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Some Remarks on Regularity of Generalized Transformation Semigroups

Phatsarapa Chanmuang and Ronnason Chinram

Department of Mathematics, Faculty of Science
Prince of Songkla University, Hat Yai
Songkhla 90112, Thailand
ronnason.c@psu.ac.th

Abstract

Let S be a semigroup. An element a in S is called *regular* if there exists $x \in S$ such that $a = axa$. A semigroup S is called *regular* if every element in S is regular. Let $T(X, Y)$ denotes the set of all full transformation from X to Y. For $\theta \in T(Y, X)$, let $(T(X, Y), \theta)$ be a semigroup $(T(X, Y), *)$ where $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in T(X, Y)$. We have the semigroup $(T(X, Y), \theta)$ is regular if and only if one of the following statments holds : $(i)|X| = 1$, $(ii)|Y| = 1$ and $(iii)\theta$ is a bijective function.

In this paper, we give some remark of regularity of $(T(X, Y), \theta)$.

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1 Introduction and Preliminaries

Let S be a semigroup. An element a in S is called *regular* if there exists $x \in S$ such that $a = axa$. Let Reg $S = \{a \in S \mid a \text{ is regular}\}$. A semigroup S is called *regular* if every element in S is regular.

Let X be a nonempty set and $T(X)$ be the full transformation semigroup on X. It is well-known that $T(X)$ is regular (see [2]). To generalize this, let X and Y be nonempty sets and

$$T(X, Y) \text{ denotes the set of all full transformation from } X \text{ to } Y.$$

For $\theta \in T(Y, X)$, let $(T(X, Y), \theta)$ be a semigroup $(T(X, Y), *)$ where $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in T(X, Y)$. The following theorem charecterize when the semigroups $(T(X, Y), \theta)$ are regular.

Theorem 1.1 *For $\theta \in T(Y, X)$, the semigroup $(T(X, Y), \theta)$ is regular if and only if one of the following statments holds.*
$(i)|X| = 1.$
$(ii)|Y| = 1.$
$(iii)\theta$ *is a bijective function.*

Proof. See Theorem 2.1 in [1], choose $k = |Y|$. ■

The following corollary follows by Theorem 1.1.

Corollary 1.2 *Assume $|X|, |Y| > 1$. We have if $|X| \neq |Y|$, then $(T(X, Y), \theta)$ is not a regular semigroup for all $\theta \in T(Y, X)$.*

In this paper, we give some remark of regularity of generalized transformation semigroups $(T(X, Y), \theta)$.

2 Main Results

The following theorem give some interesting remark of a regular element in the semigroup $(T(X, Y), \theta)$.

Theorem 2.1 *Let $\alpha \in T(X, Y)$. If α is a regular element of $(T(X, Y), \theta)$, then $|ran\ \alpha| \leq |ran\ \theta|$.*

Proof. Suppose α is a regular element of $(T(X, Y), \theta)$. Then there exists $\beta \in T(X, Y)$ such that $\alpha = \alpha\theta\beta\theta\alpha$. Then

$$|\text{ran } \alpha| = |(X)\alpha| = |(X)\alpha\theta\beta\theta\alpha| \leq |((Y)\theta)\alpha| = |(\text{ran } \theta)\alpha| \leq |\text{ran } \theta|.$$

Hence the theorem is proved. ■

The following corollary and Theorem 2.1 are equivalent.

Corollary 2.2 *Let $\alpha \in T(X, Y)$. If $|ran\ \alpha| > |ran\ \theta|$, then α is not a regular element of the semigroup $(T(X, Y), \theta)$.*

The next theorem shows that $|\text{ran } \theta|$ is the maximum of $\{|\text{ran } \alpha| \mid \alpha$ is regular in $(T(X, Y), \theta)\}$.

Theorem 2.3 *There exists a regular element α in the semigroup $(T(X, Y), \theta)$ such that $|ran\ \alpha| = |ran\ \theta|$.*

Proof. For each $x \in \text{ran } \theta$, we have there exists $y_x \in Y$ such that $(y_x)\theta = x$. Let $x_0 \in \text{ran } \theta$. Define $\alpha : X \to Y$ by

$$(x)\alpha = \begin{cases} y_x & \text{if } x \in \text{ ran } \theta \\ y_{x_0} & \text{if } x \notin \text{ ran } \theta \end{cases}$$

Then $|\text{ran } \alpha| = |\text{ran } \theta|$. Now, to show that $\alpha = \alpha\theta\alpha\theta\alpha$, let $x \in X$.

Case 1 : $x \in$ ran θ. We have

$$(x)\alpha\theta\alpha\theta\alpha = (y_x)\theta\alpha\theta\alpha = (x)\alpha\theta\alpha = (y_x)\theta\alpha = (x)\alpha.$$

Case 2 : $x \notin$ ran θ. We have

$$(x)\alpha\theta\alpha\theta\alpha = (y_{x_0})\theta\alpha\theta\alpha = (x_0)\alpha\theta\alpha = (y_{x_0})\theta\alpha = (x_0)\alpha = y_{x_0} = (x)\alpha.$$

Then α is regular and $|\text{ran } \alpha| = |\text{ran } \theta|$.

Hence the theorem is proved. ■

The next two theorems show the minimum of the number of regular elements of $(T(X,Y),\theta)$ is $|Y|$.

Theorem 2.4 *The semigroup $(T(X,Y),\theta)$ contains at least $|Y|$ regular elements, that is $|Reg\ (T(X,Y),\theta)| \geq |Y|$.*

Proof. For each $y \in Y$, define $\alpha_y : X \to Y$ by

$$(x)\alpha_y = y \text{ for all } x \in X.$$

So $\alpha_y \in T(X,Y)$ and ran $\alpha_y = \{y\}$. We have for all $\beta \in T(X,Y)$,

$$\text{ran } \alpha_y\theta\beta\theta\alpha_y \subseteq \text{ ran } \alpha_y.$$

Then ran $\alpha_y\theta\beta\theta\alpha_y = \{y\}$. Therefore for $x \in X$,

$$(x)\alpha_y\theta\beta\theta\alpha_y = y = (x)\alpha_y,$$

this implies $\alpha_y\theta\beta\theta\alpha_y = \alpha_y$. So for each $y \in Y$, we have α_y is regular.

Hence the theorem is proved. ■

Theorem 2.5 *If $|ran\ \theta| = 1$ then $|Reg\ (T(X,Y),\theta)| = |Y|$.*

Proof. Since the number of elements in $T(X,Y)$ such that $|\text{ran } \alpha| = 1$ is $|Y|$, by Theorem 2.1 and the proof of Theorem 2.4, the theorem holds. ■

Corollary 2.6 *Let X and Y be finite sets. If $|ran\ \theta| = 1$, then the number of irregular elements in $(T(X,Y),\theta)$ is $|Y|^{|X|} - |Y|$.*

Proof. We have $|T(X,Y)| = |Y|^{|X|}$. By Theorem 2.5, the theorem holds. ■

Let X and Y be finite sets. Theorem 2.7 shows the upper bound of the number of regular elements of $(T(X,Y),\theta)$ is $n^{|X|}\binom{|Y|}{n}$.

Theorem 2.7 *Let X and Y be finite sets. Assume that $|ran\ \theta| = n$. Then $|Reg\ (T(X,Y),\theta)| \leq n^{|X|}\binom{|Y|}{n}$.*

Proof. Since the number of elements α in $T(X,Y)$ such that $|\text{ran }\alpha| \leq n$ is $n^{|X|}\binom{|Y|}{n}$, by Theorem 2.1, $|\text{Reg }(T(X,Y),\theta)| \leq n^{|X|}\binom{|Y|}{n}$. ■

The last theorem give some sufficient condition for an element in $(T(X,Y),\theta)$ to be regular.

Theorem 2.8 *Let $\alpha \in T(X,Y)$. If α is a regular element in $(T(X,Y),\theta)$, then $|ran\ \alpha| = |ran\ \alpha\theta|$ and $|ran\ \alpha| = |ran\ \theta\alpha|$.*

Proof. Suppose that $|\text{ ran }\alpha| \neq |\text{ran }\alpha\theta|$ or $|\text{ran }\alpha| \neq |\text{ran }\theta\alpha|$.

Case 1 : $|\text{ran }\alpha| \neq |\text{ran }\alpha\theta|$. Since $|((X)\alpha)\theta| \leq |(X)\alpha|, |\text{ran }\alpha| > |\text{ran }\alpha\theta|$, that is $|(X)\alpha\theta| < |(X)\alpha|$. So for all $\beta \in T(X,Y)$,

$$|(X)\alpha\theta\beta\theta\alpha| = |((X)\alpha\theta)\beta\theta\alpha| \leq |(X)\alpha\theta| < |(X)\alpha|,$$

this implies that α is not regular.

Case 2 : $|\text{ran }\alpha| \neq |\text{ ran }\theta\alpha|$. Since $|((Y)\theta)\alpha| \leq |(X)\alpha|, |\text{ran }\alpha| > |\text{ran }\theta\alpha|$, that is $|(Y)\theta\alpha| < |(X)\alpha|$. So for all $\beta \in T(X,Y)$,

$$|(X)\alpha\theta\beta\theta\alpha| = |((X)\alpha\theta\beta)\theta\alpha| \leq |(Y)\theta\alpha| < |(X)\alpha|,$$

this implies that α is not regular. ■

Remark 2.9 Let $X = \{1,2,3\}, Y = \{a,b,c\}$ and $\theta = \begin{pmatrix} a & b & c \\ 1 & 1 & 2 \end{pmatrix}$. By Theorem 2.8, we have that $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ a & a & b \end{pmatrix}$ is not regular in $(T(X,Y),\theta)$ because $|\text{ran }\alpha\theta| = |\{1\}| = 1 \neq 2 = |\text{ran }\alpha|$. This remark shows that the element in $(T(X,Y),\theta)$ such that $|\text{ran }\alpha| \leq |\text{ran }\theta|$ need not be regular.

References

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