

A Note on Atom Structures of Relation and Cylindric Algebras

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Abstract

RRA stands for the class of representable relation algebras, while **RCA_n** stands for the class of cylindric algebras of dimension n .

We give a simple new proof to the following.

1. There exist two atomic relation algebras with the same atom structure, only one of which is representable.
2. **RRA** is not closed under completions and is not atom-canonical.
3. There exists a non-representable relation algebra with a dense representable subalgebra
4. **RRA** is not Sahlqvist axiomatizable.
5. There exists an atomic relation algebra with no complete representation.

And same for **RCA_n** for $2 < n < \omega$, in place of **RA**.

Our proof substantially simplifies the proof of Hodkinson in [4].

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The results of this note are known, but the proof is new and is much simpler than Hodkinson's proof in [4]. We follow the notation of [3]. In particular, we assume familiarity of constructing relation algebra atom structures by defining the forbidden triples of atoms.

1 The atom structure

Let $n > 2$. let $k = (n - 1)(n - 2)/2$. We define a relation algebra atom structure α with atoms $1'$, the sole identity atom, and $[j, i]$, $j < n$ and $i \in \mathbb{N}$. All atoms are self converse. For an atom $[j, i]$ j is its colour. $1'$ has no colour. To define composition, we list the forbidden triples of atoms. Consider the graph (\mathbb{N}, E) with nodes \mathbb{N} and i, l is an edge i.e $(i, l) \in E$ if $0 < |i - l| < k$. The forbidden triples are:

$$(1', a, b), (a, 1', b), (a, b, 1'), a \neq b$$

$$([s, i], [s, j], [s, l])$$

where $s < n$ and $i, j, l \in \mathbb{N}$ are independent in the graph (\mathbb{N}, E) , that is $(x, y) \notin E$ for $x, y \in \{i, j, l\}$.

Let $\mathfrak{Cm}\alpha$ be the complex algebra over α . It is easy to check that $\mathfrak{Cm}\alpha$ is a relation algebra (that is α is a relation algebra atom structure).

Theorem 1.1. $\mathfrak{Cm}\alpha$ is not representable

Proof. Assume for contradiction that $g : \mathfrak{Cm}\alpha \rightarrow \mathfrak{B}$ is an embedding into a proper relation set algebra \mathfrak{B} with base set X . Each $h(a)$ ($a \in \mathfrak{Cm}\alpha$) is a binary relation on X , and h respects the relation algebra operations. For $Y \subseteq \mathbb{N}$ and $s < n$, set

$$[s, Y] = \{[s, l] : l \in Y\}.$$

For $r \in \{0, \dots, k - 1\}$, $k\mathbb{N} + r$ denotes the set $\{kq + r : q \in \mathbb{N}\}$. Let

$$J = \{1', [s, k\mathbb{N} + r] : r < k, s < n\}.$$

Then $\sum J = 1$ in $\mathfrak{Cm}\alpha$. As J is finite, we have for any $x, y \in X$ there is a $P \in J$ with $(x, y) \in h(P)$. Since $\mathfrak{Cm}\alpha$ is infinite then X is infinite. By Ramsey's Theorem, there are distinct $x_i \in X$ ($i < \omega$) and $P \in J$ such that $(x_i, x_j) \in h(P)$ for all $i < j < \omega$. Clearly $P \neq 1'$. Also $(P; P).P \neq 0$. This follows from that if $x_0, x_1, x_2 \in X$, $a, b, c \in \mathfrak{Cm}\alpha$, $(x_0, x_1) \in h(a)$, $(x_1, x_2) \in h(b)$, and $(x_0, x_2) \in h(c)$, then $(a; b).c \neq 0$. A non-zero element a of $\mathfrak{Cm}\alpha$ is monochromatic, if $a \leq 1'$, or $a \leq [s, \mathbb{N}]$ for some $s < n$. Now P is monochromatic, it follows from the definition of α that $(P; P).P = 0$. This contradiction shows that $\mathfrak{Cm}\alpha$ is not representable. ■

We have that $\mathfrak{Cm}H$ is an n dimensional cylindric algebra. Let $\mathfrak{Tm}H$ be the subalgebra generated by the atoms of $\mathfrak{Cm}H$. Then we have $\mathfrak{Tm}H \in \mathbf{CA}_n$. We will go further. We show that $\mathfrak{Tm}H$ is in fact representable.

We define an atom structure $\alpha' = \alpha \cup \{[i, \infty] : i < n\}$ by adding n new elements to α . We list the forbidden triples. These are as before

$$(1', a, b), (a, 1', b), (a, b, 1'), a \neq b$$

$$([s, i], [s, j], [s, l])$$

where $s < n$ and $i, j, l \in \mathbb{N}$ are independent ; and the new

$$([s, i], [s, j], [s, \infty])$$

if $i = j$ or $|i - j| \geq k$.

Let H , as before, be the set of atomic α networks. Let $\mathfrak{Tm}H$ be the cylindric term algebra over H and let $\text{Uf}\mathfrak{Tm}(H)$ be the atom structure consisting of ultrafilters of $\mathfrak{Tm}H$. Let H' be the set of atomic α' networks. We will show that as first order structures $H' \cong \text{Uf}\mathfrak{Tm}H$, and that $\mathfrak{Cm}H'$ is representable.

Notation, cf. [3] 13.30. For $\bar{a} \in {}^{<n}n$ we write $\mathfrak{s}_{\bar{a}}$ for an arbitrary string of substitutions w such that $\hat{w} = \bar{a}$.

In more detail. Let $n \geq 3$ and $i, j < n$. We define a string of substitutions \mathfrak{s}_{ij} :

$$\begin{aligned} \mathfrak{s}_{ij} &= \mathfrak{s}_i^0 \mathfrak{s}_j^1, \text{ if } j \neq 0 \\ \mathfrak{s}_{ij} &= \mathfrak{s}_0^1 \mathfrak{s}_i^0 \text{ iff } j = 0, i \neq 1 \\ \mathfrak{s}_{ij} &= \mathfrak{s}_0^2 \mathfrak{s}_1^0 \mathfrak{s}_2^1 \text{ iff } j = 0, i = 1 \end{aligned}$$

For every ultrafilter μ in $\text{Uf}\mathfrak{Tm}H$ and each $i, j < n$ either

- there is $a \in \alpha$, $\mathfrak{s}_{ij}a \in \mu$, or
- there is $c < n$ and for all cofinite $S \subseteq \mathbb{N}$, $\mathfrak{s}_{ij}[c, S] \in \mu$. (Note that here \mathfrak{s}_{ij} is string of substitutions mapping $(0, 1)$ to (i, j) .)

For each $\mu \in \text{Uf}\mathfrak{Tm}H$ define an α' labelled graph N_μ with nodes n labelled

$$N_\mu(i, j) = a \in \alpha \text{ if } \mathfrak{s}_{ij}a \in \mu$$

and

$$N_\mu(i, j) = [c, \infty] \text{ if } \mathfrak{s}_{ij}[c, S] \in \mu \text{ for all cofinite } S.$$

It is easy to see that for all $\mu \in \text{Uf}\mathfrak{Tm}H$ we have $N_\mu \in H'$. Now we have to work in the other direction:

Definition 1.2. Let $N \in H'$

- Write $A(N)$ for the set of edges (i, j) of N such that $N(i, j) \in \alpha$ and write $E(N)$ for all other edges. Note that $(i, j) \in E(N) \implies \exists l < n, N(i, j) = [l, \infty]$.
- Let $S \subseteq \mathbb{N}$. For any $\rho : E(N) \rightarrow S$, let N_ρ be the symmetrically labelled graph with nodes $\{0, 1, \dots, n - 1\}$ satisfying

$$\begin{aligned} N_\rho(i, j) &= N(i, j) \text{ if } (i, j) \in A(N) \\ &= [c, \rho(i, j)] \text{ if } (i, j) \in E(N), N(i, j) = [c, \infty] . \end{aligned}$$

- Let

$$[N, S] = \{N_\rho : \rho \text{ is an injection } E(N) \rightarrow S\}$$

- For $k < \omega$, let

$$[N, S, k] = \{N_\rho \in [N, S] : Rg(\rho) \subseteq [s_0, s_0 + k - 1] \subseteq S, \text{ for some } s_0 \in S\}$$

Lemma 1.3. *Let $N \in H'$ and let S be a cofinite subset of \mathbb{N} . Every $N_\rho \in [N, S, k]$ is a network, hence is in H .*

Proof. Let $N_\rho \in [N, S, k]$ and let $i, j, k < n$. ρ is an injection $E(N) \rightarrow [s_0, s_0 + k - 1] \subseteq S$, for some $s_0 \in S$. We have to show that $(N_\rho(i, j), N_\rho(j, k), N_\rho(i, k))$ is not forbidden. If one of the edges is labelled by the identity, the result holds as N_ρ is symmetric and all tuples $(1', x, x)$ are consistent. Suppose none of the labels are the identity. If the three colours of the labels are not all the same then the triangle by definition is consistent. So assume that all three edges have colour $c < n$. If all three edges belong to $A(N)$, then consistency is assured since $N \in H'$ and N_ρ agrees with N on such edges. If exactly one of the three edges is in $E(N)$, say the labels in N are $([c, \infty], [c, p], [c, q])$ for some $p, q \in \mathbb{N}$, then by the consistency of N we have $0 < |p - q| < k$. The triple of atoms labelling the triangle $(i, j, k) \in N_\rho$ is $([c, \rho(i, j)], [c, \rho(j, k)], [c, p])$ for some $p \in \mathbb{N}$, and this is consistent since $0 < |p - q| < k$. If two or more of the three atoms belong to $E(N)$, the triple of atoms in N_ρ is $([c, \rho(i, j)], [c, \rho(j, k)], [c, p])$ for some $p \in \mathbb{N}$. Since ρ is injective, we know that $\rho(i, j) \neq \rho(j, k)$ and since $Rg(\rho) \subseteq [s_0, s_0 + k - 1]$, we know that $|\rho(i, j) - \rho(j, k)| < k$ hence the triangle is consistent. Thus $N_\rho \in H$. ■

Since $|E(N)| \leq k$ and S is cofinite, injections $\rho : E(N) \rightarrow [s_0, s_0 + k - 1] \subseteq S$ exist. Hence

Lemma 1.4. *Let $N \in H'$ and let S be a cofinite subset of \mathbb{N} . $[N, S, k] \subseteq H$ is non-empty.*

For $S \subseteq \mathbb{N}$ $M, N \in H$ write $M \sim_S N$ iff for all $i, j < n$ we either have $M(i, j) = N(i, j)$ or $M(i, j) = [k, s], N(i, j) = [k, s']$ (some $k < n, s, s' \in S$.)

Lemma 1.5. *For $t \in \mathfrak{Tm}H$ there is a cofinite subset S of \mathbb{N} such that for*

$$M, N \in H, M \sim_S N \implies (M \in t \iff N \in t)$$

Proof. By structured induction over t . ■

Hence

Lemma 1.6. *For $N \in H'$, for all $t \in \mathfrak{Tm}H$ there is a cofinite subset S of \mathbb{N} such that if $[N, S, k] \leq t$ then $[N, S] \leq t$*

For $N \in H'$ define

$$\eta_N = \{t \in \mathfrak{Tm}H : \exists \text{ cofinite } S, [N, S, k] \leq t\}$$

Lemma 1.7. *For all $N \in H'$, η_N is an ultrafilter of $\mathfrak{Tm}H$.*

Proof. By lemma 1.6 ■

Lemma 1.8. *The map $\mu \rightarrow N_\mu$ and the map $N \rightarrow \eta_N$ are inverses of each other.*

Proof. Straightforward ■

Lemma 1.9. *For any cofinite subset S of \mathbb{N} if $N1, N2 \in H'$, $i < n$ and $N1 \equiv_i N2$, then for all $N1_{\rho_1} \in [N1, S, k]$ there is $N2_{\rho_2} \in [N2, S, k]$ with $N1_{\rho_1} \equiv_i N2_{\rho_2}$*

Proof. Write $E(N1 - i)$ for $\{(j, l) : i \notin \{j, l\} \subseteq E(N1)\}$. Since $N1 \equiv_i N2$ we have $E(N1 - i) = E(N2 - i)$. Let $N1_{\rho_1} \in [N1, S, k]$. ρ_1 is an injection from $E(N1)$ into $[s_0, s_0+k-1] \subseteq S$, for some $s_0 \in S$. Let $\rho_2 : E(N2) \rightarrow [s_0, s_0+k-1]$ be any injection extending $\rho_1 \upharpoonright E(N1 - i)$. Since $|E(N2)| \leq k$, this is possible. Then $N1_{\rho_1} \equiv_i N2_{\rho_2} \in [N2, S, k]$ ■

Lemma 1.10. $H' \cong \text{Uf}(\mathfrak{Tm}H)$

Proof. The isomorphism is

$$N \rightarrow \mu_N.$$

This is a bijection, by Lemma 1.8. Lemma 1.9 shows that it is an isomorphism. ■

Lemma 1.11. $\mathfrak{Cm}H'$ is representable.

Proof. We build an atomic representation step by step by a game. Suppose some finite structure M has been built. Let \mathcal{M} be the set of all atomic networks labelled by atoms of α' , each with n nodes. Every pair is related by an atom all triangles are consistent. Player \forall picks some $N \in \mathcal{M}$ some $i < n$ and $n - 1$ sequence \bar{m} from M such that $M(\bar{m}) \cong (N \setminus i)$. \exists has to add a new point i to M such that $M(\bar{m}, i) \cong N$. This involves labelling all edges (j, i) incident with the new node i where $j \in M \setminus \text{ran}(\bar{m})$. She picks a colour c different from the colours of $N(i, l)$ all $l < n$ and she labels the edges with $[c, \infty]$. Note that there are at most $n - 1$ colours for $N(i, l)$ and since we have n colours this is possible. This is a consistent labelling. (Point is that new “monochromatic” triangles are of the form $([i, x], [i, \infty], [i, \infty])$ which is now not forbidden.) Let N be the limit of this play as \forall makes all possible moves. Then N is the desired representation. That is $\mathfrak{Cm}H'$ embeds in the set algebra with unit nN . ■

We now have:

Theorem 1.12. $\mathfrak{Tm}H \in \mathbf{RCA}_n$

Proof. We have $\mathfrak{Tm}H \subseteq \mathfrak{Cm}(\mathbf{Uf}\mathfrak{Tm}H) \cong \mathfrak{Cm}H' \in \mathbf{RCA}_n$ ■

Using our construction we formulate and prove the (known) results in the abstract. For undefined terminology in the coming corollaries, the reader is referred to [4] or [3].

Theorem 1.13. (1) *There exist two atomic relation algebras with the same atom structure, only one of which is representable.*

(2) **RRA** *is not closed under completions and is not atom-canonical.*

(3) *There exists a non-representable relation algebra with a dense representable subalgebra*

(4) **RRA** *is not Sahlqvist axiomatizable.*

(5) *There exists an atomic relation algebra with no complete representation.*

Proof.

(1) $\mathfrak{Tm}\alpha$ and $\mathfrak{Cm}\alpha$ have the same atom structure. $\mathfrak{Tm}\alpha$ is representable and $\mathfrak{Cm}\alpha$ is not.

(2) $\mathfrak{Cm}\alpha$ is the completion of $\mathfrak{Tm}\alpha$. $\mathfrak{Cm}(\mathbf{AtRRA})$ is not contained in **RRA**. Thus **RRA** is not atom-canonical.

(3) $\mathfrak{Tm}\alpha$ is dense in $\mathfrak{Cm}\alpha$.

(4) **RRA** is a conjugated variety that is not closed under completions, hence it is not Sahlqvist axiomatizable.

(5) $\mathfrak{Tm}\alpha$ has no complete representation; else $\mathfrak{Cm}\alpha$ would be representable ■

The analogous result holds for \mathbf{RCA}_n when $2 < n < \omega$ by using (the representable algebra) $\mathfrak{Tm}H$ and (its completion; the non-representable algebra) $\mathfrak{Cm}H$. We note that the algebraic Theorem 1.13 is related to omitting types for finite variable fragments of first order logic with n variables in [2], see also [1] for a related result.

References

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