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M^*- Condition and Co-prime Modules

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Abstract

In this paper we introduce the M^*- condition in the category of $\sigma[M]$ and give some equivalent for modules which satisfy in M^*- condition. We will proceed by introducing the notion of coprime M^*-module as dual notion of prime M-module. Finally we investigate the basic properties of such modules.

Keywords: M-module, M-generated, M^*- condition, M-coprime, Co-prime Module

1 Introduction

Let R be an associative ring with unit and R-Mod denotes the category of unital left R-module. By M we mean a left R-module. A module X in R-Mod, the category of left R-modules, is said to be M-generated if there exists an R-epimorphism from a direct sum of M on to X. Dually a module X is said to be M-cogenerated if there exists an R-monomorphism from N to a direct product of M. The category $\sigma[M]$ of modules subgenerated by M is defined to be the full subcategory of R-Mod that contains all modules X, such that X is isomorphic to a submodule of an M-generated module. For more notions on the category $\sigma[M]$, see [5]. We have two *preradicals (trace and reject)* in R-Mod which are:

$$Tr_M(N) = \Sigma\{Imf | f \in Hom(N, M)\}$$

and

$$Re_N(M) = \cap\{kerf | f \in Hom(N, M)\},$$

for $N \in R - mod$
for more basic notion(see [5]). In [2] some properties of M- *ideals* and M-*prime* submodules related to subcategory of $\sigma[M]$ studies. In this paper we

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introduce the concept of M^*- condition and M-*coprime* as dual notion of these class of modules and obtain some basic results.

2 M^*-Condition Modules

Definition 1.1. Let M be an R-module. The submodule N of M has M^*-condition if there is a class C of modules in $\sigma[M]$ such that

$$N = Tr_M(C).$$

Note that although the definition of M^*- condition is given relative to the subcategory $\sigma[M]$; it is easy to check that N satisfy M^*-condition if and only if $N = Tr_M(C)$ for some class C in $R - Mod$.

Theorem 1.2 Let M be an R-module. For a submodule $N \subseteq M$ the following conditions are equivalent :
i) N satisfies M^*- condition ;
ii) $Im(g) \subseteq N$ for all $g \in Hom_R(N, M)$;
iii) $N = Tr_M(N)$.

Proof. *i*) $\Rightarrow$ *ii*) Assume that $N = Tr_M(C)$ for the class C of module in $\sigma[M]$. Then $Im(g) \subseteq N$ for all $g \in Hom_R(N, M)$.
ii) $\Rightarrow$ *iii*) Suppose that $Im(g) \subseteq N$ for all $g \in Hom_R(N, M)$. Then $Tr_M(N) \subseteq N$, and hence $N \subseteq Tr_N(C)$, since $Tr_M(N)$ is the unique largest submodule of M generated by N.
iii) $\Rightarrow$ *i*) This follows immediately from definition of M^*- condition.

Theorem 1.3 Let $\{N_\alpha\}_{\alpha \in I}$ be a family of submodules of M where every N_α satisfies M^*- condition. Then $\sum_{\alpha \in I} N_\alpha$ satisfies M^*- condition .

Proof. Suppose that every submodule N_α of M satisfies M^*. Let $g \in Hom_R(\Sigma N_\alpha, M)$, then $goi_\alpha = g_\alpha$ where i_α is the inclusion map from N_α into ΣN_α. Hence $Im(goi_\alpha) \subseteq N_\alpha$, since N_α satisfies M^*- condition(Theorem 1.2). Then $Im(g) = Im \sum g_\alpha = \sum Img_\alpha \subseteq \sum N_\alpha$ for all α. Therefore ΣN_α satisfies M^*- condition.

Theorem 1.4 Let N be a submodule of M with M^*- condition and $N \subseteq K$ be a submodule of M. Then N satisfies K^* - condition.

Proof. If $g \in Hom_R(N, K)$ then consider iog where i is the inclusion from K into M. We have $Im(iog) \subseteq N \subseteq K$, since N satisfy condition $M^\star$. Thus $Im(g) \subseteq N$; and hence N satisfies $K^\star$ - condition.

Definition 1.5 Let N be a submodule of M. For every R- module X define

$$N.X = Tr_X(C) = \sum\{Imf \mid f : W \longrightarrow X\},$$

where C is the class of modules W such that $Im(f) \subseteq N$ for all $f \in Hom_R(W, M)$. It follows immediately from the definition that $N.X = X$ if and only if $Im(f) \subseteq N$ for all $f \in Hom_R(X, M)$.

Remark 1.6 Let N be a submodule of M. It is easy to see that for every $R-$ module X, $N.X$ is the largest submodule Y of X such that $N.Y = Y$.

Proposition 1.7 Let M and X be R- modules. Then for all $g \in Hom_R(X, M)$, $\cap\ Ker(g) \subseteq N.X$.

Proof. We have $N(N.X) = N.X$, then $Im(f) \subseteq N$ for all $f \in Hom_R(N.X, M)$, then $Ker(f) = Ker(goi) \subseteq N.X$ where i is the inclusion map from $N.X$ into X, since $Ker(f) \subseteq N.X$.
Thus $i^{-1}Ker(g) \subseteq N.X$, and hence $\cap Ker(g) \subseteq N.X$ for all $g \in Hom_R(X, M)$.

Proposition 1.8 Let X and X_0 be R-module and X_0 be a submodule of X. Then $N.X_0 \subseteq N.X$, and for every R -homomorphism $f : X \to X_0$,$f(N.X) \subseteq N.X_0$.

Proof. Let $y \in Tr_{X_0}(C)$. Then there exists an $f \in Hom_R(C, X_0)$ such that $y \in Imf$. Now consider iof, where i is the inclusion map from X_0 into X. Since $y \in Im(iof) = Im(f)$, thus $y \in Tr_X(C)$.
Let $f(x) \in f(Tr_X(C))$. Then $x \in Tr_X(C) = \sum Im(h)$ for all $h \in Hom_R(C, X)$. Thus there exists $g \in Hom_R(W, X)$ such that $x \in Im(g)$ and $W \in C$, and hence; $x = g(y)$ for some $y \in W$.Thus $f(x) = fog(y)$. Therefore $f(x) \in Imfog \subseteq Tr_{X_0}(C)$.

Theorem 1.9 Let M be an R module and N be a submodule of M. Then for any R- module X we have $N.X = X$ if and only if $Tr_M(X) \subseteq N$.

Proof. We know that $N.X = X$ if and only if $Im(f) \subseteq N$ for all $f \in Hom_R(X, M)$, and this occurs if and only if $Tr_M(X) \subseteq N$.

Corollary 1.10 Let M be an R module. If N is a submodule of M,and $N.M = N$, then N satisfies M^*- condition.

Proof. We have $Tr_M(C) = N$ for all $g \in Hom_R(W, M)$ such that $Im(g) \subseteq N$ and $W \in C$ by Theorem 1.9, since $N.M = N$. Thus $Im(g) \subseteq N$ for $g \in Hom_R(N, M)$. Therefore N satisfy M^*- condition.

Theorem 1.11 Let N and K be submodules of M. Then the following hold:
i) If $N \subseteq K$, then $N.X \subseteq K.X$, for all X modules;
ii) If $N.K$ satisfy M^*- condition then N.K satisfies K^*- condition;
iii) $N.M$ is the largest submodule M contained N that satisfy M^*- condition;
iv) $N.K \subseteq N + K$.

Proof. i) We have $N.X = Tr_X(C)$ where C is the class of modules W such that $Im(f) \subseteq N$ for all $f \in Hom_R(W, M)$. Since $N \subseteq K$, then $Imf \subseteq K$, and hence $Tr_X(C) \subseteq K.X$. Therefore $N.X \subseteq K.X$.

ii) Clearly $N.K \subseteq K$. Since by hypothesis $N.K$ satisfies in M^*- condition, it follows, from Theorem 1.4 that $N.K$ satisfy K^*- condition.

iii) We have $N.M = Tr_M(C)$, where C is the class of modules in $\sigma[M]$. Thus $N.M$ satisfy condition M^*. We certainly have $N.M \subseteq M$, because $N.M = Tr_M(C)$, where C is the class of modules W such that $Im(f) \subseteq N$ for all $f \in Hom_R(W, M)$. Thus $Tr_M(C) \subseteq N$. Suppose that $L \subseteq N$, satisfies in M^*- condition and i, the inclusion map from L into M. If $x \in L$ then $i(x) \in L \subseteq N$. Thus $x \in Imi$, and hence $x \in N.M$.

iv) It is clear that $N.K \subseteq K$. On the other hand $N.K \subseteq N.M \subseteq N$ by (ii). Thus $N.K \subseteq N + K$.

3 M- Coprime Modules

In this section we introduce and study the notion of M- coprime modules and investigates the basic properties of such modules.
Definition 2.1 A module X is said to be M- *coprime* if $Hom_R(X, M) \neq 0$ and $Tr_M(X) = Tr_M(X/Y)$ for all submodules $Y \subseteq X$ such that $Hom_R(X/Y, M) \neq 0$.

Theorem 2.2 The following conditions are equivalent for any left R-module X such that $Hom_R(X, M) \neq 0$

i) X is an M- coprime module;

ii)If $N.(X/Y) = (X/Y)$ then $N.X = X$ for any submodule $N \subseteq M$ and any submodule $Y \subseteq X$ with $M.(X/Y) \neq 0$;

iii) If $N.(X/Y) = (X/Y)$ then $N.X = X$ for any submodule N, such that satisfy condition M^*.

Proof. $i) \Rightarrow ii)$ Let N be any submodule of M ,and Y be any submodule of X with $M.(X/Y) \neq 0$. Then $Hom_R(X/Y, M) \neq 0$ and so $Tr_M(X) = Tr_M(X/Y)$; since X is M- coprime. By assumption we obtain $N.(X/Y) = (X/Y)$ and so $N.X = X$. (1.9).

$ii) \Rightarrow iii)$ this is clear.

$iii) \Rightarrow i)$ Suppose that Y is nonzero submodule of X with $Hom_R(X/Y, M) \neq 0$ and $N = Tr_M(X/Y)$. By assumption there exists $0 \neq f \in Hom_R(\frac{X}{Y}, M)$.By using Theorem 1.9 we have $N(X/Y) = (X/Y)$, so $N.X = X$. Then $Tr_M(X) \subseteq N$. Thus $Tr_M(X) \subseteq Tr_M(X/Y)$, since $N = Tr_M(X/Y)$. Hence $Tr_M(X) = Tr_M(X/Y)$, showing that X is M- coprime module.

Theorem 2.3 Let X be an R-module such that $Hom_R(X, M) \neq 0$. If module X is M- coprime, then for each $m \in M - Tr_M(X)$ and each $0 \neq f \in Hom_R(X, M)$ $m \notin Im(g)$ for all $g \in Hom_R(X/Y, M)$.

Proof. Let Y be a submodule of X and $m \in M - Tr_M(C)$. Since X is M-coprime module ,then $Tr_M(X) = Tr_M(X/Y)$ such that $Hom_R(X, M) \neq 0$ and $Hom_R(X/Y, M) \neq 0$. By assumption $m \notin Tr_M(X)$ and so

$$m \notin Tr_M(X/Y) \Rightarrow \text{for all } g \in Hom_R(X/Y, M),\ m \notin Img.$$

Corollary 2.4 Let X be an M- coprime module and Y be submodule of X. A module X/Y is M- coprime if and only if $Hom_R(X/Y, M) \neq 0$.

Proof. ($\Rightarrow$)Easy.

For the converse, follows from Theorem 2.2(ii), since $Hom_R(X/Y, M) \neq 0$. Also for all submodule Z/Y of X/Y , $N(X/Z) = (X/Z)$ we have $NX = X$, since X is M- coprime. By Theorem 1.9 we obtain the $Tr_M(X) \subseteq N$, proving that $N(X/Y) = (X/Y)$. Thus X/Y is an M- coprime.

Corollary 2.5 Let X be an R-module such that from X, $Hom_R(X, M) \neq 0$ and $M.X \subseteq Ann_X(M)$. Then the following condition are equivalent:
i) X is an M- coprime module;

ii) $X/M.X$ is an M- coprime module;

iii) $X/Tr_M(X)$ is an M- coprime module.

Proof. $i) \Rightarrow ii)$ It follows from Corollary 2.4.

$ii) \Rightarrow iii)$ It follows from definition of $tr_M(X)$.

$iii) \Rightarrow i)$It follows from Corollary 2.4.

Theorem 2.6 For a nonzero module X the following conditions are equivalent.

i) Each factor module X generates X;

ii) X is M- coprime for every module M such that $\sigma[M] = \sigma[X]$ and X is cogenerated;

iii) X is M- coprime for every module M such that $Hom_R(X, M) \neq 0$.

Proof. $i) \Rightarrow iii)$ If Y is a nonzero submodule of X then for any module M we have $Tr_M(X) \subseteq Tr_M(X/Y)$, since X/Y is generator for X. Thus X is $M-$ coprime module if and only if $Hom_R(X, M) \neq 0$.

$iii) \Rightarrow ii)$ Trivial.

$ii) \Rightarrow i)$ Given any nonzero submodule Y of X; consider the module $M = X \oplus Y$,then X is $M-$ cogenerator and $\sigma[M] = \sigma[X]$ since Y belongs to $\sigma[X]$. By assumption X is $M-$ coprime, and so $Tr_M(X) = Tr_M(X/Y) = X$. This implies that $Tr_M(X/Y) = X$, and therefore X/Y generates X.

Theorem 2.7 The module nonzero M is an $M-$ coprime module if and only if every factor module of M, generates M.

Proof. Assume that M is an $M-$ coprime module. Then for any nonzero submodule N with $Hom_R(M/N, M) \neq 0$ we have $Tr_M(M) = Tr_M(M/N) = M$. It follows that M/N generates M.

Conversely, assume that factor module M, generates M, then for every nonzero submodule N we have $Tr_M(M/N) = M$.Thus $Tr_M(M) = Tr_M(M/N)$ and $Hom_R(M/N, M) \neq 0$ and so M is an $M-$ coprime .

Recall that an R- module X is said to be *comonoform* if each nonzero homomorphism $f : X \to X/Y$ is epimorphism.

Corollary 2.9 If M is comonoform module, then M is an $M-$ coprime module.

Theorem 2.10 If M is quasi-injective, then a submodule N of M satisfies M^*-condition if and only if it is a fully invariant submodule of M.

Proof. Suppose that M is quasi-injective, and N is a fully invariant submodule of M. If $f \in Hom(N, M)$,and i is the inclusion from N into M ,then f lifts to $\hat{f} \in End(M)$, with $f = \hat{f}oi$,since M is quasi-injective. Therefore $\hat{f}(N) \subseteq N$ since N is fully invariant, and so $f(N) = \hat{f}oi(N) = \hat{f}(N) \subseteq (N)$. Showing that N satisfies M^*- condition.

Theorem 2.11 Assume that M is quasi-injective, and let K and N be submodules of M. Then the following hold:

i)Assume that $N \subseteq K$ and K satisfies M^*-condition. If N satisfies K^*-condition and K is a direct summand of M, then N satisfies M^*-condition.

ii)If N and K satisfy M^*-condition, then $N \cap K$ satisfies M^*-condition.

Proof. i) Suppose that $f \in End(M)$, then we have the following diagram

$$K \to M \to M \to K$$

where i is inclusion map from K to M and πoi is projection from M onto K. Then $\pi ofoi(N) \subseteq N$, and $\pi of(N) \subseteq N$. Since K is direct summand of M and $f(N) \subseteq K$, we have $\pi oiof(N) = f(N)$. Therefore $f(N) \subseteq N$. By Proposition 2.10 N satisfies M^*-condition.

ii) Suppose that $f \in End(M)$. Since N and K satisfy M^*-condition, they are fully invarant submodule of M.Therefore $f(N) \subseteq N$ and $f(K) \subseteq K$. Thus $f(N \cap K) \subseteq f(N) \cap f(K) \subseteq N \cap K$. Therefore $N \cap K$ satisfies M^*-condition.

Definition 2.12 Let P satisfies M^*-condition. We call P is said to be a prime M^* if there exists an M- coprime module X such that $P = Tr_M(X)$.

Definition 2.13 Let P be a M^*- prime. We say that P is a M^*-prime associated to the module X if $P = Tr_M(Y)$ for an M- coprime submodule Y of X.

Theorem 2.14 Let M be an artinian module, and X be any module. If $Hom(X, M) \neq 0$, then X has an associated M^*- prime.

Proof. Consider the set of all proper submodule of M that satisfies condition M^* and are Trace of submodules of X. Since $Hom(X, M) \neq 0$, this set contains $Tr_M(X)$, and so it is nonempty.

Since M is artinian, there exists a minimal element P in the set, with $P = Tr_M(Y)$. If $\acute{Y}$ is any submodule of Y, then $Tr_M(Y/\acute{Y}) \subseteq Tr_M(Y)$, and so by definition, Y is M- coprime.

Definition 2.15 Let P satisfies M^*- condition. The R- module P is said to be a M^*- primitive if $P = Tr_M(S)$ for a simple module S.

Theorem 2.16 Let P satisfies M^*-condition.

i)If P is a minimal, then P is M^*-prime;

ii)If P is M^*- primitive, then P is M^*- prime .

Proof. i) Suppose that P is minimal and satisfies M^*- condition. If K is any nonzero submodule of P with $Hom(P/K, M) \neq 0$, then $Tr_M(P/K) \subseteq P = Tr_M(P)$ and $Hom(P, M) \neq 0$, so the minimality of P forces $Tr_M(P) = Tr_M(\frac{P}{K})$. Thus P is M- coprime module with $Tr_M(P) = P$.

ii)If P is M^*- primitive, then $P = Tr_M(S)$ for a simple left R- module S. It follows immediately from Corollary 2.9 that S is an M- coprime module.

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