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Algebraic Solutions of the Matrix Equations $X + A^T X^{-1} A = Q$ and $X - A^T X^{-1} A = Q$

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Abstract

Two nonrecursive algebraic methods for computing the extreme solutions of the matrix equations $X+A^T X^{-1} A = Q$ and $X-A^T X^{-1} A = Q$ are proposed. The first method is based on the nonrecursive algebraic solutions of the corresponding matrix equations with $Q = I$. The second method is based on the algebraic solution of the corresponding discrete time Riccati equations. Both methods avoid convergence problems and provide simple formulas for computing the accurate solutions of these matrix equations.

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1 Introduction

The central issue of this paper is to compute nonrecursive algebraic solutions of the matrix equations:

$$X + A^T X^{-1} A = Q \tag{1}$$

and

$$X - A^T X^{-1} A = Q \tag{2}$$

where Q is a $n \times n$ Hermitian positive definite matrix, A is a $n \times n$ matrix and A^T denotes the transpose of A.

These equations arise in many applications in various research areas including control theory, ladder networks, dynamic programming, stochastic filtering and statistics: see [2], [11] for references concerning equation (1) and [6] for references concerning equation (2). These equations have been studied recently by many authors [2]-[9], [11]-[12]: the theoretical properties such as necessary and sufficient conditions for the existence of a positive definite solution have been investigated and numerical methods for solving these equations have been proposed; the available methods are recursive algorithms based mainly on the fixed point iteration and on applications of the Newton's algorithm or the cyclic reduction method.

Concerning equation (1), a solution X of (1) is a $n \times n$ Hermitian positive definite matrix, as stated in [11]. It is well known [5], [9] that if (1) has a positive definite solution X, then there exist minimal and maximal solutions X_+ and X_-, respectively, such that $0 < X_- \leq X \leq X_+$ for any positive definite solution X. Here, if X and Y are Hermitian matrices, then $X \leq Y$ $(X < Y)$ means that $Y - X$ is a nonnegative definite (positive definite) matrix.

Concerning equation (2), it is well known [6], [9] that there always exists a unique positive definite solution, which is the maximal solution X_+ of (2), and if A is nonsingular, then there exists a unique negative definite solution, which is the minimal solution X_- of (2).

The minimal and maximal solutions X_+ and X_- are referred as the extreme solutions of (1) and (2).

Furthermore, it is well known [9] that the minimal solution of $X + A^T X^{-1} A = Q$ and the maximal solution of the following equation

$$Y + A Y^{-1} A^T = Q \tag{3}$$

satisfy the relation:

$$X_- = Q - Y_+ \tag{4}$$

Thus, it becomes obvious that the minimal solution of (1) can be derived through the maximal solution of (1): the minimal solution of (1) can be computed using (4) via the maximal solution of the equation (3), which is of type (1).

It is also well known [9] that if A is nonsingular, then the minimal solution of $X - A^T X^{-1} A = Q$ and the maximal solution of the following equation,

$$Y - A Y^{-1} A^T = Q \tag{5}$$

satisfy the relation:

$$X_- = Q - Y_+ \tag{6}$$

Thus, it becomes obvious that the minimal solution of (2) can be derived through the maximal solution of (2): the minimal solution of (2) can be computed using (6) via the maximal solution of the equation (5), which is of type (2). Moreover, the relation between the solutions of (1) and (2) is described in [3]: The solution X of $X - A^T X^{-1} A = Q$ can be computed as:

$$X = Y - AQ^{-1}A^T \tag{7}$$

where Y is the solution of the following equation, which is of type (1):

$$Y + B^T Y^{-1} B = R \tag{8}$$

with

$$B = AQ^{-1}A \tag{9}$$

$$R = Q + A^T Q^{-1} A + AQ^{-1} A^T \tag{10}$$

Thus, it becomes obvious that the solution of (2) can be derived through the solution of (1): the extreme solutions of (2) can be computed using (7) via the extreme solutions of (8), which is of type (1).

Hence, it becomes clear that the extreme solutions of (1) and (2) can be derived through the maximal solution of (1).

In this paper, nonrecursive algebraic methods for computing the extreme solutions of the matrix equations (1) and (2) are proposed. Simulation results are given to illustrate the efficiency of the proposed methods.

2 Nonrecursive algebraic solutions

2.1 Reduction Method

2.1.1 Algebraic solution for $X + A^T X^{-1} A = Q$

It is known [5] that the matrix equation (1) can be reduced to a corresponding matrix equation with $Q = I$, where I is the identity matrix. The proposed method takes advantage of this reduction and provides an algebraic solution of (1) via the algebraic solution of the reduced corresponding equation.

Multiplying both sides of (1) with $Q^{-1/2}$ on the right and on the left (recall that Q is a positive definite matrix and hence it is nonsingular) we have:

$$Q^{-1/2} X Q^{-1/2} + Q^{-1/2} A^T X^{-1} A Q^{-1/2} = I \tag{11}$$

Setting

$$Y = Q^{-1/2} X Q^{-1/2} \tag{12}$$

in (11) we take:

$$Y + L^T Y^{-1} L = I \tag{13}$$

where $L = Q^{-1/2} A Q^{-1/2}$.

It is well known [4] and [5] that if L is normal with numerical radius $r(L) \leq \frac{1}{2}$, then the maximal and minimal solutions of (13) are given by:

$$Y_+ = \frac{1}{2}\left[I + (I - 4L^T L)^{1/2}\right] \tag{14}$$

$$Y_- = \frac{1}{2}\left[I - (I - 4L^T L)^{1/2}\right] \tag{15}$$

Recall that L is normal, if the relation $LL^T = L^T L$ holds, and that the numerical radius of a $n \times n$ matrix M is defined by:

$$r(M) = max\{\, |x^* M x| \;:\;\; x^* x = 1 \,\}$$

and, more specifically, the numerical radius of a normal matrix N is derived by

$$r(N) = max\{\, \sqrt{\lambda_i} \;:\;\; \lambda_i \in \sigma(N^T N) \,\},$$

where $\sigma(N^T N)$ denotes the eigenvalues of $N^T N$.

Rewriting (12) as:

$$X = Q^{1/2}\, Y\, Q^{1/2}$$

the extreme solutions of (1) are derived by (14) and (15) as:

$$X_+ = Q^{1/2}\, Y_+\, Q^{1/2} \tag{16}$$

$$X_- = Q^{1/2}\, Y_-\, Q^{1/2} \tag{17}$$

The proposed solution is not a recursive method, and hence it avoids convergence problems. The method requires the matrix $L = Q^{-1/2} A Q^{-1/2}$ to be normal with numerical radius $r(L) \leq \frac{1}{2}$.

2.1.2 Algebraic solution for $X - A^T X^{-1} A = Q$

Multiplying both sides of (2) with $Q^{-1/2}$ on the right and on the left (recall that Q is a positive definite matrix and hence it is nonsingular) we have:

$$Q^{-1/2} X Q^{-1/2} - Q^{-1/2} A^T X^{-1} A Q^{-1/2} = I \tag{18}$$

Setting

$$Y = Q^{-1/2} X Q^{-1/2} \tag{19}$$

in (18) we take:

$$Y - L^T Y^{-1} L = I \tag{20}$$

where $L = Q^{-1/2} A Q^{-1/2}$. We are going to prove that: if L is normal and nonsingular, then the maximal and minimal solutions of (20) are given by:

$$Y_+ = \frac{1}{2}\left[I + (I + 4L^T L)^{1/2}\right] \tag{21}$$

$$Y_- = \frac{1}{2}\left[I - (I + 4L^T L)^{1/2}\right] \tag{22}$$

It is obvious that the matrix $I + 4L^T L$ is Hermitian and positive definite, since $(I + 4L^T L)^T = I + 4L^T L$ and for every vector $x \neq 0$ with $x \in \mathbb{R}^n$ holds

$$x^T (I + 4L^T L) x = x^T x + 4x^T L^T L x = \|x\|^2 + 4\|Lx\|^2 > 0,$$

where $\|x\|$ denotes the norm of vector x. Moreover, if L is a normal matrix, then we take:

$$\begin{aligned}
L(I + 4L^T L) &= (L^2)^{1/2} (I + 4L^T L)^{1/2} (I + 4L^T L)^{1/2} \\
&= [L(L + 4LL^T L)(I + 4L^T L)]^{1/2} \\
&= [L(I + 4L^T L) L (I + 4L^T L)]^{1/2} \\
&= [(L + 4LL^T L) L (I + 4L^T L)]^{1/2} \\
&= [(I + 4L^T L) L^2 (I + 4L^T L)]^{1/2}
\end{aligned}$$

Thus we have:

$$L(I + 4L^T L) = (I + 4L^T L)^{1/2} L (I + 4L^T L)^{1/2} \tag{23}$$

Since $(I + 4L^T L)^{1/2}$ is also a positive definite matrix, hence it is nonsingular, we are able to multiply both sides of (23) with $(I + 4L^T L)^{-1/2}$ and we derive:

$$L(I + 4L^T L)^{1/2} = (I + 4L^T L)^{1/2} L \tag{24}$$

From (21), (22) and (24) we take:

$$Y_+ L = L Y_+ \tag{25}$$

and

$$Y_- L = L Y_- \tag{26}$$

It is easy to verify that Y_+ in (21) is a solution of (20). Indeed, multiplying on the right the equation (20) with the nonsingular matrix Y_+ (recall that Y_+ is a sum of the positive definite matrices I and $(I+4L^TL)^{1/2}$, and hence it is also positive definite) and using (25) we take:

$$\begin{aligned} Y_+^2 - L^T Y_+^{-1} L Y_+ = Y_+ \;\;&\Rightarrow\;\; Y_+^2 - L^T Y_+^{-1} Y_+ L = Y_+ \\ &\Rightarrow\;\; Y_+^2 - L^T L = Y_+ \;\Rightarrow\; Y_+ - L^T Y_+^{-1} L = I \end{aligned}$$

Thus we have shown that Y_+ in (21) is a positive definite solution of (20). On the other hand, it is well known [6], [9] that there always exists a unique positive definite solution of (20), which is the maximal solution. Then we conclude that Y_+ in (21) is the maximal solution of (20).

Concerning the minimal solution of (20), it is known [9] that: if L is nonsingular, the unique negative definite solution of (20) is given by

$$Y_- = I - Z_+, \tag{27}$$

where Z_+ is the maximal solution of the equation :

$$Z - L Z^{-1} L^T = I \tag{28}$$

Using (21), it is clear that: if L is normal, then the maximal solution of (28) is given by:

$$Z_+ = \frac{1}{2}\left[I + (I + 4LL^T)^{1/2}\right] = \frac{1}{2}\left[I + (I + 4L^TL)^{1/2}\right] \tag{29}$$

Consequently, substituting (29) in (27) we take (22) and thus, we conclude that Y_- in (22) is the minimal solution of (20). This completes the proof that, if L is normal and nonsingular, then the maximal and minimal solutions of (20) are given by (21) and (22), respectively.

Rewriting (19) as:

$$X = Q^{1/2}\, Y\, Q^{1/2}$$

the extreme solutions of (2) are derived by (21) and (22) as:

$$X_+ = Q^{1/2}\, Y_+\, Q^{1/2} \tag{30}$$
$$X_- = Q^{1/2}\, Y_-\, Q^{1/2} \tag{31}$$

The proposed solution is not a recursive method, and hence it avoids convergence problems. The method requires the matrix $L = Q^{-1/2} A Q^{-1/2}$ to be normal and nonsingular.

2.2 Riccati Equation Solution Method

2.2.1 Algebraic solution for $X + A^T X^{-1} A = Q$

Working as in [5] and [6] we are able to derive a Riccati equation, which is equivalent to the matrix equation (1). The proposed method takes advantage of this relation and provides an algebraic solution of (1) via the algebraic solution of the related Riccati equation. In fact, the matrix equation (1) can be written as:

$$X = Q - A^T X^{-1} A$$

Then we have:

$$\begin{aligned} X &= Q - A^T \left[Q - A^T X^{-1} A\right]^{-1} A = Q - A^T A^{-1} \left[A^{-T} Q A^{-1} - X^{-1}\right]^{-1} A^{-T} A \\ &= Q + A^T A^{-1} \left[-A^{-T} Q A^{-1} + X^{-1}\right]^{-1} A^{-T} A \\ &= Q + \left(A^T A^{-1}\right) \left[X^{-1} + (-A^{-T} Q A^{-1})\right]^{-1} \left(A^T A^{-1}\right)^T \qquad (32) \end{aligned}$$

By (32), we lead to the equivalent related Riccati equation is:

$$\begin{aligned} P &= q + fPf^T - fPh^T(hPh^T + r)^{-1} hPf^T \\ &= q + f(P^{-1} + h^T r^{-1} h)^{-1} f^T \qquad (33) \end{aligned}$$

with

$$\begin{aligned} f &= A^T A^{-1} \qquad (34) \\ q &= Q \qquad (35) \\ b &= h^T r^{-1} h = -A^{-T} Q A^{-1} \qquad (36) \end{aligned}$$

It becomes obvious that the matrix equation (1) is equivalent to the related Riccati equation (33) and that the two equations have equivalent solutions. Thus, the unique maximal solution of (1) coincides with the unique positive definite solution of the related Riccati equation:

$$X_+ = P \qquad (37)$$

It is clear that we will be able to solve (1), if we know the solution of the related Riccati equation. The solution of the related Riccati equation can be derived using the algebraic solution proposed in [1] and [10]. More specifically, from the Riccati equation's parameters the following symplectic matrix is formed:

$$\Phi = \begin{bmatrix} a^{-1} & a^{-1} b \\ c a^{-1} & a^T + c a^{-1} b \end{bmatrix} \qquad (38)$$

where

$$\begin{aligned} a &= f^T && (39)\\ b &= h^T r^{-1} h && (40)\\ c &= q && (41) \end{aligned}$$

and a is nonsingular.

Thus, substituting (34)-(36) in (39)-(41), by (38) the following symplectic matrix is derived :

$$\Phi = \begin{bmatrix} A^{-1}A^T & -A^{-1}QA^{-1} \\ QA^{-1}A^T & A^TA^{-1} - QA^{-1}QA^{-1} \end{bmatrix} \tag{42}$$

Moreover, in (42) the matrix Φ can be written as:

$$\Phi = W\ell W^{-1} \tag{43}$$

where ℓ is the matrix, which contains the eigenvalues of matrix Φ, i.e.,

$$\ell = \begin{bmatrix} \Lambda & O \\ O & \Lambda^{-1} \end{bmatrix}$$

with Λ the diagonal matrix, which contains the eigenvalues of matrix Φ, that lie outside the unit circle. In (43), W is the matrix, which contains the corresponding eigenvectors of matrix Φ and denotes

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}. \tag{44}$$

Then, the solution of the Riccati equation is given in terms of the eigenvectors of matrix Φ:

$$P = W_{21}W_{11}^{-1}$$

Finally, the maximal solution of (1) is computed from (37) as

$$X_+ = W_{21}W_{11}^{-1} \tag{45}$$

with W_{11}, W_{21} are defined by (44).

In the Appendix, it is shown that the minimal solution of (1) is:

$$X_- = W_{22}W_{12}^{-1} \tag{46}$$

The proposed solution is not a recursive method, and hence it avoids convergence problems. The method requires A to be nonsingular.

2.2.2 Algebraic solution for $X - A^T X^{-1} A = Q$

Working as in [6] we are able to derive a Riccati equation, which is equivalent to the matrix equation (2). The proposed method takes advantage of this relation and provides an algebraic solution of (2) via the algebraic solution of the related Riccati equation. In fact, the matrix equation (2) can be written as:

$$X = Q + A^T X^{-1} A$$

Then we have:

$$\begin{aligned} X &= Q + A^T \left[Q + A^T X^{-1} A\right]^{-1} A = Q + A^T A^{-1} \left[A^{-T} Q A^{-1} + X^{-1}\right]^{-1} A^{-T} A \\ &= Q + \left(A^T A^{-1}\right) \left[X^{-1} + A^{-T} Q A^{-1}\right]^{-1} \left(A^T A^{-1}\right)^T \qquad (47) \end{aligned}$$

By (47), we lead to the equivalent related Riccati equation, which is formed as the same way as in (33):

$$\begin{aligned} P &= q + fPf^T - fPh^T \left(hPh^T + r\right)^{-1} hPf^T \\ &= q + f\left(P^{-1} + h^T r^{-1} h\right)^{-1} f^T \qquad (48) \end{aligned}$$

with

$$\begin{aligned} f &= A^T A^{-1} \qquad (49) \\ q &= Q \qquad (50) \\ b &= h^T r^{-1} h = A^{-T} Q A^{-1} \qquad (51) \end{aligned}$$

It becomes obvious that the matrix equation (2) is equivalent to the related Riccati equation (48) and that the two equations have equivalent solutions. Thus, the unique maximal solution of (2) coincides with the unique positive definite solution of the related Riccati equation:

$$X_+ = P \qquad (52)$$

The solution of the related Riccati equation can be derived using the algebraic solution proposed in [1] and [10], following the previous procedure.

Thus, substituting (49)-(51) in (39)-(41), by (38) the following symplectic matrix is derived :

$$\Phi = \begin{bmatrix} A^{-1} A^T & A^{-1} Q A^{-1} \\ Q A^{-1} A^T & A^T A^{-1} + Q A^{-1} Q A^{-1} \end{bmatrix} \qquad (53)$$

Moreover, in (53) the matrix Φ can be written as:

$$\Phi = W \ell W^{-1} \qquad (54)$$

where ℓ is the matrix, which contains the eigenvalues of matrix Φ, i.e.,

$$\ell = \begin{bmatrix} \Lambda & O \\ O & \Lambda^{-1} \end{bmatrix}$$

with Λ the diagonal matrix, which contains the eigenvalues of matrix Φ that lie outside the unit circle. In (54), W is the matrix, which contains the corresponding eigenvectors of matrix Φ, and denotes

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}. \tag{55}$$

Then, the solution of the Riccati equation is given in terms of the eigenvectors of matrix Φ :

$$P = W_{21}W_{11}^{-1}$$

Finally, the maximal solution of (2) is computed from (52) as

$$X_+ = W_{21}W_{11}^{-1} \tag{56}$$

with W_{11}, W_{21} are defined by (55).

In the Appendix, it is shown that the minimal solution of (2) is:

$$X_- = W_{22}W_{12}^{-1} \tag{57}$$

The proposed solution is not a recursive method, and hence it avoids convergence problems. The method requires A to be nonsingular.

All algebraic nonrecursive solutions proposed in this section are summarized in Tables 1 and 2: the Reduction Method is presented in Table 1 and the Riccati Equation Solution Method is presented in Table 2.

Table 1. Reduction Method

$X + A^T X^{-1} A = Q$	$X - A^T X^{-1} A = Q$
$L = Q^{-1/2} A Q^{-1/2}$ $Y_+ = \frac{1}{2}\left[I + \left(I - 4L^T L\right)^{1/2}\right]$ $Y_- = \frac{1}{2}\left[I - \left(I - 4L^T L\right)^{1/2}\right]$ $X_+ = Q^{1/2} Y_+ Q^{1/2}$ $X_- = Q^{1/2} Y_- Q^{1/2}$	$L = Q^{-1/2} A Q^{-1/2}$ $Y_+ = \frac{1}{2}\left[I + \left(I + 4L^T L\right)^{1/2}\right]$ $Y_- = \frac{1}{2}\left[I - \left(I + 4L^T L\right)^{1/2}\right]$ $X_+ = Q^{1/2} Y_+ Q^{1/2}$ $X_- = Q^{1/2} Y_- Q^{1/2}$
Requirements $L = Q^{-1/2} A Q^{-1/2}$ is normal with numerical radius $r(L) \leq \frac{1}{2}$	Requirements $L = Q^{-1/2} A Q^{-1/2}$ is normal and nonsingular

Table 2. Riccati Equation Solution Method

$X + A^T X^{-1} A = Q$	$X - A^T X^{-1} A = Q$
$\Phi = \begin{bmatrix} A^{-1}A^T & -A^{-1}QA^{-1} \\ QA^{-1}A^T & A^TA^{-1} - (QA^{-1})^2 \end{bmatrix}$ $\Phi = W\ell W^{-1}$ $\ell = diag\,(\Lambda,\, \Lambda^{-1})$ $W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$ $X_+ = W_{21}W_{11}^{-1}$ $X_- = W_{22}W_{12}^{-1}$ Requirements A is nonsingular	$\Phi = \begin{bmatrix} A^{-1}A^T & A^{-1}QA^{-1} \\ QA^{-1}A^T & A^TA^{-1} + (QA^{-1})^2 \end{bmatrix}$ $\Phi = W\ell W^{-1}$ $\ell = diag\,(\Lambda,\, \Lambda^{-1})$ $W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$ $X_+ = W_{21}W_{11}^{-1}$ $X_- = W_{22}W_{12}^{-1}$ Requirements A is nonsingular

3 Simulation results

Simulation results are given to illustrate the efficiency of the proposed methods. The proposed methods compute accurate solutions as verified trough the following simulation examples.

Example 3.1 This example concerns the equation $X + A^T X^{-1} A = Q$ and is taken from [8]. Consider the equation (1) with

$$A = \begin{bmatrix} 0.20 & 0.20 & 0.10 \\ 0.20 & 0.15 & 0.15 \\ 0.10 & 0.15 & 0.25 \end{bmatrix}, \quad Q = I.$$

In this example, A is Hermitian, thus $L = Q^{-1/2}AQ^{-1/2} = A$ is Hermitian and hence L is normal with $r(L) = r(Q^{-1/2}AQ^{-1/2}) = r(A) = \frac{1}{2}$, thus the Reduction Method can be applied. In addition, A is nonsingular due to the fact that it is Hermitian, thus the Riccati Equation Solution Method can be applied. Both methods compute the same maximal and minimal solutions of (1):

$$X_+ = \begin{bmatrix} 0.8265 & -0.1684 & -0.1582 \\ -0.1684 & 0.8316 & -0.1633 \\ -0.1582 & -0.1633 & 0.8214 \end{bmatrix} \text{ and } X_- = \begin{bmatrix} 0.1735 & 0.1684 & 0.1582 \\ 0.1684 & 0.1684 & 0.1633 \\ 0.1582 & 0.1633 & 0.1786 \end{bmatrix}$$

Example 3.2 This example concerns the equation $X + A^T X^{-1} A = Q$ with

$$A = \begin{bmatrix} 1.6 & 0.4 \\ 0.4 & 2.6 \end{bmatrix}, \quad Q = \begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix}.$$

In this example, $L = Q^{-1/2}AQ^{-1/2} = A$ is normal with $r(L) = r(Q^{-1/2}AQ^{-1/2}) = 0.2643 < \frac{1}{2}$, thus the Reduction Method can be applied. In addition, A is nonsingular, thus the Riccati Equation Solution Method can be applied. Both methods compute the same maximal and minimal solutions of (1):

$$X_+ = \begin{bmatrix} 19.8527 & -0.1480 \\ -0.1480 & 9.2602 \end{bmatrix} \text{ and } X_- = \begin{bmatrix} 0.1473 & 0.1480 \\ 0.1480 & 0.7398 \end{bmatrix}$$

Example 3.3 This example concerns the equation $X - A^T X^{-1} A = Q$ and is taken from [8] and [9]. Consider the equation (2) with

$$A = \begin{bmatrix} 5 & 0 \\ -2 & 2 \end{bmatrix}, \quad Q = \frac{1}{9}\begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}.$$

In this example, $L = Q^{-1/2}AQ^{-1/2}$ is normal and nonsingular, thus the Reduction Method can be applied. In addition, A is nonsingular, thus the Riccati Equation Solution Method can be applied. Both methods compute the same maximal and minimal solutions of (2):

$$X_+ = \frac{1}{9}\begin{bmatrix} 50 & -10 \\ -10 & 20 \end{bmatrix} \quad \text{and} \quad X_- = \begin{bmatrix} -5 & 1 \\ 1 & -2 \end{bmatrix}$$

Example 3.4 This example concerns the equation $X - A^T X^{-1} A = Q$ and is taken from [8] and [9]. Consider the equation (2) with

$$A = \begin{bmatrix} 50 & 20 \\ 10 & 60 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}.$$

In this example, $L = Q^{-1/2}AQ^{-1/2}$ is *not* normal, thus the Reduction Method cannot be applied. On the other hand, A is nonsingular, thus the Riccati Equation Solution Method can be applied; the computed maximal and minimal solutions of (2) are:

$$X_+ = \begin{bmatrix} 51.7994 & 16.0999 \\ 16.0999 & 62.2516 \end{bmatrix} \quad \text{and} \quad X_- = \begin{bmatrix} -48.7004 & -14.0819 \\ -14.0819 & -58.3596 \end{bmatrix}$$

4 Conclusions

Nonrecursive algebraic methods for computing the extreme solutions of the matrix equations $X + A^T X^{-1} A = Q$ and $X - A^T X^{-1} A = Q$ are proposed. The Reduction Method is based on the nonrecursive algebraic solutions of the corresponding reduced matrix equations with $Q = I$. The Riccati Equation Solution Method is based on the algebraic solution of the corresponding discrete time Riccati equations.

Concerning equation $X + A^T X^{-1} A = Q$, the Reduction Method is valid, when $L = Q^{-1/2} A Q^{-1/2}$ is normal with numerical radius $r(L) \leq 1/2$, while the Riccati Equation Solution Method is valid, when A is nonsingular.

Concerning equation $X - A^T X^{-1} A = Q$, the Reduction Method is valid, when $L = Q^{-1/2} A Q^{-1/2}$ is normal and nonsingular, while the Riccati Equation Solution Method is valid, when A is nonsingular.

Both methods avoid convergence problems and provide simple formulas for computing the accurate solutions of these matrix equations, as verified trough simulation experiments. It is clear [3]-[6] that all available algebraic methods for solving the matrix equations $X + A^T X^{-1} A = Q$ and $X - A^T X^{-1} A = Q$ can be used to solve special cases of corresponding Riccati equations. In this paper algebraic solutions of the corresponding Riccati equations are used to solve any of the above matrix equations.

5 Appendix

5.1 Riccati Equation Solution Method: Minimal solutions

5.1.1 Minimal solution for $X + A^T X^{-1} A = Q$

In order to compute the maximal solution of $X + A^T X^{-1} A = Q$ using the Riccati Equation Solution Method, by (42) the following symplectic matrix is formed :

$$\Phi = \begin{bmatrix} A^{-1}A^T & -A^{-1}QA^{-1} \\ QA^{-1}A^T & A^TA^{-1} - QA^{-1}QA^{-1} \end{bmatrix}, \tag{A.1}$$

where A is nonsingular. Then, the matrix Φ can be written as:

$$\Phi = W \ell W^{-1} \tag{A.2}$$

where ℓ is the matrix, which contains the eigenvalues of matrix Φ :

$$\ell = \begin{bmatrix} \Lambda & O \\ O & \Lambda^{-1} \end{bmatrix}$$

with Λ the diagonal matrix, which contains the eigenvalues of matrix Φ that lie outside the unit circle. In (A.2), W is the matrix, which contains the corresponding eigenvectors of matrix Φ :

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

By (45), the maximal solution of $X + A^T X^{-1} A = Q$ becomes :

$$X_+ = W_{21} W_{11}^{-1}$$

It is well known [9] that the minimal solution of $X + A^T X^{-1} A = Q$ and the maximal solution of $Y + AY^{-1}A^T = Q$ satisfy the relation:

$$X_- = Q - Y_+ \tag{A.3}$$

It is obvious that the maximal solution of $Y + AY^{-1}A^T = Q$ can be derived using the Riccati Equation Solution Method. In fact, in order to compute the maximal solution of $Y + AY^{-1}A^T = Q$ using the Riccati Equation Solution Method, thus the following symplectic matrix is formed:

$$\widetilde{\Phi} = \begin{bmatrix} A^{-T}A & -A^{-T}QA^{-T} \\ QA^{-T}A & AA^{-T} - QA^{-T}QA^{-T} \end{bmatrix}$$

Then, the matrix $\widetilde{\Phi}$ can be written as:

$$\widetilde{\Phi} = \widetilde{W}\,\widetilde{\ell}\,\widetilde{W}^{-1} \tag{A.4}$$

where $\widetilde{\ell}$ is the matrix, which contains the eigenvalues of matrix $\widetilde{\Phi}$:

$$\widetilde{\ell} = \begin{bmatrix} \widetilde{\Lambda} & O \\ O & \widetilde{\Lambda}^{-1} \end{bmatrix}$$

with $\widetilde{\Lambda}$ the diagonal matrix, which contains the eigenvalues of matrix $\widetilde{\Phi}$ that lie outside the unit circle. In (A.4), $\widetilde{W}$ is the matrix, which contains the eigenvectors of matrix $\widetilde{\Phi}$ and denotes:

$$\widetilde{W} = \begin{bmatrix} \widetilde{W}_{11} & \widetilde{W}_{12} \\ \widetilde{W}_{21} & \widetilde{W}_{22} \end{bmatrix}$$

As in (45), the maximal solution of $Y + AY^{-1}A^T = Q$ becomes :

$$Y_+ = \widetilde{W}_{21} \widetilde{W}_{11}^{-1} \tag{A.5}$$

At this point we make the following basic remark: Φ is similar to $\widetilde{\Phi}$. In fact, Φ is similar to Φ^{-1}, because these matrices have the same eigenvalues. Furthermore, Φ^{-1} is similar to $\widetilde{\Phi}$, because there exists the matrix

$$N = \begin{bmatrix} I & O \\ Q & -I \end{bmatrix}$$

such that

$$N\widetilde{\Phi} = \Phi^{-1}N. \tag{A.6}$$

Note that holds $N = N^{-1}$, consequently, Φ and $\widetilde{\Phi}$ have the same eigenvalues, hence $\widetilde{\ell} = \ell$. Combining (A.2), (A.4) and (A.6), we have:

$$\begin{aligned} \widetilde{\Phi} &= N^{-1}\Phi^{-1}N \Rightarrow \\ \widetilde{W}\,\widetilde{\ell}\,\widetilde{W}^{-1} &= N^{-1}\left(W\ell W^{-1}\right)^{-1}N \\ &= N^{-1}W\ell^{-1}W^{-1}N = N^{-1}W\left(J\ell J\right)W^{-1}N \\ &= \left(N^{-1}WJ\right)\ell\left(JW^{-1}N\right) = \left(N^{-1}WJ\right)\ell\left(N^{-1}WJ^{-1}\right)^{-1} \end{aligned} \tag{A.7}$$

where

$$J = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$$

with $J = J^{-1}$.

By (A.7) and the relationships $\widetilde{\ell} = \ell$ and $N = N^{-1}$, arises $\widetilde{W}J = NW$, from where we have:

$$W_{22}W_{12}^{-1} + \widetilde{W}_{21}\widetilde{W}_{11}^{-1} = W_{22}W_{12}^{-1} + \left(QW_{12} - W_{22}\right)W_{12}^{-1} = Q \tag{A.8}$$

Combining the relations in (A.3), (A.5) and (A.8), we derive the minimal solution of equation $X + A^TX^{-1}A = Q$ as:

$$X_{-} = Q - Y_{+} = Q - \widetilde{W}_{21}\widetilde{W}_{11}^{-1} = W_{22}W_{12}^{-1} \tag{A.9}$$

5.1.2 Minimal solution for $X - A^TX^{-1}A = Q$

In order to compute the maximal solution of $X - A^TX^{-1}A = Q$ using the Riccati Equation Solution Method, by (53) the following symplectic matrix is formed :

$$\Phi = \begin{bmatrix} A^{-1}A^T & A^{-1}QA^{-1} \\ QA^{-1}A^T & A^TA^{-1} + QA^{-1}QA^{-1} \end{bmatrix}, \tag{B.1}$$

where A is nonsingular. Then, the matrix Φ can be written as:

$$\Phi = W\ell W^{-1} \tag{B.2}$$

where ℓ is the matrix, which contains the eigenvalues of matrix Φ :

$$\ell = \begin{bmatrix} \Lambda & O \\ O & \Lambda^{-1} \end{bmatrix}$$

with Λ the diagonal matrix, which contains the eigenvalues of matrix Φ that lie outside the unit circle. In (B.2), W is the matrix, which contains the corresponding eigenvectors of matrix Φ :

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

By (56), the maximal solution of $X - A^T X^{-1} A = Q$ becomes :

$$X_+ = W_{21} W_{11}^{-1}$$

It is well known [9] that the minimal solution of $X - A^T X^{-1} A = Q$ and the maximal solution of $Y - AY^{-1}A^T = Q$ satisfy the relation:

$$X_- = Q - Y_+ \tag{B.3}$$

It is obvious that the maximal solution of $Y - AY^{-1}A^T = Q$ can be derived using the Riccati Equation Solution Method. In fact, in order to compute the maximal solution of $Y - AY^{-1}A^T = Q$ using the Riccati Equation Solution Method, thus the following symplectic matrix is formed:

$$\widetilde{\Phi} = \begin{bmatrix} A^{-T}A & A^{-T}QA^{-T} \\ QA^{-T}A & AA^{-T} + QA^{-T}QA^{-T} \end{bmatrix}$$

Then, the matrix $\widetilde{\Phi}$ can be written as:

$$\widetilde{\Phi} = \widetilde{W}\,\widetilde{\ell}\,\widetilde{W}^{-1} \tag{B.4}$$

where $\widetilde{\ell}$ is the matrix, which contains the eigenvalues of matrix $\widetilde{\Phi}$:

$$\widetilde{\ell} = \begin{bmatrix} \widetilde{\Lambda} & O \\ O & \widetilde{\Lambda}^{-1} \end{bmatrix}$$

with $\widetilde{\Lambda}$ the diagonal matrix, which contains the eigenvalues of matrix $\widetilde{\Phi}$ that lie outside the unit circle and $\widetilde{W}$ is the matrix, which contains the eigenvectors of matrix $\widetilde{\Phi}$ and denotes:

$$\widetilde{W} = \begin{bmatrix} \widetilde{W}_{11} & \widetilde{W}_{12} \\ \widetilde{W}_{21} & \widetilde{W}_{22} \end{bmatrix}$$

As in (56), the maximal solution of $Y - AY^{-1}A^T = Q$ becomes :

$$Y_+ = \widetilde{W}_{21} \widetilde{W}_{11}^{-1} \tag{B.5}$$

At this point we make the following basic remark: Φ is similar to $\widetilde{\Phi}$. In fact, Φ is similar to Φ^{-1}, because these matrices have the same eigenvalues. Furthermore, Φ^{-1} is similar to $\widetilde{\Phi}$, because there exists the matrix

$$N = \begin{bmatrix} I & O \\ Q & -I \end{bmatrix}$$

such that

$$N\widetilde{\Phi} = \Phi^{-1}N. \tag{B.6}$$

Note that holds $N = N^{-1}$, consequently, Φ and $\widetilde{\Phi}$ have the same eigenvalues, hence $\widetilde{\ell} = \ell$. Combining (B.2), (B.4) and (B.6), we have:

$$\begin{aligned} \widetilde{\Phi} &= N^{-1}\Phi^{-1}N \Rightarrow \\ \widetilde{W}\,\widetilde{\ell}\,\widetilde{W}^{-1} &= N^{-1}\left(W\ell W^{-1}\right)^{-1}N \\ &= N^{-1}W\ell^{-1}W^{-1}N = N^{-1}W\left(J\ell J\right)W^{-1}N \\ &= \left(N^{-1}WJ\right)\ell\left(JW^{-1}N\right) = \left(N^{-1}WJ\right)\ell\left(N^{-1}WJ^{-1}\right)^{-1} \end{aligned} \tag{B.7}$$

where

$$J = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$$

with $J = J^{-1}$.

By (B.7) and the relationships $\widetilde{\ell} = \ell$ and $N^{-1} = N$, arises $\widetilde{W}J = NW$, from where we have:

$$W_{22}W_{12}^{-1} + \widetilde{W}_{21}\widetilde{W}_{11}^{-1} = W_{22}W_{12}^{-1} + \left(QW_{12} - W_{22}\right)W_{12}^{-1} = Q \tag{B.8}$$

Combining the relations in (B.3), (B.5) and (B.8), we derive the minimal solution of equation $X - A^TX^{-1}A = Q$ as:

$$X_- = Q - Y_+ = Q - \widetilde{W}_{21}\widetilde{W}_{11}^{-1} = W_{22}W_{12}^{-1} \tag{B.9}$$

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