma$-class has stronger consequences than imposing the same condition on an $\mathcal{H}_\gamma$-class.

Theorem 2.7 *Let M be an $le-\Gamma$-semigroup. If a $\mathcal{B}_\gamma$-class B of M satisfies the Green's condition, then*

1. *B is γ-regular;*

2. *the representative γ-bi-ideal element b_B of B is γ-idempotent and $b_B = b_B\gamma e\gamma b_B$;*

3. *B is γ-intra-regular.*

Proof. (1) Let $x, y, x\gamma y \in B$. Since from Lemma 1.7 $\mathcal{B}_\gamma \subseteq \mathcal{H}_\gamma$, we can apply Theorem 2.6(3) to obtain the inequality $x \leq x\gamma e$. Multiplying it through on the left by y and utilizing Lemma 1.10 we get

$$x\gamma y \leq x\gamma e\gamma y = x\gamma e\gamma x = x\gamma y\gamma e\gamma x\gamma y,$$

whence

$$x \vee x\gamma e\gamma x = x\gamma y \vee x\gamma y\gamma e\gamma x\gamma y = x\gamma y\gamma e\gamma x\gamma y = x\gamma e\gamma x.$$

Therefore we have $x \leq x\gamma e\gamma x$, that is, x is γ-regular. Now Proposition 2.1 implies that the whole class B is γ-regular.

(2) From Lemma 1.10 and the fact that x is γ-regular we obtain

$$x\gamma e\gamma x = x\gamma y\gamma e\gamma x \leq x\gamma e\gamma x\gamma y\gamma e\gamma x = x\gamma e\gamma x\gamma x\gamma e\gamma x \leq x\gamma e\gamma x.$$

This shows that for $b_B = x\gamma e\gamma x$ we have $b_B = b_B\gamma b_B$. Thus, b_B can play the role of x (as well as those of y and $x\gamma y$) in the above reasoning, whence $b_B = b_B\gamma e\gamma b_B$.

(3) From the Proposition 2.3, B is γ-intra-regular because it contains an γ-idempotent element (namely, b_B).

From the Theorem 2.7(2) we obtain the nontrivial part of the following

Corollary 2.8 *Let M be an $le-\Gamma$-semigroup. The $\mathcal{B}_\gamma$-class B of M satisfies the Green's condition if and only if B contains an γ-idempotent.*

The following Proposition is an application of the Theorem 2.7 and gives us a sufficient condition under which γ-bi-ideal elements and γ-quasi-ideal elements coincide.

Proposition 2.9 *Let M be an $le - \Gamma$-semigroup. If for each $\mathcal{H}_\gamma$-class H of M the $\mathcal{B}_\gamma$-class B_q of the representative γ-quasi-ideal element $q = q_H$ satisfies the Green's condition, then the γ-quasi-ideal elements and the γ-bi-ideal elements of M coincide.*

Proof. Since q is the greatest element of H, it is also the greatest element of its $\mathcal{B}_\gamma$-class B_q. By Theorem 2.7(1) B_q is γ-regular whence by Theorem 2.7(2) its greatest element q is such that $q = q\gamma q$. Since H satisfies the Green's condition, Theorem 2.6(6) implies that H is γ-regular, whence by Proposition 2.5, $H = B_q$. Thus, the relations $\mathcal{H}_\gamma$ and $\mathcal{B}_\gamma$ coincide whence the set of γ-quasi-ideal elements (which are precisely the greatest elements of the $\mathcal{H}_\gamma$-classes by Theorem 2.6(2)) and the set of γ-bi-ideal elements (which are precisely the greatest elements of the $\mathcal{B}_\gamma$-classes) coincide.

3 $\mathcal{B}_\gamma$-classes being Γ-subsemigroups

In this section we investigate and give various conditions ensuring that an $\mathcal{B}_\gamma$-class of an $le - \Gamma$-semigroup M forms a group or a subsemigroup in $M_\gamma = (M, \circ)$. We have also included here some known results from [2] concerning the same question in $\mathcal{H}_\gamma$ case comparing with the results in $\mathcal{B}_\gamma$ case.

In [2] it is proved the following result

Proposition 3.1 [*2, Proposition* 2.3] *Let M be a $le - \Gamma$-semigroup. An $\mathcal{H}_\gamma$-class H of M is a subgroup of $M_\gamma = (M, \circ)$ if and only if H consists of a single γ-idempotent.*

Now we prove the following result which is similar with the above result.

Proposition 3.2 *Let M be an $le - \Gamma$-semigroup. A $\mathcal{B}_\gamma$-class B of M is a subgroup of $M_\gamma = (M, \circ)$ if and only if B consists of a single γ-idempotent.*

Proof. $\Rightarrow$. Let B be a sub-Γ-group of M, that is B_γ is a group, $\gamma \in \Gamma$. We denote by i the identity element of B_γ. Then $i \leq b_B$. By Theorem 2.7, $b_B\gamma b_B \in B$, whence $b_B\gamma b_B \leq b_B = b_B\gamma i \leq b_B\gamma b_B$. Thus, $b_B = b_B\gamma b_B$ and $b_B = i$.

Let z be an arbitrary element of B. We denote by z^{-1} the inverse element of z in B_γ. Then $z^{-1} \leq b_B$. Then we obtain $b_B = z\gamma z^{-1} \leq z\gamma b_B = z$. On the other hand, $z \leq b_B$, thus $z = b_B$.

$\Leftarrow$. This part is obvious.

In [2] it is proved the following theorem which gives sufficient and necessary conditions under which a $\mathcal{H}_\gamma$-class of Γ-semigroup M is a subsemigroup of semigroup $M_\gamma = (M, \circ)$.

In general $\mathcal{H}_\gamma$-classes and $\mathcal{B}_\gamma$-classes of an $le - \Gamma$-semigroup M satisfying Green's condition are not subsemigroups of semigroups $M_\gamma = (M, \circ)$ as the next example shows.

Example 3.3 *Consider the $le - \Gamma$-semigroup M of all order-preserving transformation of the finite chain $X = \{1 < 2 < ... < n\}$, where $\Gamma = \{id_X\}$ and for two mappings f, g the mapping $f id_X g$ denote the usual mapping composition f and g and for $a \in X$*

$$(f \vee g)(a) = max\{f(a), g(a)\}, (f \wedge g)(a) = min\{f(a), g(a)\}$$

(the maximum and the minimum are considered with respect to order in X); and the greatest element e is the transformation that sends every $a \in X$ to the number n.

It can be easily checked that in this $le - \Gamma$-semigroup the relation $\mathcal{H}_\gamma$ and $\mathcal{B}_\gamma$ coincide, whence

$$(f, g) \in \mathcal{B}_\gamma \Leftrightarrow f(n) = g(n).$$

For $n = 3$ the transformation $f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$ are $\mathcal{B}_\gamma$-related and f is an γ-idempotent whence the $\mathcal{B}_\gamma$-class B of these elements satisfies the Green's condition, but $g\gamma g = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \notin B$.

Theorem 3.4 [*2*, *Theorem* 2.4] *Let H be an $\mathcal{H}_\gamma$-class of M. Then the following are equivalent:*

1. *H is a subsemigroup of $M_\gamma = (M, \circ)$;*
2. *$\forall x \in H, x\gamma x \in H$;*
3. *H satisfies Green's condition and for all $x \in H$, $x\gamma q = q\gamma q = q\gamma x$, where $q = q_H$.*

The following theorem gives sufficient and necessary conditions under which a $\mathcal{B}$-class is a subsemigroup of semigroup $M_\gamma = (M, \circ)$.

Theorem 3.5 *Let B be an $\mathcal{B}_\gamma$-class of M. Then the following are equivalent:*

1. *B is a subsemigroup of $M_\gamma = (M, \circ)$;*
2. *$\forall x \in B, x\gamma x \in B$;*
3. *$b = b_B$ is a γ-zero of B; that is $\forall x \in B, x\gamma b_B = b_B = b_B\gamma x$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Since B satisfies the Green's condition, Theorem 2.7(2) implies that $b = b\gamma e\gamma b$. By Lemma 1.10 we have for all $x \in B$, $b\gamma e\gamma b = x\gamma e\gamma x$, whence $b = x\gamma e\gamma x$ and $x\gamma b = x\gamma b = x\gamma(x\gamma e\gamma x) = x\gamma x\gamma e\gamma x$. By Lemma 1.10 $x\gamma x\gamma e = x\gamma e$, consequently $x\gamma b = x\gamma e\gamma x = b$. Symmetrically one can prove that $b\gamma x = b$.

(3) $\Rightarrow$ (1). Take arbitrary $x, y \in B$. By Lemma 1.10 and Theorem 2.7(2) we have $y\gamma e\gamma x = x\gamma e\gamma x = b\gamma e\gamma b = b$. By Lemma 1.11 $x, y \leq b$, whence $x\gamma y \leq b\gamma b = b$. Using these observation and the fact that b is a γ-zero of B, we obtain the following equalities:

$$b\gamma(x\gamma y) = x\gamma y \vee x\gamma y\gamma e\gamma x\gamma y = x\gamma y \vee x\gamma b\gamma y = x\gamma y \vee b\gamma y = x\gamma y \vee b = b$$

Thus, $x\gamma y \in B$, that is, B is a Γ-subsemigroup of M.

Lemma 3.6 *Let M be an $le-\Gamma$-semigroup and B be a $\mathcal{B}_\gamma$-class satisfying Green's condition. If $b = b_B$ is an γ-ideal element, then b is a γ-zero of B.*

Proof. Since b is an γ-ideal element, we have $b\gamma e \leq b$ and $e\gamma b \leq b$. On the other hand, since the $\mathcal{H}_\gamma$-class H_b satisfies the Green's condition, by Theorem 2.6(3) we have that $b\gamma e \geq b$ and $e\gamma b \geq b$. As a consequence we have $b\gamma e = b = e\gamma b$. Now from Lemma 1.10 we have $\forall x \in B, x\gamma e = b\gamma e$ and $e\gamma x = e\gamma b$ and since by Theorem 2.7(2) $b = b\gamma e\gamma b$, we get

$$x\gamma b = x\gamma e\gamma b = b\gamma e\gamma b = b \text{ and } b\gamma x = b\gamma e\gamma x = b\gamma e\gamma b = b$$

This shows that b is an γ-zero of B.

As an immediate consequence of Lemma 3.6 and Theorem 3.5 we have

Corollary 3.7 *Let M be an $le-\Gamma$-semigroup. If a $\mathcal{B}_\gamma$-class B of M satisfies the Green's condition and b_B is an γ-ideal element, then B is a subsemigroup of $M_\gamma = (M, \circ)$.*

Remark 2. Note that in duo $le-\Gamma$-semigroups, γ-bi-ideal elements are γ-ideal elements. Clearly, Corollary 3.7 implies that in duo $le-\Gamma$-semigroups the $\mathcal{B}_\gamma$-classes satisfying the Green's condition are subsemigroups.

Corollary 3.8 *Let M be an $le-\Gamma$-semigroup. The $\mathcal{B}_\gamma$-class B_e of the greatest element e of M is a subsemigroup of $M_\gamma = (M, \circ)$ if and only if e is an γ-idempotent.*

Proof. If B_e is a Γ-subsemigroup, then it satisfies the Green's condition and therefore, by Theorem 2.7(2) the element e, which is the representative γ-bi-ideal element of B_e, is γ-idempotent. Conversely, if e is γ-idempotent, then B_e satisfies the Green's condition. Since e is an γ-ideal element, Corollary 3.7 applies.

In [2] it is proved the following result.

Proposition 3.9 [*2, Proposition* 2.9] *Let H be an $\mathcal{H}_\gamma$-class of M such that its representative γ-quasi-ideal element $q = q_H$ is minimal in the set of all γ-quasi-ideal elements of M. Then $H = (q] = \{a \in M | a \leq q\}$ and H is a subsemigroup of $M_\gamma = (M, \circ)$.*

The following Theorem gives another sufficient condition, under which a $\mathcal{B}_\gamma$-class or an $\mathcal{H}_\gamma$-class of a $le-\Gamma$-semigroup M is a subsemigroup of $M_\gamma = (M, \circ)$.

Theorem 3.10 *Let M be an $le-\Gamma$-semigroup M. If $b \in M$ is minimal in the set of all γ-bi-ideal elements of M, then*

1. *$B_b = (b] = \{x \in M | x \leq b\}$ and B_b is a subsemigroup of $M_\gamma = (M, \circ)$.*
2. *$H_b = \{x \in M | x \leq b\gamma e \wedge e\gamma b\}$ and H_b is a subsemigroup of $M_\gamma = (M, \circ)$.*

Proof. (1) Let $x \leq b$. From the minimality of b and since $b_\gamma(x) = x \vee x\gamma e\gamma x \leq b \vee b\gamma e\gamma b = b$, we obtain that $b_\gamma(x) = b$ or equivalently, $x \in B_b$. Therefore we get that $B_b = \{x \in M | x \leq b\}$. As $b\gamma e\gamma b$ is an γ-bi-ideal element less than or equal to b, and since b is minimal, we have $b = b\gamma e\gamma b$. This implies that $b\gamma b = b\gamma b\gamma e\gamma b \leq ba\gamma e\gamma b = b$. But $b\gamma b$ is again an γ-bi-ideal element since $b\gamma b\gamma e\gamma b\gamma b \leq b\gamma b\gamma e\gamma b = b\gamma b$ and again the minimality of b implies that $b = b\gamma b$. If now $x_1, x_2 \in B_b$, then $x_1\gamma x_2 \leq b\gamma b = b$ which shows that B_b is a subsemigroup of $M_\gamma = (M, \circ)$.

(2) From (1) we see that the $\mathcal{H}_\gamma$-class H_b satisfies the Green's condition. Let $q = b\gamma e \wedge e\gamma b$ be the representative γ-quasi-ideal element of this class. Then we have

$$\begin{aligned} q\gamma q &= q\gamma e\gamma q && \text{by Theorem 2.6(4)} \\ &= b\gamma e\gamma b && \text{by Lemma 1.10} \\ &= b && \text{by Theorem 2.7(2)} \end{aligned}$$

For all element $x \in H_b$, the element $x\gamma b$ is an γ-bi-ideal element since we have

$$\begin{array}{rcll} x\gamma b\gamma e\gamma x\gamma b & = & x\gamma b\gamma e\gamma b\gamma b & \text{by Lemma 1.10} \\ & = & x\gamma b\gamma b = x\gamma b & \text{by Theorem 2.7(2)} \end{array}$$

But $x \leq q = b\gamma e \wedge e\gamma b$ implies that $x\gamma b \leq b\gamma e\gamma b = b$, whence from the minimality of b we obtain $x\gamma b = b$ or equivalently, $x\gamma q\gamma q = q\gamma q$. Observe that by Theorem 2.6(1) $q = q\gamma e \wedge e\gamma q \geq q\gamma q$ whence $x\gamma q \geq x\gamma q\gamma q$. On the other hand, $x \leq q$ implies $x\gamma q \leq q\gamma q$. Hence the equality $x\gamma q\gamma q = q\gamma q$ implies that $x\gamma q = q\gamma q$. Similarly, one has that $q\gamma x = q\gamma q$. Now Theorem 3.4 implies that H_b is a subsemigroup of $M_\gamma = (M, \circ)$.

Finally, we show that $H_b = \{x \in M | x \leq b\gamma e \wedge e\gamma b\}$. Indeed: if $x \leq q$, then we have $x\gamma e\gamma x \leq q\gamma e\gamma q = q\gamma q = b$. As $x\gamma e\gamma x$ is an γ-bi-ideal element and $b = q\gamma q$ is minimal, we have that $x\gamma e\gamma x = q\gamma q$. Now we see that on the one hand the following hold true

$$\begin{array}{rcll} q\gamma e & \geq & x\gamma e & \text{by the choice of } x \\ & \geq & x\gamma e\gamma x\gamma e & \text{since } e \geq e\gamma x\gamma e \\ & = & q\gamma q\gamma e & \text{since } q\gamma q = x\gamma e\gamma x \\ & = & q\gamma e & \text{by Theorem 2.6(1) and Lemma 1.10} \end{array}$$

whence $x\gamma e = q\gamma e$. Similarly, $e\gamma x = e\gamma q$. On the other hand we have,

$$\begin{array}{rcll} x & \leq & q & \text{by the choice of } x \\ & \leq & q\gamma e & \text{since } q = q\gamma e \wedge e\gamma q \\ & = & x\gamma e & \text{from above,} \end{array}$$

whence $r_\gamma(x) = r_\gamma(q)$ and in a similar way we get $l_\gamma(x) = l_\gamma(q)$, which finally implies the result.

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