An Engel Type Condition with Vanishing Derivations on Lie Ideals

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Abstract
Let $R$ be a prime ring of characteristic different from 2, $Z(R)$ the center of $R$, $d$ and $\delta$ derivations of $R$, $L$ a non-central Lie ideal of $R$. We prove that if $\delta[[d(u),u],d(u)] = 0$ for all $u \in L$ then either $d = 0$ or $\delta = 0$.

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1 Introduction.
In this paper we prove a result which combines derivations of prime rings with Engel type conditions. Define $[x,y] = [x,y]_1 = xy -yx$ and $[x,y]_k = [[x,y]_{k-1},y]$, for $k \geq 2$. Then, for a fixed integer $n \geq 1$, an Engel condition is a polynomial $[x,y]_n$ in non-commuting indeterminates.
Let $R$ be a prime ring with center $Z(R)$ and $d : R \to R$ be an additive mapping of $R$ such that $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. In literature such a mapping is called derivation of $R$. In particular $d$ is an inner derivation, induced by an element $q \in R$, if $d(r) = [q,r]$ for all $r \in R$.
The relationship between Engel conditions and derivations appeared in a well known Posner’s paper [17]. More precisely Posner proved that if $[d(r),r] \in Z(R)$, for all $r \in R$, that is $[[d(x),x],y]$ is satisfied by $R$, then $R$ must be commutative. Several authors extended the theorem of Posner to the case when the condition $[d(x),x]_k$ is satisfied by some suitable subset of $R$. In [13] (Theorem 2) Lanski generalized Posner’s result by proving that if $L$ is a non-commutative Lie ideal of $R$ which satisfies $[[d(x),x],y]$ then $char(R) = 2$ and $R$ satisfies $s_4$, the standard identity of degree 4. Later in [12] Lanski extended his own result to the case when the Lie ideal $L$ satisfies the condition $[d(x),x]_k$,
for a fixed integer $k \geq 1$. Also in this case Lanski obtained the conclusion that $\text{char}(R) = 2$ and $R$ satisfies $s_4$.

Recently in [3] Cheng studied derivations of prime rings that satisfy certain special Engel type conditions: he showed that if $R$ is a prime ring of characteristic different from 2 which satisfies the condition $[[d(x), x], d(x)]$, then it must be commutative.

Our purpose here is to continue this line of investigation by studying the set $S = \{[[d(x), x], d(x)], x \in L\}$, where $d$ is a derivation defined on $R$ and $L$ is a non-central Lie ideal of $R$. An approach that can be used in studying $S$ is to examine its size and a reasonable criteria for studying the size of $S$ is to examine its centralizer $C_R(S) = \{r \in R : [r, s] = 0, \forall s \in S\}$. If $S$ is rather large we would expect that $C_R(S) = Z(R)$. Our first goal is to confirm this fact by proving the following:

**Theorem 1** Let $R$ be a non-commutative prime ring of characteristic different from 2, $Z(R)$ the center of $R$. If $\delta$ and $d$ are both inner derivations of $R$ such that $\delta[[d(u), u], d(u)] = 0$ for all $u \in [R, R]$, then either $d = 0$ or $\delta = 0$.

Finally, according to Theorem 1, it would seem natural to ask oneself what happens in the more general case when $d$ and $\delta$ are not necessarily inner derivations and $S = \{[[d(x), x], d(x)], x \in L\}$ for a non-central Lie ideal $L$ of $R$ such that $\delta(s) = 0$, for all $s \in S$. In order to analyse this case, our main result has the following flavour:

**Theorem 2** Let $R$ be a prime ring of characteristic different from 2, $Z(R)$ the center of $R$, $d$ and $\delta$ derivations of $R$, $L$ a non-central Lie ideal of $R$. If $\delta[[d(u), u], d(u)] = 0$ for all $u \in L$ then either $d = 0$ or $\delta = 0$.

## 2 The Matrix Case.

We begin with the case when $R$ is a ring of matrices over a field and $d$ and $\delta$ are inner derivations. As above, for any elements $s, t$ in a ring, we shall denote $[s,t]_2$ the triple commutator $[[s,t],t]$, and we shall use this notation through the rest of the paper. We have:

**Lemma 1** Let $R = M_k(F)$ be the ring of $k \times k$ matrices over the field $F$ of characteristic different from 2, with $k > 1$, $a, b$ elements of $R$ such that $[a, [[[b, u]_2, [b, u]]] = 0$, for all $u \in [R, R]$. Then either $a \in Z(R)$ or $b \in Z(R)$.

**Proof.** The first aim is to prove that, if $b$ is not a diagonal matrix, then $a$ must be a central matrix. We will divide the proof in two cases: $k = 2$ and $k \geq 3$. 

Case 1: $k = 2$

Say $a = \sum_{ij} a_{ij}e_{ij}$, $b = \sum_{ij} b_{ij}e_{ij}$, where $a_{ij}, b_{ij} \in F$, and $e_{ij}$ are the usual matrix units. Suppose that $b$ is not a diagonal matrix, for example let $b_{21} \neq 0$.

Let $u = [r_1, r_2] = [e_{11}, e_{12}] = e_{12}$. Thus

$$0 = [a, [[b, u]_2, [b, u]]] = -4ae_{12}be_{12} - 4e_{12}be_{12}a$$

and right multiplying by $e_{12}$:

$$e_{12}be_{12}be_{12}ae_{12} = 0 \quad \text{that is} \quad b_{21}a_{21} = 0.$$

From $b_{21} \neq 0$ and $\text{char}(R) \neq 2$, we have $a_{21} = 0$. Moreover again by (1), it follows $(ae_{12} - e_{12}a) = 0$. Thus, left multiplying by $e_{11}$ and right multiplying by $e_{22}$, we also have $e_{11}ae_{12} - e_{12}ae_{22} = 0$, that is $a_{11} = a_{22}$.

In a similar way, for $v = e_{21} \in [R, R]$ we have $b_{12}a_{12} = 0$. Then, if $b_{12} \neq 0$ we get $a_{12} = 0$ and $a$ is a central matrix. In this case we are done. Now suppose $a_{12} \neq 0$ and $b_{12} = 0$, hence we are in the following situation:

$$a = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{11} \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix}.$$

Let now $v = [e_{11} + e_{12}, e_{21} + e_{22}] = e_{11} + e_{12} - e_{21} - e_{22}$. By calculation one has:

$$[b, v] = \begin{bmatrix} -b_{21} \\ 2b_{21} - b_{22} + b_{11} \\ b_{11} - b_{22} \end{bmatrix}$$

$$[b, v]_2 = 2 \begin{bmatrix} -b_{11} - b_{21} + b_{22} \\ b_{11} + b_{21} - b_{22} \\ b_{11} + b_{21} - b_{22} \end{bmatrix}$$

and if denote $X = -b_{11} - b_{21} + b_{22}$, it follows

$$0 = \begin{bmatrix} a, [[b, v]_2, [b, v]] \end{bmatrix} = 2 \begin{bmatrix} -a_{12}X \\ 0 \\ a_{12}X \end{bmatrix}$$

which implies

$$X = -b_{11} - b_{21} + b_{22} = 0$$

(2).

Analogously for $w = [e_{12}, e_{22} + e_{21}] = e_{11} - e_{22} + e_{12} \in [R, R]$, we have

$$[b, v] = \begin{bmatrix} -b_{21} \\ 2b_{21} \\ b_{21} \end{bmatrix}, \quad [b, v]_2 = 2 \begin{bmatrix} -b_{21} \\ -b_{11} - b_{21} + b_{22} \\ 2b_{21} \end{bmatrix}.$$

Now denote $Y = b_{11} - b_{22}$ and $Z = -b_{11} - b_{21} + b_{22}$, then

$$[[b, v]_2, [b, v]] = 4 \begin{bmatrix} b_{21}(Z - Y) \\ 0 \\ b_{21}(Z - Y) \end{bmatrix}$$
and
\[ 0 = [a, [[b, v], [b, v]]] = 4 \begin{bmatrix} 0 & -2(Z - Y)a_{12}b_{21} \\ 0 & 0 \end{bmatrix} \]
which implies
\[ 0 = Z - Y = -2b_{11} - b_{21} + 2b_{22} \quad (3). \]
The equalities (2) and (3) imply the contradiction \( b_{21} = 0 \). Hence, if \( a \) is not central then \( b \) must be a diagonal matrix in \( R = M_2(F) \).

**Case 2: \( k \geq 3 \)**

As above, for \( u = e_{ij} \), we have
\[ 0 = [a, [[b, u], [b, u]]] = -4ae_{ij}be_{ij}be_{ij} + 4e_{ij}be_{ij}be_{ij}a. \]
Left multiplying by \( e_{ii} \) and right multiplying by \( e_{jj} \) we get
\[ e_{ij}be_{ij}ae_{ii} = 0 \quad \text{that is} \quad b_{ji}a_{ji} = 0, \quad \forall j \neq i, l. \quad (4) \]
Analogously, left multiplying by \( e_{pp} \), with \( p \neq i \),
\[ e_{pp}ae_{ij}be_{ij} = 0 \quad \text{that is} \quad a_{pi}b_{ji} = 0 \quad \forall i \neq j, p. \quad (4') \]
Suppose \( b \) is not a diagonal matrix. Let \( i \neq j \) such that \( b_{ji} \neq 0 \). Hence
\[ a_{pi} = 0, \quad \forall p \neq i, \quad \text{and} \quad a_{jl} = 0, \quad \forall l \neq j \quad (5) \]
and
\[ a_{ii} = a_{jj} \quad (5'). \]
Consider now any \( q \neq i, j \) and \( v = (e_{ij} + e_{qj}) = [e_{ij} + e_{qj}, e_{jj}] \in [R, R] \). So,
\[ [a, [[b, e_{ij} + e_{qj}], [b, e_{ij} + e_{qj}]]] = 0 \]
and left multiplying by \( e_{hh} \), with \( h \neq i, q \), we have
\[ -4ae_{ij} + e_{qj}b(e_{ij} + e_{qj})b(e_{ij} + e_{qj}) - 4(e_{ij} + e_{qj})b(e_{ij} + e_{qj})b(e_{ij} + e_{qj})a = 0 \quad (6) \]
By (6), and using both (4') and (5), we obtain
\[ a_{hq}b_{ji} = 0, \quad \text{that is} \quad a_{hq} = 0 \quad \forall h \neq i, q \quad \forall q \neq i, j \quad (7). \]
The facts (7) and (5) imply that:
if \( b_{ji} \neq 0 \) then the non-zero entries of the matrix \( a \) are just in the \( i \)-th row, in \( j \)-th column or in the main diagonal (assertion A).

As above, we assume \( b_{ji} \neq 0 \) and let \( m \neq i, j \). Denote by \( \sigma_m \) and \( \tau_m \) the following automorphisms of \( R \):
\[ \sigma_m(x) = (1 + e_{jm})x(1 - e_{jm}) = x + e_{jm}x - xe_{jm} - e_{jm}xe_{jm} \]
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\[ \tau_m(x) = (1 - e_{jm})x(1 + e_{jm}) = x - e_{jm}x + xe_{jm} - e_{jm}xe_{jm} \]

and say \( \sigma_m(b) = \sum \sigma_{rs}e_{rs}, \tau_m(b) = \sum \tau_{rs}e_{rs} \) where \( \sigma_{rs}, \tau_{rs} \in F \). We have

\[ \sigma_{ji} = b_{ji} + b_{mi} \quad \text{and} \quad \tau_{ji} = b_{ji} - b_{mi}. \]

If there exists \( m \) such that \( \sigma_{ji} = b_{ji} + b_{mi} = 0 \) or \( \tau_{ji} = b_{ji} - b_{mi} = 0 \) then \( b_{mi} = -b_{ji} \neq 0 \) or \( b_{mi} = b_{ji} \neq 0 \). Therefore \( b_{ji} \neq 0 \) and \( b_{mi} \neq 0 \), and so, using assertion (A), the non-zero entries of the matrix \( a \) are just in the i-row or in the main diagonal, since \( m \neq j \). Hence

\[ a = \sum_{r, r \neq i} a_{rr}e_{rr} + \sum_s a_{is}e_{is}, \quad a_{rs} \in F \quad (8) \]

Now assume that \( \sigma_{ji} \neq 0 \) and \( \tau_{ji} \neq 0 \), for all \( m \neq i, j \), and recall that, for any \( F \)-automorphism \( \varphi \) of \( R \),

\[ [\varphi(a), [[\varphi(b), u], [\varphi(b), u]]] = 0 \quad \text{for all} \quad u \in [R, R]. \]

Thus in this case by assertion (A), for any \( m \neq i, j \), the non-zero entries of the matrices \( \sigma_m(a) \) and \( \tau_m(a) \) are just in the i-th row, in j-th column or in the main diagonal. In particular, since

\[ \sigma_m(a) = a + e_{jm}a - ae_{jm} - e_{jm}ae_{jm} \]

\[ \tau_m(a) = a - e_{jm}a + ae_{jm} - e_{jm}ae_{jm} \]

then both the above matrices have zero in the \((j, m)\) entry that is

\[ a_{jm} + a_{mm} - a_{jj} - a_{mj} = 0 \]

\[ a_{jm} - a_{mm} + a_{jj} - a_{mj} = 0, \quad \forall m \neq i, j. \]

Moreover by assertion (A) \( a_{jm} = 0 \), because \( m \neq i, j \) and so

\[ a_{mm} - a_{jj} = a_{mj} = a_{jj} - a_{mm} \]

which implies \( a_{mj} = 0 \), for all \( m \neq i, j \). At this point we can write again the matrix \( a \) as follows:

\[ a = \sum_{r, r \neq i} a_{rr}e_{rr} + \sum_s a_{is}e_{is}. \quad (8') \]

In other words, by (8) and (8'), we have:

if \( b_{ji} \neq 0 \) then the non-zero entries of the matrix \( a \) are just in the i-th row or in the main diagonal (assertion B).
Let again $b_{ji} \neq 0$ and $m \neq i, j$. Denote

$$\lambda_m(x) = (1 + e_{mi})x(1 - e_{mi}) = x + e_{mi}x - x e_{mi} - e_{mi}xe_{mi}$$

$$\mu_m(x) = (1 - e_{mi})x(1 + e_{mi}) = x - e_{mi}x + x e_{mi} - e_{mi}xe_{mi}$$

and say $\lambda_m(b) = \sum \lambda_{rs} e_{rs}$, $\mu(b) = \sum \mu_{rs} e_{rs}$ with $\lambda_{rs}, \mu_{rs} \in F$. We have that

$$\lambda_{ji} = b_{ji} - b_{jm} \quad \text{and} \quad \mu_{ji} = b_{ji} + b_{jm}.$$ 

If there exists $m \neq i, j$ such that $\lambda_{ji} = b_{ji} - b_{jm} = 0$ or $\mu_{ji} = b_{ji} + b_{jm} = 0$ then $b_{jm} = b_{ji} \neq 0$ or $b_{jm} = -b_{ji} \neq 0$. Thus, by assertion (B), $a$ is just a diagonal matrix because $b_{ji} \neq 0$, $b_{jm} \neq 0$ and $m \neq i, j$.

On the other hand, if $\lambda_{ji} \neq 0$ and $\mu_{ji} \neq 0$, for all $m \neq i, j$, then the non-zero entries of the matrices $\lambda_m(a)$ and $\mu_m(a)$ are just in the $i$-th row and in the main diagonal. In particular, since

$$\lambda_m(a) = a + e_{mi}a - ae_{mi} - e_{mi}ae_{mi}$$

$$\mu_m(a) = a - e_{mi}a + ae_{mi} - e_{mi}ae_{mi}$$

then both the matrices have zero in the $(m, i)$ entry, that is

$$a_{mi} + a_{ii} - a_{mm} - a_{im} = 0$$

$$a_{mi} - a_{ii} + a_{mm} - a_{im} = 0, \quad \forall m \neq i, j.$$ 

Moreover by assertion (B) $a_{mi} = 0$, because $m \neq i, j$, and so

$$a_{mm} - a_{ii} = a_{im} = a_{ii} - a_{mm}$$

which implies $a_{im} = 0$, for all $m \neq i, j$. Finally in any case, if $b_{ji} \neq 0$, we can write the matrix $a$ as follows:

$$a = \sum_r a_{rr} e_{rr} + a_{ij} e_{ij}.$$ 

Now Assume $w = [e_{ij} + e_{iq}, e_{jj} + e_{qq}] = e_{ij} + e_{iq} \in [R, R]$, for all $q \neq i, j$. In this case we have

$$[b, w]_2 = -2(e_{ij} + e_{iq})b(e_{ij} + e_{iq})$$

and

$$[[b, w], [b, w]] = -4(e_{ij} + e_{iq})b(e_{ij} + e_{iq})b(e_{ij} + e_{iq})$$

By the above argument, $a = \sum_r a_{rr} e_{rr} + a_{ij} e_{ij}$, moreover

$$0 = [a, [[b, w]_2, [b, w]]] = -4a_{ii}(e_{ij} + e_{iq})b(e_{ij} + e_{iq})b(e_{ij} + e_{iq}) +$$
\[4a_{jj}(e_{ij} + e_{iq})b(e_{ij} + e_{iq})be_{ij} + 4a_{qq}(e_{ij} + e_{iq})b(e_{ij} + e_{iq})be_{iq} \quad (9)\]

In particular the \((i, q)\) entry of the matrix \((9)\) is zero, that is

\[4(a_{qq} - a_{ii})(b_{ji} + b_{qi})^2 = 0 \quad (9').\]

Notice that if \(b_{qi} \neq 0\), by \((5')\) it follows \(a_{qq} = a_{ii}\). On the other hand, when \(b_{qi} = 0\) and since \(b_{ji} \neq 0\), by \((9')\) we also have \(a_{qq} = a_{ii}\). From this and \((5')\), in any case \(a_{ii} = a_{jj} = a_{qq} = \alpha \in F\), for all \(q \neq i, j\), that is \(a = \alpha I + a_{ij}e_{ij}\). Since \(a\) and \(a' = a_{ij}e_{ij}\) induce the same inner derivation, without loss of generality we replace \(a\) by \(a'\), then we obtain that \([a', [b, u]_2, [b, u]]] = 0\), for all \(u \in [R, R]\). Moreover we note that \(b_{ij} = 0\), if not by \((5)\) it follows \(a_{ij} = 0\) and \(a\) is a central matrix.

Let now \(\chi \in Aut_F\?(R)\) with \(\chi(x) = (1 + e_{ji})x(1 - e_{ji})\).

Of course \([\chi(a'), [[\chi(b), u]_2, [\chi(b), u]]] = 0\), for all \(u \in [R, R]\). By calculation we have that

\[
\chi(a') = a_{ij}(e_{ij} + e_{jj} - e_{ii} - e_{ji})
\]

\[
\chi(b) = b + e_{ji}b - be_{ji}.
\]

If the \((j, i)\)-entry of the matrix \(\chi(b)\) is not zero, by \((5')\) the \((i, i)\)-entry of the matrix \(\chi(a')\) is equal to the \((j, j)\)-one. This means that \(-a_{ij} = a_{ij}\) and, since \(\text{char}(R) \neq 2\), it follows \(a_{ij} = 0\), that is \(a' = 0\) and \(a\) is central.

Consider the case when the \((j, i)\)-entry of \(\chi(b)\) is zero. By calculation we have

\[b_{ji} + b_{ii} - b_{jj} = 0 \quad (10).\]

Finally define \(\varphi \in Aut_F\?(R)\) with \(\varphi(x) = (1 - e_{ji})x(1 + e_{ji})\).

Again \([\varphi(a'), [[\varphi(b), u]_2, [\varphi(b), u]]] = 0\), for all \(u \in [R, R]\) and

\[
\varphi(a') = a_{ij}(e_{ij} - e_{jj} + e_{ii} - e_{ji})
\]

\[
\varphi(b) = b - e_{ji}b + be_{ji} - e_{ji}be_{ji}.
\]

Following the same above calculations, we obtain that either \(a\) is central or the \((j, i)\)-entry of the matrix \(\varphi(b)\) is zero, that is

\[b_{ji} - b_{ii} + b_{jj} = 0 \quad (10').\]

As a consequence of these facts, if \(a\) is not central, then by \((10)\) and \((10')\) we have the contradiction \(b_{ij} = 0\).

Thus we conclude that if \(b\) is not diagonal then \(a\) must be central.

Therefore, we can assume that \(b\) is a diagonal matrix in \(M_k(F)\) also in the case \(k \geq 3\).

Finally, for any \(\psi \in Aut_F(R)\), we have \([\psi(a), [[\psi(b), u]_2, [\psi(b), u]]] = 0\), for all
\( u \in [R, R] \), and so, by the previous cases, \( \psi(b) \) must be a diagonal matrix in \( M_k(F) \) for any \( k \geq 2 \).

In particular, for any \( r \neq s \), if \( \psi(x) = (1 + e_{rs})x(1 - e_{rs}) \), then

\[
\psi(b) = b + e_{rs}b - be_{rs} - e_{rs}be_{rs} = b + (b_{ss} - b_{rr})e_{rs}.
\]

This means \( b_{rr} = b_{ss} \), for all \( r \neq s \), that is \( b \) must be central. \( \Box \)

3 The proof of the Theorems.

Before beginnig the proof of the main theorem, for the sake of completeness, we prefer to recall some basic notations, definitions and some easy consequences of the result of Kharchenko [11] about the differential identities on a prime ring \( R \). We refer to Chapter 7 in [2] for a complete and detaleid description of the theory of generalized polynomial identities involving derivations.

We denote by \( Q \) the Martindale quotients ring of \( R \) and let \( C = Z(Q) \) be the extended centroid of \( R \) ([2], Chapter 2).

It is well known that any derivation of a prime ring \( R \) can be uniquely extended to a derivation of its Martindale quotients ring \( Q \), and so any derivation of \( R \) can be defined on the whole \( Q \) ([2], pg. 87).

Now, we denote by \( \text{Der}(Q) \) the set of all derivations on \( Q \). By a derivation word we mean an additive map \( \Delta \) of the form \( \Delta = d_1d_2...d_m \), with each \( d_i \in \text{Der}(Q) \). Then a differential polynomial is a generalized polynomial, with coefficients in \( Q \), of the form \( \Phi(\Delta_j(x_i)) \) involving noncommutative indeterminates \( x_i \) on which the derivations words \( \Delta_j \) act as unary operations. The differential polynomial \( \Phi(\Delta_j(x_i)) \) is said a differential identity on a subset \( T \) of \( Q \) if it vanishes for any assignment of values from \( T \) to its indeterminates \( x_i \).

Let \( D_{\text{int}} \) be the \( C \)-subspace of \( \text{Der}(Q) \) consisting of all inner derivations on \( Q \) and let \( d \) and \( \delta \) be two non-zero derivations on \( R \). By Theorem 2 in [11] we have the following result (see also Theorem 1 in [15]):

**Fact 1** Let \( R \) be a prime ring of characteristic different from 2, if \( d \) and \( \delta \) are \( C \)-linearly independent modulo \( D_{\text{int}} \) and \( \Phi(\Delta_j(x_i)) \) is a differential identity on \( R \), where \( \Delta_j \) are derivations words of the following form \( \delta, d, \delta^2, \delta d, d^2 \), then \( \Phi(y_{ji}) \) is a generalized polynomial identity on \( R \), where \( y_{ji} \) are distinct indeterminates.

As a particular case, we have:

**Fact 2** If \( d \) is a non-zero derivation on \( R \) and

\[
\Phi(x_1, ..., x_n, d(x_1), ..., d(x_n), d^2(x_1), ..., d^2(x_n))
\]

is a differential identity on \( R \), then one of the following holds:
1. either \( d \in D_{\text{int}} \);

2. or \( R \) satisfies the generalized polynomial identity

\[
\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n).
\]

We study now the case when \( \delta \) and \( d \) are both \( Q \)-inner derivations:

**Theorem 1** Let \( R \) be a non-commutative prime ring of characteristic different from 2, \( Z(R) \) the center of \( R \). If \( \delta \) and \( d \) are both \( Q \)-inner derivations such that \( \delta([d(u), u], d(u)) = 0 \) for all \( u \in [R, R] \), then either \( d = 0 \) or \( \delta = 0 \).

**Proof.** Let \( \delta \) be the inner derivation induced by the element \( a \in Q \), and \( d \) the one induced by \( b \in Q \). Trivially \( a \) and \( b \) are not in the extended centroid \( C \), which is the center of \( Q \). These assumptions say that \( R \) satisfies the generalized polynomial identity

\[
\left[ a, \left[ [b, [x, y]]_2, [b, [x, y]] \right] \right]
\]

which is explicitly:

\[
G(x, y) = a \left( b[x, y]^2b[x, y] - b[x, y]^3b - b[x, y]b[x, y]^2 - b[x, y]^3b + 2b[x, y]^2b[x, y] \right) +
\]

\[
a \left( [x, y]b^2[x, y] - [x, y]^2b[x, y]b - 4[x, y]b[x, y]b[x, y] + 2[x, y]b[x, y]^2b +
\]

\[
[x, y]b^2[x, y]^2 + [x, y]b[x, y]^2b \right) -
\]

\[
b \left( [x, y]^2b[x, y] - [x, y]^3b - [x, y]b[x, y]^2 - [x, y]^3b + 2[x, y]^2b[x, y] \right) a -
\]

\[
\left( [x, y]b^2[x, y] - [x, y]^2b[x, y]b - 4[x, y]b[x, y]b[x, y] +
\]

\[
2[x, y]b[x, y]^2b + [x, y]b^2[x, y]^2 + [x, y]b[x, y]^2b \right) a.
\]

Suppose \( a \notin C \) and \( b \notin C \). If \( \{a, b, 1\} \) are linearly \( C \)-independent, then by [4] \( G(x, y) \) is a non-trivial generalized polynomial identity for \( R \). On the other hand, if there exist \( \alpha, \beta \in C \) such that \( a = \alpha b + \beta \), then \( R \) satisfies the identity

\[
G(x, y) = (\alpha b + \beta) \left( b[x, y]^2b[x, y] - b[x, y]^3b - b[x, y]b[x, y]^2 - b[x, y]^3b + 2b[x, y]^2b[x, y] \right) +
\]

\[
(\alpha b + \beta) \left( [x, y]b^2[x, y] - [x, y]^2b[x, y]b - 4[x, y]b[x, y]b[x, y] +
\]

\[
2[x, y]b[x, y]^2b + [x, y]b^2[x, y]^2 + [x, y]b[x, y]^2b \right) -
\]
\begin{equation}
\begin{aligned}
&b(\[x, y]b[x, y] - \[x, y]b[x, y] + \[x, y]b[x, y] + 2\[x, y]b[x, y]) (\alpha b + \beta) - \\
&(\[x, y]b^2[x, y] - \[x, y]b[x, y]b - 4\[x, y]b[x, y]b[x, y] + 2\[x, y]b[x, y]b^2) (\alpha b + \beta)
\end{aligned}
\end{equation}

which is again a non-trivial generalized polynomial identity since \(b \notin C\).

By a theorem due to Beidar (Theorem 2 in [1]) this generalized polynomial identity is also satisfied by \(Q\). In case \(C\) is infinite, we have

\[\[a, [b, [r_1, r_2]]_2, [b, [r_1, r_2]]]\] = 0

for all \(r_1, r_2 \in Q \otimes_C \overline{C}\), where \(\overline{C}\) is the algebraic closure of \(C\). Since both \(Q\) and \(Q \otimes_C \overline{C}\) are centrally closed ([6], Theorems 2.5 and 3.5), we may replace \(R\) by \(Q\) or \(Q \otimes_C \overline{C}\) according as \(C\) is finite or infinite. Thus we may assume that \(R\) is centrally closed over \(C\) which is either finite or algebraically closed and \([a, [b, [r_1, r_2]], [b, [r_1, r_2]]]\] = 0, for all \(r_1, r_2 \in R\). By Martindale’s theorem [16], \(R\) is a primitive ring having a non-zero socle with \(C\) as the associated division ring. In light of Jacobson’s theorem ([10], pag 75) \(R\) is isomorphic to a dense ring of linear transformations on some vector space \(V\) over \(C\).

Assume first that \(V\) is finite-dimensional over \(C\). Then the density of \(R\) on \(V\) implies that \(R \cong M_k(C)\), the ring of all \(k \times k\) matrices over \(C\). Since \(R\) is not commutative we assume \(k \geq 2\). In this case the conclusion follows by Lemma 1.

Assume next that \(V\) is infinite-dimensional over \(C\). As in lemma 2 in [16], the set \([R, R]\) is dense on \(R\) and so from

\[G(r_1, r_2) = [a, [b, [r_1, r_2]]_2, [b, [r_1, r_2]]]\] = 0

for all \(r_1, r_2 \in R\), we have

\([a, [b, r]_2, [b, r]]\] = 0

for all \(r \in R\), that is \(R\) satisfies the generalized identity

\[G'(x) = [a, [b, x]_2, [b, x]].\]

Here our first aim is to prove that, for any idempotent element \(e \in H = \text{soc}(R)\), the ring \(eRe\) satisfies the generalized identity

\[G''(x) = [eae, [ebe, x]_2, [ebe, x]].\]
To do this, we divide the argument in three steps and of course we assume that both $a$ and $b$ are not central:

**Step 1.**
Suppose that $\{1, a, b\}$ are linearly $C$-dependent, that is there exist $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 a + \alpha_2 b + \alpha_3 = 0$. Since $b \notin C$, we may assume that $\alpha_1 \neq 0$ and so we may write $a = \beta b + \gamma$ for suitable elements $\beta, \gamma \in C$, with $\beta \neq 0$. In this case, the inner derivation induced by $a$ is the following: $[a, x] = \beta[b, x]$. Notice that $[a, [b, x]]$ is a generalized identity for $R$, moreover $b \in C$ if and only if $\beta b \in C$. Thus we may replace $b$ by $\beta b$ without loss of generality. In other words we may assume that $[a, x] = [b, x]$, that is $d = \delta$ and $R$ satisfies the following 

$$[b, [b, x], [b, x]].$$

Suppose that there exists $v \in V$ such that $\{v, vb, vb^2\}$ are linearly $C$-independent. By the density of $R$ there exists $r \in R$ such that $vr = 0$, $vbr = v$, $vb^2r = v$. Hence the following contradiction follows:

$$0 = v[b, [b, r], [b, r]] = -4v \neq 0.$$

Therefore $\{v, vb, vb^2\}$ are linearly $C$-dependent for all $v \in V$ and standard arguments show that in this case there exist $\beta_1, \beta_2 \in C$ such that $b^2 = \beta_1 b + \beta_2$. From this and $a = \beta b + \gamma$ we have that $ba = ab = (\beta \beta_1 + \gamma)b + \beta \beta_2$. Hence, in the representation of $G'(x)$ the monomials which occur are the following:

1. $\alpha x^i bx^j bx^k$, for suitable $\alpha \in C$ and $i, j, k \in \{0, 1, 2, 3\}$;
2. $\beta x^i bx^j bx^k bx^l$, for suitable $\beta \in C$ and $i, j, k, l \in \{0, 1, 2\}$.

Let now $e^2 = e \in H$. Of course $R$ satisfies

$$eG'(exe)e = e[a, [b, exe], [b, exe]]e.$$

Notice that, in the representation of $eG'(exe)e$ the monomials which occur are the following:

1. $\alpha (exe)^i (exe)^j (exe)^k$, for suitable $\alpha \in C$ and $i, j, k \in \{0, 1, 2, 3\}$;
2. $\beta (exe)^i (exe)^j (exe)^k (exe)^k (exe)l$, for suitable $\beta \in C$ and $i, j, k, l \in \{0, 1, 2\}$. 

Hence the ring $eRe$ satisfies $G''(x)$.
In light of previous argument, in all that follows we may assume that $\{1, a, b\}$ are linearly $C$-independent, in particular there exists at least $v \in V$ such that $\{v, va, vb\}$ are linearly $C$-independent.

**Step 2.**
Now we will prove that in the representation of $G'(x)$, the coefficients which occur are $\{1, a, b, ba\}$.
First suppose that there exists $v \in V$ such that $\{v, va, vb, vb^2, vab\}$ are linearly $C$-independent. By the density of $R$ that there exists $r \in R$ such that

$$(vb)r = 0, vr = v, (vab)r = 0, (va)r = 0, (vb^2)r = v.$$  

From this we get the contradiction $0 = vG'(r) = 2(va) \neq 0$.
Thus $\{v, va, vb, vb^2, vab\}$ are linearly $C$-dependent for all $v \in V$, in other words there exist $\alpha, \beta, \gamma, \lambda, \mu \in C$ such that $aa + \beta b + \gamma ab + \lambda b^2 + \mu = 0$. Of course, since we suppose that $\{1, a, b\}$ are not linearly $C$-dependent, we assume that at least one of $\gamma$ and $\lambda$ is not zero.

**Case 1:** $\gamma \neq 0$.
We may write $ab = \alpha'a + \beta'b + \lambda'b^2 + \mu'$, for some $\alpha', \beta', \lambda', \mu' \in C$. Suppose there exists $v \in V$ such that $\{v, va, vb, vb^2\}$ are linearly $C$-independent. By the density there exists $r \in R$ such that

$$(va)r = 0, (vb)r = 0, vr = v, (vb^2)r = v$$  

and so we also have $v(ab)r = (\mu' + \lambda')v$. By calculations it follows that

$$0 = vG'(r) = 2(-\mu'v + va)$$  

which is a contradiction since $v, va$ are linearly independent. Thus, for all $v \in V$, $\{v, va, vb, vb^2\}$ are linearly $C$-dependent and so there exist $\omega_1, \omega_2, \omega_3 \in C$ such that $b^2 = \omega_1a + \omega_2b + \omega_3$. This also implies that $ab = (\alpha' + \lambda'\omega_1)a + (\beta' + \lambda'\omega_2)b + \lambda'\omega_3\mu$, and Step 2 is done.

**Case 2:** $\lambda \neq 0$.
In this case we have $b^2 = \alpha''a + \beta''b + \gamma''ab + \mu''$, for suitable $\alpha'', \beta'', \gamma'', \mu'' \in C$. Since $\{1, a, b\}$ are linearly $C$-independent, at least one of $b^2$ and $\gamma''$ is not zero. Notice that in case $b^2 = 0$ and $\gamma'' \neq 0$, it follows that $ab = \alpha'''a + \beta'''b + \mu'''$ for $\alpha''', \beta''', \mu''' \in C$ and we are done. Thus we assume $b^2 \neq 0$.
If there exists $v \in V$ such that $\{v, va, vb, vab\}$ are linearly $C$-independent, then by the density there exists $r \in R$ such that

$$(va)r = 0, (vb)r = 0, vr = v, (vab)r = v$$
so that $vb^2r = (\gamma'' + \mu'')v$. This implies the contradiction

$$0 = vG'(r) = -2(vb + (\gamma'' + \mu'')va) \neq 0.$$  

As above, this means that $\{v, va, vb, vab\}$ are linearly $C$-dependent for all $v \in V$ and so there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$ such that $\alpha_1a + \alpha_2b + \alpha_3ab + \alpha_4 = 0$. Moreover $\alpha_3 \neq 0$ since $\{1, a, b\}$ are $C$-independent. Therefore we may write

$$b^2 = (\alpha'' + \gamma'' \beta_1)a + (\beta'' + \gamma'' \beta_2)b + (\gamma'' \beta_3 + \mu'')$$  

and we are done again.

In other words, Step 2 implies that there exist $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6 \in C$ such that

$$b^2 = \omega_1a + \omega_2b + \omega_3; \quad ab = \omega_4a + \omega_5b + \omega_6.$$  

**Step 3.**

Finally we will prove that in any case, for all $e^2 = e \in H$, the ring $eRe$ satisfies $G''(x)$.

Notice that, if $ba$ is also linearly $C$-dependent from $\{1, a, b\}$, we have that the coefficients which occur in the representation of $G'(x)$ are $\{1, a, b\}$ and, following the same argument in Step 1, $eRe$ satisfies $G''(x)$, for all $e^2 = e \in H$. Thus in what follows we suppose that there exists $v \in V$ such that $\{v, va, vb, vba\}$ are linearly $C$-independent. We will obtain a number of contradictions.

By the density there exists $r \in R$ such that $(va)r = 0$, $(vb)r = 0$, $vr = v$ and from this we also have $vb^2r = \omega_3v$, $vabr = \omega_6v$. By calculations it follows that $0 = vG'(r) = -2\omega_6(vb) - 2\omega_3(va)$, which is a contradiction unless when $\omega_3 = \omega_6 = 0$. In this last case

$$b^2 = \omega_1a + \omega_2b; \quad ab = \omega_4a + \omega_5b.$$  

Moreover, since there exists $v \in V$ such that $\{v, va, vb\}$ are linearly $C$-independent, and by the density of $R$, there exists $s \in R$ such that $(va)s = 0$, $(vb)s = v$, $vs = 0$. It follows that $vabs = \omega_5v$ and $vb^2s = \omega_2v$. Again by calculation we have that $0 = vG''(s) = 2(\omega_5 - \omega_2 + 1)v - 2\omega_5(vb)$ which is a contradiction unless when $\omega_5 = 0$ and $\omega_2 = 1$. In this last case we write

$$b^2 = \omega_1a + b; \quad ab = \omega_4a.$$  

Again by the density of $R$, there exists $t \in R$ such that $(vba)t = 0$, $(vb)t = 0$, $(va)t = v$, $vt = v$, so that $vb^2t = \omega_1v$. Under these choices we get $0 = (vb)G'(t) = 2\omega_1(vba)$, which implies $\omega_1 = 0$. Hence we may write $b^2 = b$ and $ab = \omega_4a$. From these last we get $\omega_4(ab) = ab^2 = ab$, that is $\omega_4 = 1$ and so $ab = a$. 


Finally by the density of $R$ there exists $r' \in R$ such that $(va)r' = v$, $vr' = v$, $(vb)r' = v$ and we get again a contradiction: $0 = vG'(r') = 2(va) - 4(vb)a$.

All the previous argument says that in $G'(x)$ the coefficients which occur are $1, a, b$. Therefore, as above, one can see that $eRe$ satisfies $G''(x)$, for all $e^2 = e \in H$.

Since $a, b \notin C$, they don’t centralize the non zero ideal $H$ of $R$. Thus there exist $h_1, h_2 \in H$ such that $[a, h_1] \neq 0$, $[b, h_2] \neq 0$. Moreover, because of the infinite dimensionality, $H$ does not satisfy the polynomial $[x_1, x_2]$ that is there exist $h_3, h_4 \in H$ such that $[h_3, h_4] \neq 0$. By Litoff’s theorem in [7] there exists $e^2 = e \in H$ such that

$$ah_1, h_1a, bh_2, h_2b \in eRe \quad \text{and also} \quad h_1, h_2, h_3, h_4 \in eRe$$

moreover $eRe$ is a central simple algebra finite dimensional over its center. Since $[h_3, h_4] \neq 0$, then $eRe$ is not commutative and so $eRe \cong M_t(C)$, for $t \geq 2$. We know that $[eae, [ebe, x]]$ is a generalized polynomial identity for $eRe$, then by the finite dimensional case, we have that either $eae \in Z(eRe)$ or $ebe \in Z(eRe)$. In the first case we have the contradiction

$$ah_1 = eah_1 = eaeh_1 = h_1eae = h_1ae = h_1a.$$ 

In the second one we get

$$bh_2 = ebeh_2 = h_2ebe = h_2be = h_2b$$

a contradiction again. $\Box$

Now we are ready to prove our main result:

**Theorem 2** Let $R$ be a prime ring of characteristic different from $2$, $Z(R)$ the center of $R$, $d$ and $\delta$ derivations of $R$, $L$ a non-central Lie ideal of $R$. If $\delta[[d(u), u], d(u)] = 0$ for all $u \in L$ then either $d = 0$ or $\delta = 0$.

**Proof.** By first we note that we may assume that $R$ is not commutative, since $L$ is not central. Moreover, since $\text{char}(R) \neq 2$, there exists a non-central two-sided ideal $I$ of $R$ such that $[I, I] \subseteq L$ (see p. 4-5 in [8]; Lemma 2 and Proposition 1 in [5]). Therefore $\delta[[d(u), u], d(u)] = 0$ for all $u \in [I, I]$. Moreover by [15] $R$ and $I$ satisfy the same differential polynomial identities, that is $\delta[[d(u), u], d(u)] = 0$ for all $u \in [R, R]$.

First suppose that $\delta$ and $d$ are $C$-independent modulo $D_{\text{int}}$. 
By assumption $R$ satisfies the differential identity
\[
\delta \left[ d([x, y]), [x, y], d([x, y]) \right] =
\]
\[
\left[ \left[ \delta d(x), y \right] + \left[ d(x), \delta(y) \right] + \left[ \delta(x), d(y) \right] + [x, \delta\delta(y)], [x, y] \right] + [d(x), y] + [x, d(y)] +
\]
\[
\left[ \left[ [d(x), y] + [x, d(y)], [\delta(x), y] + [x, \delta(y)] \right], [d(x), y] + [x, d(y)] \right] +
\]
\[
\left[ \left[ [d(x), y] + [x, d(y)], [x, y] \right], \delta d(x), y] + [d(x), \delta(y)] + [\delta(x), d(y)] + [x, \delta d(y)] \right](11).
\]

By Kharchenko’s theorem [11] $R$ satisfies the polynomial identity
\[
\left[ \left[ x_5, y \right] + [x_1, x_4] + [x_3, x_2] + [x, x_6], [x, y], [x_1, y] + [x, x_2] \right] +
\]
\[
\left[ \left[ [x_1, y] + [x, x_2], [x_3, y] + [x, x_4] \right], [x_1, y] + [x, x_2] \right] +
\]
\[
\left[ \left[ [x_1, y] + [x, x_2], [x, y] \right], [x_5, y] + [x_1, x_4] + [x_3, x_2] + [x, x_6] \right].
\]

In particular $R$ satisfies any blendend component
\[
\left[ \left[ [x, x_6], [x, y] \right], [x_1, y] \right] + \left[ \left[ [x_1, y], [x, y] \right], [x, x_6] \right] \quad (11').
\]

Since $R$ satisfies a polynomial identity, there exists $M_k(F)$, the ring of all matrices over a suitable field $F$, such that $R$ and $M_k(F)$ satisfy the same polynomial identities (see [9], Theorem 2 p.54 and Lemma 1 p.89). If $k = 1$ then we have the contradiction that $R$ is commutative. In case $k \geq 2$, we choose in (11')

\[ x = e_{11}, y = -e_{21}, x_1 = -e_{22}, x_6 = e_{12} \]

and get the contradiction $-2e_{21} = 0$.

Let now $\delta$ and $d$ $C$-dependent modulo $D_{\text{int}}$. There exist $\gamma_1, \gamma_2 \in C$, such that $\gamma_1\delta + \gamma_2d \in D_{\text{int}}$, and, by Theorem 1, in case both $d$ and $\delta$ are inner derivations we are done.

Now suppose that at most one of the two derivations can be inner. We will show that in this case we have a number of contradictions.

Suppose $\gamma_1 = 0$ and $\gamma_2 \neq 0$; then, for some non-central element $q \in Q$, $d = dq$ is the inner derivation induced by $q$ and $\delta$ is an outer derivation. By (11) $R$ satisfies
which is
\[
\left[\left[\delta(q), x\right] + [q, \delta(x)], y\right] + [[q, x], \delta(y)] + [\delta(x), [q, y]] + [x, \delta(q), y] + [q, \delta(y)]]\right], [x, y],
\]
\[
[[q, x], y] + [x, [q, y]]
\]
\[
\left[\left[[q, x], y\right] + [x, [q, y]], [\delta(x), y] + [x, \delta(y)]]\right], [x, x] + [x, [q, y]]
\]
\[
\left[[\delta(q), x] + [q, \delta(x)], y\right] + [[q, x], \delta(y)] + [\delta(x), [q, y]] + [x, \delta(q), y] + [q, \delta(y)]]\right], [x, y],
\]
\[
[[q, x], y] + [x, [q, y]] + [x, \delta(q), y] + [q, \delta(y)]]\right]
\]

As above, by Kharchenko’s result, \( R \) satisfies the generalized polynomial identity
\[
\left[[\delta(q), x] + [q, x_1], y\right] + [[q, x], x_2] + [x_1, [q, y]] + [x, [\delta(q), y] + [q, x_2]], [x, y],
\]
\[
[[q, x], y] + [x, [q, y]]
\]
\[
\left[[q, x], y\right] + [x, [q, y]], [x_1, y] + [x, x_2], [[q, x], y] + [x, [q, y]]
\]
\[
\left[[[q, x], y] + [x, [q, y]], [x, y]\right], [\delta(q), x] + [q, x_1], y + [[q, x], x_2] + [x_1, [q, y]] + [x, [\delta(q), y] + [q, x_2]
\]

and in particular \( R \) satisfies the component in the indeterminates \( x, y, x_2 \).
\[ F(x, y, x_2) = \left( \left( [[q, x], x_2] + [x, [q, x_2]], [x, y], [[q, x], y] + [x, [q, y]] \right) + \right. \\
\left. \left( [[q, x], y] + [x, [q, y]], [x, x_2], [[q, x], y] + [x, [q, y]] \right) + \right. \\
\left. \left( [[q, x], y] + [x, [q, y]], [x, y], [[q, x], y] + [x, [q, y]] \right) \right) (12). \]

Since \( q \notin C \), the previous one is a non-trivial generalized polynomial identity for \( R \). By Martindale's theorem [16], \( R \) is a primitive ring having a non-zero socle with \( C \) as the associated division ring. As we said above, \( R \) is isomorphic to a dense ring of linear transformations on some vector space \( V \) over \( C \).

Assume first that \( V \) is finite-dimensional over \( C \), that is \( R \cong M_k(C) \), the ring of all \( k \times k \) matrices over \( C \). Of course we assume that \( k \geq 2 \) and denote \( q = \sum q_{ij}e_{ij} \), with \( q_{ij} \in C \). In this case, in (12) substitute \( x = e_{rr}, y = -e_{sr}, x_2 = e_{rs} \), for any \( r \neq s \). By calculations it follows that \( -4e_{11}qe_{21}qe_{22} = 0 \), which implies \( q_{rs} = 0 \), for all \( r \neq s \). Therefore \( q \) is a diagonal matrix in \( M_k(C) \). Following the same argument in Lemma 1, we may conclude with the contradiction that \( q \) must be a central element of \( R \).

Assume next that \( V \) is infinite-dimensional over \( C \). Our aim is here to prove that

\[ F'(x, y, x_2) = \left( \left( [[q, x], x_2] + [x, [q, x_2]], [x, y], [[q, x], y] + [x, [q, y]] \right) + \right. \\
\left. \left( [[q, x], y] + [x, [q, y]], [x, x_2], [[q, x], y] + [x, [q, y]] \right) + \right. \\
\left. \left( [[q, x], y] + [x, [q, y]], [x, y], [[q, x], y] + [x, [q, y]] \right) \right) \]

is a generalized identity for \( eRe \), for all \( e^2 = e \in H \).

Suppose first that \( \{1, q, q^2\} \) are linearly \( C \)-dependent. Since \( q \notin C \), we may write \( q^2 = \alpha q + \beta \), for some \( \alpha, \beta \in C \). Thus, in the representation of \( F(x, y, x_2) \), the terms which occur are the following:

1. \( \alpha f_1(Z_1, Z_2)q^2f_2(Y_1, Y_2)f_3(T_1, T_2) \), where \( \alpha \in C \), \( f_1, f_2, f_3 \) are suitable polynomials over \( C \), and \( Z_1, Z_2, Y_1, Y_2, T_1, T_2 \in \{x, y, x_2\} \);

2. \( \beta qf_4(Z_1, Z_2)q^2f_5(Y_1, Y_2) \), where \( \beta \in C \), \( f_4, f_5 \) are suitable polynomials over \( C \), and \( Z_1, Z_2, Y_1, Y_2 \in \{x, y, x_2\} \);

3. \( \gamma f_6(Z_1, Z_2)q^2f_7(Y_1, Y_2)q \), where \( \gamma \in C \), \( f_6, f_7 \) are suitable polynomials over \( C \), and \( Z_1, Z_2, Y_1, Y_2 \in \{x, y, x_2\} \).
Let \( e^2 = e \in H \). Of course \( R \) satisfies \( eF(exe, eye, ex_2e)e \). Notice that, in the representation of \( eF(exe, eye, ex_2e)e \), the terms which occur are the following:

1. \( \alpha f_1(Z_1, Z_2)eqe f_2(Y_1, Y_2)eqe f_3(T_1, T_2) \), where \( \alpha \in C \), \( f_1, f_2, f_3 \) are suitable polynomials over \( C \), and \( Z_1, Z_2, Y_1, Y_2, T_1, T_2 \in \{exe, eye, ex_2e\} \);

2. \( \beta eqe f_4(Z_1, Z_2)eqe f_5(Y_1, Y_2) \), where \( \beta \in C \), \( f_4, f_5 \) are suitable polynomials over \( C \), and \( Z_1, Z_2, Y_1, Y_2 \in \{exe, eye, ex_2e\} \);

3. \( \gamma f_6(Z_1, Z_2)eqe f_7(Y_1, Y_2)eqe \), where \( \gamma \in C \), \( f_6, f_7 \) are suitable polynomials over \( C \), and \( Z_1, Z_2, Y_1, Y_2 \in \{exe, eye, ex_2e\} \).

In other words, we have that \( eRe \) satisfies \( F'(x, y, x_2) \).

Let now \( \{1, q, q^2\} \) linearly \( C \)-independent, therefore there exists \( v \in V \) such that \( \{v, vq, vq^2\} \) are linearly \( C \)-independent. Since \( \dim_C V = \infty \) there exist \( w_1, w_2, w_3, w_4 \in V \) such that \( \{v, vq, vq^2, w_1, w_2, w_3, w_4\} \) are linearly \( C \)-independent. Moreover by the density of \( R \) there exist \( r_1, r_2, r_3 \in R \) such that:

\[
vr_1 = w_1, \quad vr_2 = w_2, \quad vr_3 = w_3
\]

\[
w_1r_2 = v, \quad w_1r_3 = v
\]

\[
w_2r_1 = 0, \quad w_3r_1 = 0
\]

\[
(vq)r_1 = 0, \quad (vq)r_2 = 0, \quad (vq)r_3 = 0
\]

\[
(vq^2)r_1 = w_4, \quad (vq^2)r_2 = 0, \quad (vq^2)r_3 = 0
\]

\[
w_4r_2 = 0, \quad w_4r_3 = v.
\]

From this we have

\[
v[r_1, r_2] = v, \quad v[r_1, r_3] = v
\]

\[
vq[r_1, r_2] = 0, \quad vq[r_1, r_3] = 0
\]

\[
vq^2[r_1, r_2] = 0, \quad vq^2[r_1, r_3] = v
\]

Hence by (12), for \( x = r_1, y = r_3, r_2 = x_2 \), we get the contradiction

\[
0 = vF(r_1, r_3, r_2) =
\]

\[
v\left[\left[\left[q, [r_1, r_2]\right], [r_1, r_3]\right], [q, [r_1, r_3]]\right] +
\]

\[
v\left[\left[\left[q, [r_1, r_3]\right], [r_1, r_2]\right], [q, [r_1, r_3]]\right] +
\]

\[
v\left[\left[\left[q, [r_1, r_3]\right], [r_1, r_3]\right], [q, [r_1, r_2]]\right] = 4v \neq 0.
\]
This contradiction says that in what follows we may assume that \( \{1, q, q^2\} \) are linearly \( C \)-dependent.

Since \( q \notin C \), it doesn’t centralize the non zero ideal \( H = \text{soc}(R) \) of \( R \). Thus there exist \( h_1 \in H \) such that \( [q, h_1] \neq 0 \). Moreover, because of the infinite dimensionality, \( H \) does not satisfy the polynomial \( [x_1, x_2] \) that is there exist \( h_2, h_3 \in H \) such that \( [h_2, h_3] \neq 0 \). By Litoff’s theorem in [7] there exists \( e^2 = e \in H \) such that

\[
h_1q, qh_1 \in eRe \quad \text{and also} \quad h_1, h_2, h_3 \in eRe
\]

moreover \( eRe \) is a central simple algebra finite dimensional over its center. Since \([h_2, h_3] \neq 0 \), then \( eRe \) is not commutative and so \( eRe \cong M_t(C) \), for \( t \geq 2 \). Moreover \( eRe \) satisfies the generalized identity \( F'(x, y, x_2) \) and by the finite dimensional case we have that \( eqe \in Z(eRe) \), which contradicts with the choice of \( h_1 \) in \( eRe \).

Suppose now \( \gamma_2 = 0 \) and \( \gamma_1 \neq 0 \); then, for some non-central element \( q \in Q \), \( \delta = d_q \) is the inner derivation induced by \( q \) and \( d \) is an outer derivation. By (11) we have that \( R \) satisfies the identity:

\[
\left[\left[\left[q, d(x)\right], y\right] + [d(x), [q, y]] + [q, [x, d(y)] + [x, q, d(y)]], [x, y]\right] + [d(x), y] + [x, d(y)]\right] +
\]

\[
\left[\left[\left[d(x), y\right] + [x, d(y)], [q, x] + [x, [q, y]]\right], [d(x), y] + [x, d(y)]\right] +
\]

\[
\left[\left[d(x), y\right] + [x, d(y)], [q, d(x)] + [d(x), q, y] + [q, x] + [d(y)] + [x, q, d(y)]\right].
\]

Since \( d \) is outer, in light of the above cited Kharchenko’s result, \( R \) satisfies the identity

\[
\left[\left[[q, x_1], y\right] + [x_1, [q, y]] + [q, x_2] + [x, [q, x_2]], [x, y]\right] + [x_1, y] + [x, x_2] +
\]

\[
\left[\left[[x_1, y] + [x, x_2], [q, x] + [x, [q, y]]\right], [x_1, y] + [x, x_2]\right] +
\]

\[
\left[\left[[x_1, y] + [x, x_2], [x, y]\right], [[q, x_1], y] + [x_1, [q, y]] + [q, x] + [x_2] + [x, q, x_2]\right]
\]

and in particular \( R \) satisfies the blended component

\[
\left[\left[[q, x], x_2\right] + [x, [q, x_2]], [x, y]\right] + [x, x_2]
\]
polynomial identities. Let $k$ which is with $0 < d \leq 1$. Thus, by Kharchenko’s theorem in [11] and by (13),

$$R$$

by Theorem 1. Therefore we consider the case when $d$ is an inner derivation, then also $\delta$ is inner, then we end up by Theorem 1. Therefore we consider the case when $d$ is an outer derivation. Thus, by Kharchenko’s theorem in [11] and by (13), $R$ satisfies the identity

$$\left[ [x, y], [x, x_2] \right] +$$

$$\left[ [x, x_2], [x, y] \right], \left[ [q, x], x_2 \right] + [x, [q, x_2]] \right]$$

which is

$$\left[ q, \left[ [x, y], [x, x_2] \right] \right].$$

By Theorem 6 in [14], either $q \in C$ or $\left[ [x, y], [x, x_2] \right]$ is central valued on $R$. In this last case $R$ satisfies a polynomial identity, hence there exists a suitable field $F$ such that $R$ and the matrix ring $M_k(F)$ satisfy the same polynomial identities. Let $k \geq 2$. Since $\left[ [x, y], [x, x_2] \right]$ must be central in $M_k(F)$, if we choose $x = e_{11}$, $y = e_{12}$, $x_2 = e_{12} + e_{21}$, we get the contradiction $\left[ [x, y], [x, x_2] \right] = -2(e_{11} + e_{21}) \notin Z(M_k(F)).$

Finally we may assume that both $\gamma_1$ and $\gamma_2$ are non-zero. So $\delta = \alpha d + d_q$, with $0 \neq \alpha \in C$ and $q \in Q$.

In this case $R$ satisfies

$$\left[ \left[ \alpha d^2(x) + [q, d(x)], y \right] + [d(x), \alpha d(y) + [q, y]] \right] + [\alpha d(x) + [q, x], d(y)] +$$

$$\left[ [x, \alpha d^2(y) + [q, d(y)]], [x, y] \right], [d(x), y] + [x, d(y)] \right] +$$

$$\left[ \left[ [d(x), y] + [x, d(y)], [\alpha d(x) + [q, x], y] + [x, \alpha d(y) + [q, y]] \right], [d(x), y] + [x, d(y)] \right] +$$

$$\left[ \left[ [d(x), y] + [x, d(y)], [x, y] \right], [\alpha d^2(x) + [q, d(x)], y] +$$

$$[d(x), \alpha d(y) + [q, y]] + [\alpha d(x) + [q, x], d(y)] + [x, \alpha d^2(y) + [q, d(y)]] \right]$$

(13).

Notice that if $d$ is an inner derivation, then also $\delta$ is inner, then we end up by Theorem 1. Therefore we consider the case when $d$ is an outer derivation. Thus, by Kharchenko’s theorem in [11] and by (13), $R$ satisfies the identity

$$\left[ [\alpha x_3 + [q, x_1], y] + [x_1, \alpha x_2 + [q, y]] + [\alpha x_1 + [q, x], x_2] + [x, \alpha x_4 + [q, x_2]], [x, y] \right],$$

$$[x_1, y] + [x, x_2] +$$

$$\left[ [x_1, y] + [x, x_2], [\alpha x_1 + [q, x], y] + [x, \alpha x_2 + [q, y]] \right], [x_1, y] + [x, x_2] +$$
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\[
\left[[x_1, y] + [x, x_2], [x, y]\right],
\]

\[
[\alpha x_3 + [q, x_1], y] + [x_1, \alpha x_2 + [q, y]] + [\alpha x_1 + [q, x], x_2] + [x, \alpha x_4 + [q, x_2]]
\]

(13')

and in particular \( R \) satisfies the component

\[
[q, [x, y], [x_1, y]]_2.
\]

If \( q \notin C \), as above we have that \([x, y], [x_1, y]\] is central valued on \( R \), which implies again the contradiction that \( R \) is commutative.

Hence consider the case when \( q \in C \), thus by (13') we have that \( R \) satisfies the identity

\[
\left[[\alpha x_3, y] + [x_1, \alpha x_2] + [\alpha x_1, x_2] + [x, \alpha x_4], [x, y]\right], [x_1, y] + [x, x_2] + \\
\left[[x_1, y] + [x, x_2], [\alpha x_1, y] + [x, \alpha x_2]\right], [x_1, y] + [x, x_2] + \\
\left[[x_1, y] + [x, x_2], [x, y]\right], [\alpha x_3, y] + [x_1, \alpha x_2] + [\alpha x_1, x_2] + [x, \alpha x_4]\right]
\]

(14)

and in particular for \( x_3 = x_4 = 0 \) and \( y = x_1 \) in (14), we get that \( R \) satisfies the identity

\[
2\alpha [([x_1, x_2], [x, x_1]), [x, x_2]] + 2\alpha [[[x, x_2], [x, x_1]], [x_1, x_2]]
\]

(15).

Also in this case there exists a field \( F \) such that \( M_k(F) \) and \( R \) satisfy the same polynomial identities, in particular \( M_k(F) \) satisfies (15). If suppose \( k \geq 2 \) and choose \( x = e_{21}, x_1 = e_{11}, x_2 = e_{12} \), by (15) it follows the contradiction \( 4\alpha(e_{22} - e_{11}) = 0 \).

\[\square\]

References


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