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Reducts of Polyadic Equality Algebras without the Amalgamation Property

Tarek Sayed Ahmed

Department of Mathematics, Faculty of Science
Cairo University, Giza, Egypt
rutahmed@gmail.com

Abstract

Let $G \subseteq {}^{\omega}\omega$ be a monoid such that $\{[i|j] : i, j \in \omega\} \subseteq G$. Let $RPEA_G$ be the reduct of representable polyadic equality algebras obtained by restricting the similarity type of $RPEA_\omega$ to substitutions in G and PEA_G be the class of abstract polyadic algebras obtained from PEA_ω by restricting the similarity type and axiomatization of PEA_ω to substitutions in G and to finite quantifiers. Assume that G consists only of finite transformations. Then for any K, $RPEA_G \subseteq K \subseteq PEA_G$, K fails to have the amalgamation property with respect to PEA_G.

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Polyadic algebras were introduced by Halmos [10] to provide an algebraic reflection of the study of first order logic without equality. Later the algebras were enriched by diagonal elements to permit the discussion of equality. That the notion is indeed an adequate reflection of first order logic was demonstrated by Halmos' representation theorem for locally finite polyadic algebras (with and without equality). Daigneault and Monk proved a strong extension of Halmos' theorem, namely, every polyadic algebra (without equality) is representable [7]. The latter a typical Stone-like representation theorem, shows that the notion of polyadic algebra is indeed an adequate reflection of Keisler's predicate logic (KL). KL is a proper extension of first order logic without equality, obtained when the bound on the number of variables in formulas is relaxed; and accordingly allowing the following as extra operations on formulas: Quantification on infinitely many variables and simultaneous substitution of (infinitely many) variables for variables. Adding equality to KL, proved problematic as illustrated algebraically by Johnson [8]. In op.cit, Johnson showed that the class of representable polyadic algebras with equality is not closed under ultraproducts, hence this class is not elementary, i.e. cannot be axiomatized by any set of first order sentences. However one can still hope for a nice axiomatization of the *variety generated* by the class of polyadic equality algebras. A subtle recent (negative) result in this direction is Németi - Sági's [13]: In sharp contrast to KL, the validities of KL *with equality* cannot be recaptured by any set of schemas analogous to Halmos' schemas, let alone a finite one. In particular, the variety generated by the class of representable polyadic algebras with equality cannot be axiomatized by a finite schema of equations. The latter answers a question originally raised by Craig [5]. It is interesting (and indeed natural) to ask for algebraic versions of model theoretic results, other than completeness, reflected algebraically by Stone-like representability results. Daigneault succeeded in stating and proving versions of Beth's and Craig's theorems. This was done by proving the algebraic analogue of Robinson's joint consistency theorem: *Locally finite* polyadic algebras (with and without equality) have the amalgamation property. Later Johnson removed the condition of local finiteness, proving that polyadic algebras *without* equality have the *strong* amalgamation property [9] . With this stronger result, Robinson's, Beth's and Craig's theorems hold for KL. Here we show that the interpolation property fails for many reducts of KL *with* equality, contrasting the equality free case [2]. Our proof is algebraic adressing reducts of the class of polyadic algebras to substitutions in a fixed beforehand monoid of substitutions containing the set $\{[i|j] : i, j \in \omega\}$ of all replacements.

Definition 0.1. Let $G \subseteq {}^{\omega}\omega$ be a monoid such that $\{[i|j] : i, j \in \omega\} \subseteq G$. By a G polyadic equality algebra of dimension ω, or a PEA_G for short, we understand an algebra of the form

$$\mathfrak{A} = \langle A, +, ., -, 0, 1, \mathsf{c}_{(\Gamma)}, \mathsf{s}_{\tau}, \mathsf{d}_{ij}\rangle_{i,j\in\omega,\Gamma\subseteq\omega,\tau\in G}$$

where $\mathsf{c}_{(\Gamma)}$ $(\Gamma \subseteq \omega)$ and s_{τ} $(\tau \in G)$ are unary operations on A, $\mathsf{d}_{ij} \in A$ $(i, j \in \omega)$, such that postulates below hold for $x, y \in A$, $\tau, \sigma \in G$, $\Gamma, \Delta \subseteq \omega$ and all $i, j \in \omega$.

1. $\langle A, +, ., -, 0, 1\rangle$ is a boolean algebra,
2. $\mathsf{c}_{(\Gamma)}0 = 0$
3. $x \leq \mathsf{c}_{(\Gamma)}x$
4. $\mathsf{c}_{(\Gamma)}(x \cdot \mathsf{c}_{(\Gamma)}y) = \mathsf{c}_{(\Gamma)}x \cdot \mathsf{c}_{(\Gamma)}y$
5. $\mathsf{c}_{(\Gamma)}\mathsf{c}_{(\Delta)}x = \mathsf{c}_{(\Gamma\cup\Delta)}x$

6. s_τ is a boolean endomorphism
7. $\mathsf{s}_{Id}x = x$
8. $\mathsf{s}_{\sigma\circ\tau} = \mathsf{s}_\sigma \circ \mathsf{s}_\tau$
9. if $\sigma \restriction (\omega \sim \Gamma) = \tau \restriction (\omega \sim \Gamma)$, then $\mathsf{s}_\sigma \mathsf{c}_{(\Gamma)}x = \mathsf{s}_\tau \mathsf{c}_{(\Gamma)}x$
10. If $\tau^{-1}\Gamma = \Delta$ and $\tau \restriction \Delta$ is one to one, then $\mathsf{c}_{(\Gamma)}\mathsf{s}_\tau x = \mathsf{s}_\tau \mathsf{c}_{(\Delta)}x$.
11. $\mathsf{d}_{ii} = 1$
12. $\mathsf{s}_\tau \mathsf{d}_{ij} = \mathsf{d}_{\tau(i),\tau(j)}$
13. $x \cdot \mathsf{d}_{ij} \leq \mathsf{s}_{[i|j]}x$

The class of representable algebras is defined via set- theoretic operations on sets of ω-ary sequences. Let U be a set. For $i, j < \omega$, $\Gamma \subseteq \omega$ and $\tau \in G$, we set

$$\mathsf{C}_{(\Gamma)}X = \{s \in {}^\omega U : \exists t \in X,\ t(j) = s(j) \quad \forall j \notin \Gamma\}$$
$$\mathsf{S}_\tau X = \{s \in {}^\omega U : s \circ \tau \in X\}.$$
$$\mathsf{D}_{ij} = \{s \in {}^\omega U : s_i = s_j\}.$$

For a set X, let $\mathfrak{B}(X)$ be the boolean set algebra $(\wp(X), \cap, \cup, \sim)$. The class of representable G polyadic equality algebras, or $RPEA_G$ is defined by

$$SP\{\langle \mathfrak{B}({}^\omega U), \mathsf{C}_{(\Gamma)}, \mathsf{S}_\tau, \mathsf{D}_{ij}\rangle : i, j < \omega, \tau \in G,\ \ U \text{ a set }\}.$$

Here SP denotes the operation of forming subdirect products. The class of G set algebras is always properly contained in the class of abstract G algebras, for the latter is a finite schema axiomatizable variety while the former is not. When G consists only of finite transformations, then the class of G set algebras is not finite schema axiomatizable (a classical result of Sain), whereas if it contains at least one infinitary substitution, then it is even axiomatizable; it is not closed under ultrapowers [14]. $\tau \in {}^\omega\omega$ is a finite transformation if $\{i \in \omega : \tau(i) \neq i\}$ is finite. We recall the notion of the amalgamation property.

Definition 0.2. Let $K \subseteq V$ be classes of algebras. K is said to have the amalgamation property, or AP for short, with respect to V, if for all $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2 \in K$, and all monomorphisms i_1 and i_2 of $\mathfrak{A}_0$ into $\mathfrak{A}_1$, $\mathfrak{A}_2$, respectively, there exists $\mathfrak{A} \in V$, a monomorphism m_1 from $\mathfrak{A}_1$ into $\mathfrak{A}$ and a monomorphism m_2 from $\mathfrak{A}_2$ into $\mathfrak{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$.

We consider the case when G consists only of finite substitutions and the Γ's are restricted to finite subsets of ω. In this case we will show that for any $\mathbf{K}$, $RPEA_G \subseteq \mathbf{K} \subseteq PEA_G$, $\mathbf{K}$ fails to have the amalgamation property with respect to PEA_G. For motivations of studying such reducts, initiated by Craig [5], see [1], [2], [3], [14], [15], [16].

1 Proof

We write $X \subseteq_\omega Y$ if X is a finite subset of ω. We need:

Definition 1.1. Let $\mathfrak{A} \in PEA_G$. A subset I of $\mathfrak{A}$ in an ideal iff the following conditions are satisfied:

(i) $0 \in I$

(ii) If $x, y \in I$, then $x + y \in I$

(iii) If $x \in I$ and $y \leq x$ then $y \in I$

(iv) For all $\Gamma \subseteq_\omega \omega$ and $\tau \in G$ if $x \in I$ then $\mathsf{c}_{(\Gamma)}x$ and $\mathsf{s}_\tau x \in I$.

It can be checked that ideals function properly, that is ideals correspond to congruences the usual way. For $X \subseteq \mathfrak{A}$, the ideal generated by X, $\mathfrak{Ig}^{\mathfrak{A}}X$ is the smallest ideal containing X, i.e the intersection of all ideals containing X. We let $\mathfrak{Sg}^{\mathfrak{A}}X$ and sometimes $\mathfrak{A}^{(X)}$ denote the subalgebra of $\mathfrak{A}$ generated by X. We write $A^{(X)}$ to denote the universe of $\mathfrak{A}^{(X)}$ and we may write $\mathfrak{A}^{(x)}(A^{(x)})$ in place of $\mathfrak{A}^{(\{x\})}(A^{(\{x\})}$.) In what follows, we write c_i instead of the more cumbersome $\mathsf{c}_{(\{i\})}$, and we wite s^i_j in place of $\mathsf{s}_{[i|j]}$.

Lemma 1.2. *Let* $\mathfrak{A} \in PEA_G$ *and* $X \in A$. *Then* $\mathfrak{Ig}^{\mathfrak{A}}X = \{y \in A : y \leq \mathsf{c}_{(\Gamma)}(x_0 + \ldots x_{k-1})\}:$ *for some* $x \in {}^kX$, *and* $\Gamma \subseteq_\omega \omega\}$.

Proof. Let H denote the set of elements on the right hand side. Note that H does not depend on G. It is easy to check $H \subseteq \mathfrak{Ig}^{\mathfrak{A}}X$. Conversely, assume that $y \in H$, $\Gamma \subseteq_\omega \omega$. It is clear that $\mathsf{c}_{(\Gamma)}y \in H$. H is closed under substitutions, since for any $\tau \in G$, any $x \in A$ there exists $\Gamma \subseteq_\omega \omega$ such that $\mathsf{s}_\tau x \leq \mathsf{c}_{(\Gamma)}x$. Now let $z, y \in H$. Assume that $z \leq \mathsf{c}_{(\Gamma)}(x_0 + \ldots x_{k-1})$ and $y \leq \mathsf{c}_{(\Delta)}(y_0 + \ldots y_{l-1})$, then

$$z + y \leq \mathsf{c}_{(\Gamma \cup \Delta)}(x_0 + \ldots x_{k-1} + y_0 \ldots + y_{l-1}).$$

$\Gamma \cup \Delta$ is of course finite if both Γ and Δ are finite. The Lemma is proved. ■

The following about ideals in G algebras will be frequently used.

- If $\mathfrak{A} \subseteq \mathfrak{B}$ are PEA_Gs and I is an ideal of $\mathfrak{A}$, then $\mathfrak{Ig}^{\mathfrak{B}}(I) = \{b \in B : \exists a \in I(b \leq a)\}$.
- If I and J are ideals of a PEA_G then the ideal generated by $I \cup J$ is $\{x : x \leq i + j \text{ for } i \in I, j \in J\}$.

We also use that if $\mathfrak{A} \in PEA_G$ then its cylindric reduct is a cylindric algebra. For a class K and a non-zero cardinal β, $\mathfrak{Fr}_\beta K$ denotes the free K algebra on β generators. We recall that HK denotes the class of homomorphic images of members of K and SK denotes the class of subalgebras of members of K. We now prove:

Theorem 1.3. *Let $RPEA_G \subseteq \mathbf{K} \subseteq PEA_G$. Then $\mathbf{K}$ does not have AP with respect to PEA_G*

Proof. Seeking a contradiction, assume that $\mathbf{K}$ has AP with respect to PEA_G. Let $\mathfrak{A} = \mathfrak{Fr}_4 PEA_G$. Let r, s and t be defined as follows:

$$r = \mathsf{c}_0(x \cdot \mathsf{c}_1 y) \cdot \mathsf{c}_0(x. - \mathsf{c}_1 y),$$

$$s = \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 z \cdot \mathsf{s}_1^0 \mathsf{c}_1 z \cdot -\mathsf{d}_{01}) + \mathsf{c}_0(x \cdot -\mathsf{c}_1 z),$$
$$t = \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 w \cdot \mathsf{s}_1^0 \mathsf{c}_1 w - \mathsf{d}_{01}) + \mathsf{c}_0(x \cdot -\mathsf{c}_1 w),$$

where $x, y, z,$ and w are the first four free generators of $\mathfrak{A}$. Then $r \leq s \cdot t$. Indeed, let

$$a = x \cdot \mathsf{c}_1 y \cdot -\mathsf{c}_0(x \cdot -\mathsf{c}_1 z),$$
$$b = x \cdot -\mathsf{c}_1 y \cdot -\mathsf{c}_0(x \cdot -\mathsf{c}_1 z).$$

Then we have

$$\begin{aligned} \mathsf{c}_1 a \cdot \mathsf{c}_1 b &\leq \mathsf{c}_1(x \cdot \mathsf{c}_1 y) \cdot \mathsf{c}_1(x \cdot -\mathsf{c}_1 y) \quad \text{by [11]1.2.7} \\ &= \mathsf{c}_1 x \cdot \mathsf{c}_1 y \cdot \mathsf{c}_1 x \cdot -\mathsf{c}_1 y \quad \text{by [11] 1.2.11} \end{aligned}$$

and so

$$\mathsf{c}_1 a \cdot \mathsf{c}_1 b = 0. \tag{1}$$

From the inclusion $x \cdot -\mathsf{c}_1 z \leq \mathsf{c}_0(x \cdot -\mathsf{c}_1 z)$ we get

$$x \cdot -\mathsf{c}_0(x \cdot -\mathsf{c}_1 z) \leq \mathsf{c}_1 z.$$

Thus $a, b \leq \mathsf{c}_1 z$ and hence, by [11] 1.2.9,

$$\mathsf{c}_1 a, \mathsf{c}_1 b \leq \mathsf{c}_1 z. \tag{2}$$

We now compute:

$$\begin{aligned} \mathsf{c}_0 a \cdot \mathsf{c}_0 b &\leq \mathsf{c}_0\mathsf{c}_1 a \cdot \mathsf{c}_0\mathsf{c}_1 b \quad \text{by [11] 1.2.7} \\ &= \mathsf{c}_0\mathsf{c}_1 a \cdot \mathsf{c}_1\mathsf{s}_1^0\mathsf{c}_1 b \quad \text{by [11] 1.5.8 (i), [11] 1.5.9 (i)} \\ &= \mathsf{c}_1(\mathsf{c}_0\mathsf{c}_1 a \cdot \mathsf{s}_1^0\mathsf{c}_1 b) \\ &= \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 a \cdot \mathsf{s}_1^0\mathsf{c}_1 b) \\ &= \mathsf{c}_0\mathsf{c}_1[\mathsf{c}_1 a \cdot \mathsf{s}_1^0\mathsf{c}_1 b \cdot (-\mathsf{d}_{01} + \mathsf{d}_{01}) \\ &= \mathsf{c}_0\mathsf{c}_1[(\mathsf{c}_1 a \cdot \mathsf{s}_1^0\mathsf{c}_1 b \cdot -\mathsf{d}_{01}) + (\mathsf{c}_1 a \cdot \mathsf{s}_1^0\mathsf{c}_1 b \cdot \mathsf{d}_{01})] \\ &= \mathsf{c}_0\mathsf{c}_1[(\mathsf{c}_1 a \cdot \mathsf{s}_1^0\mathsf{c}_1 b \cdot -\mathsf{d}_{01}) + (\mathsf{c}_1 a \cdot \mathsf{c}_1 b \cdot \mathsf{d}_{01})] \quad \text{by [11] 1.5.5} \\ &= \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 a \cdot \mathsf{s}_1^0\mathsf{c}_1 b \cdot -\mathsf{d}_{01}) \quad \text{by (1)} \\ &\leq \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 z \cdot \mathsf{s}_1^0\mathsf{c}_1 z \cdot -\mathsf{d}_{01}) \quad \text{by (2), [11] 1.2.7} \end{aligned}$$

We have proved that

$$\mathsf{c}_0[x\cdot \mathsf{c}_1 y\cdot -\mathsf{c}_0(x\cdot -\mathsf{c}_1 z)]\cdot \mathsf{c}_0[x\cdot -\mathsf{c}_1 y\cdot -\mathsf{c}_0(x\cdot -\mathsf{c}_1 z)]\leq \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 z\cdot \mathsf{s}^0_1\mathsf{c}_1 z.-\mathsf{d}_{01}).$$

In view of [11] 1.2.11 and axiom (C_3) of [11] 1.1.1 this gives

$$\mathsf{c}_0(x\cdot \mathsf{c}_1 y)\cdot \mathsf{c}_0(x\cdot -\mathsf{c}_1 y)\cdot -\mathsf{c}_0(x\cdot -\mathsf{c}_1 z)\leq \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 z\cdot \mathsf{s}^0_1\mathsf{c}_1 z\cdot -\mathsf{d}_{01}).$$

The conclusion now follows. Let $X_1=\{x,y\}$ and $X_2=\{x,z,w\}$. Then

$$\mathfrak{A}^{(X_1\cap X_2)}=\mathfrak{Sg}^{\mathfrak{A}}\{x\}. \tag{3}$$

We have

$$r\in A^{(X_1)} \text{ and } s,t\in A^{(X_2)}. \tag{4}$$

Let R be an ideal of $\mathfrak{A}$ such that

$$\mathfrak{A}/R\cong \mathfrak{Fr}_4\mathbf{K}_\omega. \tag{5}$$

Since $r\leq s\cdot t$ we have

$$r\in \mathfrak{Ig}^{\mathfrak{A}}\{s\cdot t\}\cap A^{(X_1)}. \tag{6}$$

Let

$$M=\mathfrak{Ig}^{\mathfrak{A}^{(X_2)}}[\{s\cdot t\}\cup (R\cap A^{(X_2)})]; \tag{7}$$

$$N=\mathfrak{Ig}^{\mathfrak{A}^{(X_1)}}[(M\cap A^{(X_1\cap X_2)})\cup (R\cap A^{(X_1)})]. \tag{8}$$

Then we have

$$R\cap A^{(X_2)}\subseteq M \text{ and } R\cap A^{(X_1)}\subseteq N. \tag{9}$$

From the first of these inclusions we get

$$M\cap A^{(X_1\cap X_2)}\supseteq (R\cap A^{(X_2)})\cap A^{(X_1\cap X_2)}=(R\cap A^{(X_1)})\cap A^{(X_1\cap X_2)}.$$

By (8) we have

$$N\cap A^{(X_1\cap X_2)}=M\cap A^{(X_1\cap X_2)}.$$

From this we get

$$(\mathfrak{A}^{(X_2)}/M)^{(X_1\cap X_2)}\cong \mathfrak{A}^{(X_1\cap X_2)}/M\cap A^{(X_1\cap X_2)} \tag{10}$$

$$=\mathfrak{A}^{(X_1\cap X_2)}/N\cap A^{(X_1\cap X_2)}\cong (\mathfrak{A}^{(X_1)}/N)^{(X_1\cap X_2)}.$$

From (5) and (9) we have $\mathfrak{A}^{(X_2)}/M$ is in $HS\mathbf{K}$. By a similar argument $\mathfrak{A}^{(X_1)}/N$ is in $HS\mathbf{K}$. By our assumption, there is an amalgam, i.e. there is a $\mathfrak{B}$, a $Y = \{y_0, y_1, y_2, y_3\}$ generating $\mathfrak{B}$ and

$$\mathfrak{B}^{(Y_1)} \cong \mathfrak{A}^{(X_1)}/N, \quad \mathfrak{B}^{(Y_2)} \cong \mathfrak{A}^{(X_2)}/M.$$

Here $Y_1 = \{y_0, y_1\}$ and $Y_2 = \{y_0, y_2, y_3\}$. Let P be the ideal of $\mathfrak{A}$ such that $\mathfrak{A}/P \cong \mathfrak{B}$. Then

$$\mathfrak{A}^{(X_2)}/P \cap A^{(X_2)} \cong (\mathfrak{A}/P)^{(X_2)} \cong \mathfrak{B}^{(Y_2)} \cong \mathfrak{A}^{(X_2)}/M.$$

Thus

$$P \cap A^{(X_1)} = N \tag{11}$$

and

$$P \cap A^{(X_2)} = M. \tag{12}$$

In view of $(4), (7), (11)$ we have $s \cdot t \in P$ and hence by (6) $r \in P$. Consequently from (4) and (12) we get $r \in N$. From (8) there exists elements

$$u \in M \cap A^{(X_1 \cap X_2)} \tag{13}$$

and $b \in R$ such that

$$r \leq u + b. \tag{14}$$

Since $u \in M$ by (7) there is a $\Gamma \subseteq_\omega \omega$ and $c \in R$ such that

$$u \leq \mathsf{c}_{(\Gamma)}(s \cdot t) + c.$$

Let $\{x', y', z', w'\}$ be the first four generators of $\mathfrak{D} = \mathfrak{Fr}_4\mathbf{K}$. Let h be the homomorphism from $\mathfrak{A}$ to $\mathfrak{D}$ be such that $h(i) = i'$ for $i \in \{x, y, w, z\}$. Notice that $kerh = R$. Then $h(b) = h(c) = 0$. It follows that

$$h(r) \leq h(u) \leq \mathsf{C}_{(\Gamma)}(h(s) \cdot h(t).)$$

Let $r' = h(r)$, $u' = h(u)$, $s' = h(s)$ and $t' = h(t)$. Let

$$\mathfrak{B} = (\wp({}^\omega\omega), \cup, \cap, \sim, \emptyset, {}^\omega\omega, \mathsf{C}_{(\Gamma)}, \mathsf{D}_{\kappa\lambda}, \mathsf{S}_\tau)_{\kappa,\lambda<\omega,\Gamma\subseteq_\omega\omega,\tau\in G}$$

that is $\mathfrak{B}$ is the G full set algebra in the space ${}^\omega\omega$. Let E be the set of all equivalence relations on ω, and for each $R \in E$ set

$$X_R = \{\varphi : \varphi \in {}^\omega\omega \quad \text{and, for all} \quad \xi, \eta < \omega, \varphi_\xi = \varphi_\eta \quad \text{iff} \quad \xi R \eta\}.$$

Let

$$C = \{\bigcup_{R\in L} X_R : L \subseteq E\}.$$

C is clearly closed under the formation of arbitrary unions, and since

$$\sim \bigcup_{R\in L} X_L = \bigcup_{R\in E\sim L} X_R$$

for every $L \subseteq E$, we see that C is closed under the formation of complements with respect to ${}^\omega\omega$. Thus C is a Boolean subuniverse (indeed, a complete Boolean subuniverse) of $\mathfrak{B}$; moreover, it is obvious that

$$X_R \quad \text{is an atom of} \quad (C, \cup, \cap, \sim, 0, {}^\omega\omega) \quad \text{for each} \quad R \in E. \tag{15}$$

For all $\kappa, \lambda < \omega$ we have $\mathsf{D}_{\kappa\lambda} = \bigcup\{X_R : (\kappa, \lambda) \in R \in E\}$ and hence $\mathsf{D}_{\kappa\lambda} \in B$. Also,

$$\mathsf{C}_{(\Gamma)} X_R = \bigcup\{X_S : S \in E, {}^2(\omega \sim \Gamma) \cap S = {}^2(\omega \sim \Gamma) \cap R\}$$

for any $\Gamma \subseteq_\omega \omega$ and $R \in E$. Thus, because $\mathsf{C}_{(\Gamma)}$ is completely additive, C is closed under the operation $\mathsf{C}_{(\Gamma)}$ for every $\Gamma \subseteq_\omega \omega$. It is easy to show that C is closed under substitutions. For any $\tau \in {}^\omega\omega$,

$$\mathsf{S}_\tau X_R = \bigcup\{X_S : S \in E, \forall i, j < \omega(iRj \longleftrightarrow \tau(i)S\tau(j)\}.$$

The set on the right may of course be empty. Therefore, we have shown that

$$C \quad \text{is a subuniverse of} \quad \mathfrak{B}. \tag{16}$$

We now show that there is a subset Y of ${}^\omega\omega$ such that

$$\begin{array}{l} X_{Id} \cap f(r') \neq 0 \ \text{ for every } \ f \in Hom(\mathfrak{D}, \mathfrak{B}) \\ \text{such that } \ f(x') = X_{Id} \ \text{ and } \ f(y') = Y, \end{array} \tag{17}$$

and also that for every $\Gamma \subseteq_\omega \omega$, there are subsets Z, W of ${}^\omega\omega$ such that

$$\begin{array}{l} X_{Id} \sim \mathsf{C}_{(\Gamma)} g(s' \cdot t') \neq 0 \ \text{ for every } \ g \in Hom(\mathfrak{D}, \mathfrak{B}) \\ \text{such that } \ g(x') = X_{Id}, g(z') = Z \ \text{ and } \ g(w') = W. \end{array} \tag{18}$$

Here $Hom(\mathfrak{A}, \mathfrak{B})$ stands for the set of all homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$. Let $\sigma \in {}^\omega\omega$ be such that $\sigma_0 = 0$, and $\sigma_\kappa = \kappa + 1$ for every non-zero $\kappa < \omega$. Let $\tau = \sigma \upharpoonright (\omega \sim \{0\}) \cup \{(0, 1)\}$. Then $\sigma, \tau \in X_{Id}$. Take

$$Y = \{\sigma\}.$$

Then

$$\sigma \in X_{Id} \cap \mathsf{C}_1 Y \quad \text{and} \quad \tau \in X_{Id} \sim \mathsf{C}_1 Y$$

and hence

$$\sigma \in \mathsf{C}_0(X_{Id} \cap \mathsf{C}_1 Y) \cap \mathsf{C}_0(X_{Id} \sim \mathsf{C}_1 Y). \tag{19}$$

Therefore, we have $\sigma \in f(r)$ for every $f \in Hom(\mathfrak{A}, \mathfrak{B})$ such that $f(x) = X_{Id}$ and $f(y) = Y$, and that (17) holds. We now want to show that for any given $\Gamma \subseteq_\omega \omega$ and $H \subseteq G$, there exist sets $Z, W \subseteq {}^\omega\omega$ such that (18) holds; it is clear that no generality is lost if we assume that $0, 1 \in \Gamma$, so we make this assumption. Take

$$Z = \{\varphi : \varphi \in X_{Id}, \varphi_0 < \varphi_1\} \cap \mathsf{C}_{(\Gamma)}\{Id\}$$

and

$$W = \{\varphi : \varphi \in X_{Id}, \varphi_0 > \varphi_1\} \cap \mathsf{C}_{(\Gamma)}\{Id\}.$$

We first show that

$$Id \in X_{Id} \sim \mathsf{C}_{(\Gamma)} g(s \cdot t) \tag{20}$$

for any $g \in Hom(\mathfrak{A}, \mathfrak{B})$ such that $g(x) = X_{Id}$, $g(z) = Z$, and $g(w) = W$; to do this we simply compute the value of $\mathsf{C}_{(\Gamma)} g(s \cdot t)$. For the purpose of this computation we make use of the following property of ordinals: if Δ is any non-empty set of ordinals, then $\bigcap \Delta$ is the smallest ordinal in Δ, and if, in addition, Δ is finite, then $\bigcup \Delta$ is the largest element ordinal in Δ. Also, in this computation we shall assume that φ always represents an arbitrary sequence in ${}^\omega\omega$. Then, setting

$$\Delta\varphi = \Gamma \sim \varphi[\Gamma \sim \{0, 1\}]$$

for every φ, we successively compute:

$$\mathsf{C}_1 Z = \{\varphi : |\Delta\varphi| = 2, \varphi_0 = \bigcap \Delta\varphi\} \cap \mathsf{C}_{(\Gamma)}\{Id\},$$

$$\begin{aligned}&(X_{Id} \sim \mathsf{C}_1 Z) \cap \mathsf{C}_{(\Gamma)}\{Id\} = \\ &\{\varphi : |\Delta\varphi| = 2, \varphi_0 = \bigcup \Delta\varphi, \varphi_1 = \bigcap \Delta\varphi\} \cap \mathsf{C}_{(\Gamma)}\{Id\},\end{aligned}$$

and, finally,

$$\begin{aligned}&\mathsf{C}_0(X_{Id} \sim \mathsf{C}_1 Z) \cap \mathsf{C}_{(\Gamma)}\{Id\} = \\ &\qquad \{\varphi : |\Delta\varphi| = 2, \varphi_1 = \bigcap \Delta\varphi\} \cap \mathsf{C}_{(\Gamma)}\{Id\}.\end{aligned} \tag{21}$$

Similarly, we obtain

$$\begin{aligned}&\mathsf{C}_0(X_{Id} \sim \mathsf{C}_1 W) \cap \mathsf{C}_{(\Gamma)}\{Id\} = \\ &\qquad \{\varphi : |\Delta\varphi| = 2, \varphi_1 = \bigcup \Delta\varphi\} \cap \mathsf{C}_{(\Gamma)}\{Id\}.\end{aligned}$$

The last two formulas together give

$$\mathsf{C}_0(X_{Id} \sim \mathsf{C}_1 Z) \cap \mathsf{C}_0(X_{Id} \sim \mathsf{C}_1 W) \cap \mathsf{C}_{(\Gamma)}\{Id\} = 0. \tag{22}$$

Continuing the computation we successively obtain:

$$\mathsf{C}_1 Z \cap \mathsf{D}_{01} = \{\varphi : |\Delta\varphi| = 2, \varphi_0 = \varphi_1 = \bigcap \Delta\varphi\} \cap \mathsf{C}_{(\Gamma)}\{Id\},$$

$$\mathsf{S}^0_1 \mathsf{C}_1 Z = \{\varphi : |\Delta\varphi| = 2, \varphi_1 = \bigcap \Delta\varphi\} \cap \mathsf{C}_{(\Gamma)}\{Id\},$$

$$\mathsf{C}_1 Z \cap \mathsf{S}^0_1 \mathsf{C}_1 Z = \{\varphi : |\Delta\varphi| = 2, \varphi_0 = \varphi_1 = \bigcap \Delta\varphi\} \cap \mathsf{C}_{(\Gamma)}\{Id\};$$

hence we finally get

$$\mathsf{C}_0 \mathsf{C}_1(\mathsf{C}_1 Z \cap \mathsf{S}^0_1 \mathsf{C}_1 Z \cap \sim \mathsf{D}_{01}) = \mathsf{C}_0 \mathsf{C}_1 0 = 0, \tag{23}$$

and similarly we get

$$\mathsf{C}_0 \mathsf{C}_1(\mathsf{C}_1 W \cap \mathsf{S}^0_1 \mathsf{C}_1 W \cap \sim \mathsf{D}_{01}) = 0. \tag{24}$$

Now take g to be any homomorphism from $\mathfrak{D}$ into $\mathfrak{B}$ such that $g(x') = X_{Id}$, $g(z') = Z$ and $g(w') = W$. Let $a = g(s' \cdot t')$. Then from the above

$$a \cap \mathsf{C}_{(\Gamma)}\{Id\} = \emptyset.$$

Then applying $\mathsf{C}_{(\Gamma)}$ to both sides of this equation we get

$$\mathsf{C}_{(\Gamma)} a \cap \mathsf{C}_{(\Gamma)}\{Id\} = \emptyset.$$

Thus (20) holds. Now there exists a finite $\Gamma \subseteq \omega$ and an interpolant $u' \in \mathfrak{D}^{(x)}$, that is

$$r' \leq u' \leq \mathsf{c}_{(\Gamma)}(s' \cdot t').$$

There also exist $Y, Z, W \subseteq {}^\omega\omega$ such that (17) and (18) hold. Take any $k \in Hom(\mathfrak{D}, \mathfrak{B})$ such that $k(x') = X_{Id}$, $k(y') = Y$, $k(z') = Z$, and $k(w') = W$. This is possible by the freeness of $\mathfrak{D}$. Then using the fact that $X_{Id} \cap k(r')$ is non-empty by (17) we get

$$X_{Id} \cap k(u') = k(x' \cdot u') \supseteq k(x' \cdot r') \neq 0.$$

And using the fact that $X_{Id} \sim \mathsf{C}_{(\Gamma)} k(s' \cdot t')$ is non-empty by (18) we get

$$X_{Id} \sim ku' = k(x' \cdot -u') \supseteq k(x \cdot -\mathsf{c}_{(\Gamma)}(s' \cdot t')) \neq 0.$$

However, in view of (15), it is impossible for X_{Id} to intersect both $k(u')$ and its complement since $k(u') \in C$ and X_{Id} is an atom; to see that $k(u')$ is indeed contained in C recall that $u' \in \mathfrak{D}^{(x')}$, and then observe that because of (16) and the fact that $X_{Id} \in C$ we must have $k[\mathfrak{D}^{(x')}] \subseteq C$.

■

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