

# Cohomology for Morphisms of Yetter-Drinfel'd and Hopf Modules

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**Abstract.** Cohomology and algebraic deformations for a morphism of Yetter-Drinfel'd modules and of Hopf modules are studied.

**Mathematics Subject Classification:** 13D10

**Keywords:** Yetter-Drinfel'd module, Hopf module, deformation

## 1. INTRODUCTION

A left-right Yetter-Drinfel'd module (a.k.a. crossed bimodule and quantum Yang-Baxter module)  $M$  over a bialgebra  $A$  is a vector space together with a left  $A$ -module action and a right  $A$ -comodule coaction that satisfy the so-called Yetter-Drinfel'd condition. They were introduced by Yetter [14], and are studied further in [7, 9, 10, 11, 12], among others. If the bialgebra  $A$  is a finite dimensional Hopf algebra, then the left-modules over its Drinfel'd double  $D(A)$  are exactly the Yetter-Drinfel'd modules over  $A$ .

Cohomology of Yetter-Drinfel'd modules was introduced by Panaite and Ştefan in [9]. For a pair of Yetter-Drinfel's modules  $M$  and  $N$  over a bialgebra  $A$ , they constructed a bicomplex  $Y^{\bullet,\bullet}(M, N)$  whose total cohomology  $H_{YD}^{\bullet}(M, N)$  is the desired cohomology. When  $M = N = K$ , the ground field,

$H_{YD}^\bullet(K, K)$  coincides with the bialgebra cohomology  $H_{GS}^\bullet(A, A)$  of Gerstenhaber and Schack [5, 6]. Moreover, this cohomology theory of Yetter-Drinfel'd modules is the right one for studying algebraic deformations in the sense of Gerstenhaber [1]. In fact, when  $N = M$ , the Yetter-Drinfel'd cohomology  $H_{YD}^\bullet(M, M)$  controls the algebraic deformations of  $M$  as a Yetter-Drinfel'd module. This is mentioned in [9], but the details were not given.

The purpose of this paper is to study cohomology and deformation of a *morphism* of Yetter-Drinfel'd modules, extending the constructions in [9]. Deformations of a morphism of associative algebras were considered by Gerstenhaber and Schack [2, 3, 4]. Our construction of the deformation complex of a morphism is formally similar to the one considered in [2, 3, 4]. In fact, the deformation complex of a morphism of Yetter-Drinfel'd modules can be constructed as a mapping cylinder, just like the case of an associative algebra morphism.

In [9], the cohomology for a pair of (left-right) Hopf modules is also constructed. A Hopf module  $(M, \omega, \rho)$  over a bialgebra  $A$  is just as above, except that the Yetter-Drinfel'd condition is replaced by a similar Hopf module condition. The cohomology  $H_{Hopf}^\bullet(M, N)$  for a pair of Hopf modules is very similar to the Yetter-Drinfel'd cohomology  $H_{YD}^\bullet(M, N)$ . When  $M = N$ ,  $H_{Hopf}^\bullet(M, M)$  controls the deformations of  $M$  as a Hopf module. (Again, this fact is mentioned in [9], but the details were not given.) We will construct the cohomology for a *morphism* of Hopf modules and use it to describe Hopf module morphism deformations.

**1.1. Organization.** In the next section, we first recall the Yetter-Drinfel'd cohomology in [9]. Then we construct the cohomology for a morphism of Yetter-Drinfel'd modules and observe that it is a mapping cylinder (Corollary 2.6). In section 3, we define deformations of a morphism of Yetter-Drinfel'd modules. We observe that the infinitesimal of a deformation is a 1-cocycle in the deformation complex and properly identify it with a cohomology class (Theorem 3.2). The Rigidity Theorem (Corollary 3.5) in that section states that the vanishing of  $H_{YD}^1$  for a morphism  $f$  implies that  $f$  is formally rigid, i.e. every deformation of  $f$  is equivalent to the trivial one. In the final section,

we carry out the same construction for a morphism of Hopf modules and list the corresponding results.

## 2. COHOMOLOGY FOR A MORPHISM OF YETTER-DRINFEL'D MODULES

Fix a ground field  $K$ . Tensor products and Hom are always taken over  $K$ . Also fix a  $K$ -bialgebra  $A = (A, \mu, \Delta)$  with multiplication  $\mu$  and comultiplication  $\Delta$ . We use the standard Sweedler's notation [13] for comultiplication:  $\Delta^p(a) = \sum a_{(1)} \otimes \cdots \otimes a_{(p)}$ . The reader is referred to [8, 13] for more information about bialgebras and Hopf algebras.

**2.1. Yetter-Drinfel'd modules.** A *left-right Yetter-Drinfel'd module* over  $A$  [14] is a vector space  $M$  together with a left  $A$ -module action

$$\omega: A \otimes M \rightarrow M$$

and a right  $A$ -comodule coaction

$$\rho: M \rightarrow M \otimes A$$

that satisfy the Yetter-Drinfel'd condition,

$$\begin{aligned} & (\text{Id}_M \otimes \mu) \circ (\rho \otimes \text{Id}_A) \circ \tau \circ (\text{Id}_A \otimes \omega) \circ (\Delta \otimes \text{Id}_M) \\ &= (\omega \otimes \mu) \circ (\text{Id}_A \otimes \tau \otimes \text{Id}_A) \circ (\Delta \otimes \rho), \end{aligned} \quad (2.1.1)$$

where  $\tau$  is the twist isomorphism

$$A \otimes M \cong M \otimes A.$$

We will simply call  $M = (M, \omega, \rho)$  a Yetter-Drinfel'd module in what follows.

For  $m \in M$ , write  $\rho(m) = \sum m_0 \otimes m_1 \in M \otimes A$ . If  $a \in A$ , we write  $a \cdot m$  (or  $am$ ) for  $\omega(a \otimes m)$ . Then the Yetter-Drinfel'd condition (2.1.1) can be written as

$$\sum (a_{(2)} \cdot m)_0 \otimes (a_{(2)} \cdot m)_1 a_{(1)} = \sum (a_{(1)} \cdot m_0) \otimes (a_{(2)} m_1)$$

for all  $a \in A$  and  $m \in M$ .

**2.2. Cohomology of Yetter-Drinfel'd modules.** Here we recall from [9] the Yetter-Drinfel'd cohomology for a pair of Yetter-Drinfel'd modules. We will closely follow the notations in [9].

Let  $N = (N, \omega_N, \rho_N)$  be another Yetter-Drinfel'd module over  $A$ . For integers  $n, p \geq 0$ , set

$$Y^{n,p}(M, N) = \text{Hom}(A^{\otimes n} \otimes M, N \otimes A^{\otimes p}).$$

For  $g \in Y^{n,p}(M, N)$ ,  $\mathbf{a} \in A^{\otimes n}$ , and  $m \in M$ , denote  $g(\mathbf{a} \otimes m) \in N \otimes A^{\otimes p}$  by

$$\sum g(\mathbf{a} \otimes m)^0 \otimes g(\mathbf{a} \otimes m)^1 \otimes \cdots \otimes g(\mathbf{a} \otimes m)^p.$$

Now we recall the maps in the bicomplex.

For  $\mathbf{a} = a^1 \otimes \cdots \otimes a^{n+1} \in A^{\otimes(n+1)}$ , write  $\mathbf{a}' = a^1 \otimes \cdots \otimes a^n$ . For  $0 \leq i \leq n+1$  and  $m \in M$ , the map

$$b_i^{n,p}: Y^{n,p}(M, N) \rightarrow Y^{n+1,p}(M, N)$$

is defined as follows. The element  $b_i^{n,p}(g)(\mathbf{a} \otimes m) \in N \otimes A^{\otimes p}$  is given by:

- $\sum a_{(1)}^1 g(a^2 \otimes \cdots \otimes a^{n+1} \otimes m)^0 \otimes \cdots \otimes a_{(p+1)}^1 g(a^2 \otimes \cdots \otimes a^{n+1} \otimes m)^p$  if  $i = 0$ ,
- $g(a^1 \otimes \cdots \otimes (a^i a^{i+1}) \otimes \cdots \otimes a^{n+1} \otimes m)$  if  $1 \leq i \leq n$ ,
- $\sum g(\mathbf{a}' \otimes a_{(p+1)}^{n+1} m)^0 \otimes g(\mathbf{a}' \otimes a_{(p+1)}^{n+1} m)^1 a_{(1)}^{n+1} \otimes \cdots \otimes g(\mathbf{a}' \otimes a_{(p+1)}^{n+1} m)^p a_{(p)}^{n+1}$  if  $i = n+1$ .

Now define the horizontal map

$$d_m^{n,p}: Y^{n,p}(M, N) \rightarrow Y^{n+1,p}(M, N)$$

by

$$d_m^{n,p} = \sum_{i=0}^{n+1} (-1)^i b_i^{n,p}.$$

To define the vertical maps, suppose that  $\mathbf{a} = a^1 \otimes \cdots \otimes a^n \in A^{\otimes n}$  and  $m \in M$ . Write  $\mathbf{a}_{(2)}$  for a generic term  $a_{(2)}^1 \otimes \cdots \otimes a_{(2)}^n$  and  $\mathbf{x}$  for  $\mathbf{a}_{(2)} \otimes m$ . For  $0 \leq i \leq p+1$ , the map

$$c_i^{n,p}: Y^{n,p}(M, N) \rightarrow Y^{n,p+1}(M, N)$$

is defined by setting  $c_i^{n,p}(g)(\mathbf{a} \otimes m)$  equal to:

- $\sum (g(\mathbf{x})^0)_0 \otimes \left( (g(\mathbf{x})^0)_1 a_{(1)}^1 \cdots a_{(1)}^n \right) \otimes g(\mathbf{x})^1 \otimes \cdots \otimes g(\mathbf{x})^p$  if  $i = 0$ ,
- $\sum g(\mathbf{a} \otimes m)^0 \otimes \cdots \otimes g(\mathbf{a} \otimes m)_{(1)}^i \otimes g(\mathbf{a} \otimes m)_{(2)}^i \otimes \cdots \otimes g(\mathbf{a} \otimes m)^p$  if  $1 \leq i \leq p$ ,

- $\sum g(a_{(1)}^1 \otimes \cdots \otimes a_{(1)}^n \otimes m_0) \otimes (a_{(2)}^1 \cdots a_{(2)}^n m_1)$  if  $i = p + 1$ .

Now define the vertical map

$$d_c^{n,p}: Y^{n,p}(M, N) \rightarrow Y^{n,p+1}(M, N)$$

by

$$d_c^{n,p} = \sum_{i=0}^{p+1} (-1)^i c_i^{n,p}.$$

Both  $d_m^{\bullet,p}$  and  $d_c^{n,\bullet}$  are differentials, and

$$d_c^{n+1,p} \circ d_m^{n,p} = d_m^{n,p+1} \circ d_c^{n,p}.$$

Therefore,  $Y^{\bullet,\bullet}(M, N)$  is a bicomplex with  $d_m^{\bullet,\bullet}$  and  $d_c^{\bullet,\bullet}$  as structure maps. The *Yetter-Drinfel'd cohomology*  $H_{YD}^{\bullet}(M, N)$  is defined to be the cohomology of the total complex  $Y^{\bullet}(M, N)$  of  $Y^{\bullet,\bullet}(M, N)$ .

For an element  $\alpha \in Y^k(M, N)$  in the total complex, write  $\alpha_{i,k-i}$  for the component of  $\alpha$  that lies in  $Y^{i,k-i}(M, N)$ . We write

$$d^k: Y^k(M, N) \rightarrow Y^{k+1}(M, N)$$

for the differential, so that

$$d^k = d_c^{i,k-i} + (-1)^{k-i} d_m^{i,k-i}$$

on  $Y^{i,k-i}(M, N)$ .

**2.3. Cohomology of a morphism.** Given two Yetter-Drinfel'd modules  $(M, \omega_M, \rho_M)$  and  $(N, \omega_N, \rho_N)$  over  $A$ , a *morphism*  $f: M \rightarrow N$  of Yetter-Drinfel'd modules is a linear map of the underlying vector spaces that is simultaneously a left  $A$ -module morphism and a right  $A$ -comodule morphism. Fix such a morphism  $f: M \rightarrow N$ . We now define the deformation complex of  $f$ .

We make the following definitions:

- $Y^0(f) = Y^0(M, M) \oplus Y^0(N, N)$ .
- $Y^1(f) = Y^1(M, M) \oplus Y^1(N, N) \oplus \text{Hom}(M, N)$ .
- $Y^k(f) = Y^k(M, M) \oplus Y^k(N, N) \oplus Y^{k-1,0}(M, N) \oplus Y^{0,k-1}(M, N)$  for  $k \geq 2$ .

Now we define the differentials

$$d_f^k: Y^k(f) \rightarrow Y^{k+1}(f)$$

as follows:

- $d_f^0(\alpha; \beta) = (d^0\alpha; d^0\beta; f \circ \alpha - \beta \circ f)$ .
- $d_f^1(\alpha; \beta; \delta) = (d^1\alpha; d^1\beta; f \circ \alpha_{1,0} - \beta_{1,0} \circ (\text{Id}_A \otimes f) - d_m^{0,0}\delta; (f \otimes \text{Id}_A) \circ \alpha_{0,1} - \beta_{0,1} \circ f - d_c^{0,0}\delta)$ .
- For  $k \geq 2$ ,

$$d_f^k(\alpha; \beta; \delta; \epsilon) = (d^k\alpha; d^k\beta; f \circ \alpha_{k,0} - \beta_{k,0} \circ (\text{Id}_{A^{\otimes k}} \otimes f) - d_m^{k-1,0}\delta; (f \otimes \text{Id}_{A^{\otimes k}}) \circ \alpha_{0,k} - \beta_{0,k} \circ f - d_c^{0,k-1}\epsilon).$$

To see that  $(Y^\bullet(f), d_f^\bullet)$  is a cochain complex, we will observe that it is actually the mapping cylinder of a certain chain map.

**2.4. Mapping cylinder.** Let  $\varphi: D \rightarrow E$  be a chain map between cochain complexes. Recall that the *mapping cylinder of  $\varphi$*  is the cochain complex  $\text{Cyl}(\varphi)$  with

$$\text{Cyl}^n(\varphi) = D^n \oplus E^{n-1}$$

and differential

$$\delta(d; e) = (\delta_D d; \varphi(d) - \delta_E e).$$

There is an associated short exact sequence

$$0 \rightarrow E^{\bullet-1} \xrightarrow{i} \text{Cyl}^\bullet(\varphi) \xrightarrow{\pi} D^\bullet \rightarrow 0,$$

where

$$i(e) = (-1)^n(0; e)$$

for  $e \in E^{n-1}$ , and an associated long exact sequence in homology.

Now consider the direct sum cochain complex

$$D^\bullet = Y^\bullet(M, M) \oplus Y^\bullet(N, N).$$

Also, let  $E^\bullet$  denote the total complex of the sub-bicomplex of  $Y^{\bullet,\bullet}(M, N)$  consisting of the 0th row  $Y^{\bullet,0}(M, N)$  and the 0th column  $Y^{0,\bullet}(M, N)$ . In

particular,

$$E^0 = \text{Hom}(M, N)$$

and

$$E^k = Y^{k,0}(M, N) \oplus Y^{0,k}(M, N)$$

for  $k \geq 1$ . Define a map

$$\varphi: D^\bullet \rightarrow E^\bullet$$

by

$$\varphi(\alpha; \beta) = (f \circ \alpha_{k,0} - \beta_{k,0} \circ (\text{Id}_{A^{\otimes k}} \otimes f)); (f \otimes \text{Id}_{A^{\otimes k}}) \circ \alpha_{0,k} - \beta_{0,k} \circ f)$$

for  $(\alpha; \beta) \in D^k$ .

**Proposition 2.5.** *The map  $\varphi: D^\bullet \rightarrow E^\bullet$  is a chain map.*

*Proof.* This follows immediately from the following elementary observations:

1.  $d_m^{k,0}(f \circ \alpha_{k,0}) = f \circ (d_m^{k,0} \alpha_{k,0})$ .
2.  $d_m^{k,0}(\beta_{k,0} \circ (\text{Id}_{A^{\otimes k}} \otimes f)) = (d_m^{k,0} \beta_{k,0}) \circ (\text{Id}_{A^{\otimes k+1}} \otimes f)$ .
3.  $d_c^{0,k}((f \otimes \text{Id}_{A^{\otimes k}}) \circ \alpha_{0,k}) = (f \otimes \text{Id}_{A^{\otimes k+1}}) \circ (d_c^{0,k} \alpha_{0,k})$ .
4.  $d_c^{0,k}(\beta_{0,k} \circ f) = (d_c^{0,k} \beta_{0,k}) \circ f$ .

Statements (1) and (4) use the compatibilities of  $f$  with the left  $A$ -actions and the right  $A$ -coactions, respectively. The other two statements are direct from the definitions.  $\square$

It is clear that for the resulting mapping cylinder  $(\text{Cyl}^\bullet(\varphi), \delta)$ ,  $\text{Cyl}^k(\varphi) = Y^k(f)$  and  $\delta^k = d_f^k$  for all  $k$ .

**Corollary 2.6.**  *$(Y^\bullet(f), d_f^\bullet)$  is a cochain complex.*

Define the *Yetter-Drinfel'd cohomology of  $f$* , denoted  $H_{YD}^\bullet(f)$ , to be the cohomology of the cochain complex  $Y^\bullet(f)$ .

Using the associated long exact sequence in cohomology for the mapping cylinder, we obtain:

**Corollary 2.7.** *For  $k \geq 3$ , if  $H_{YD}^k(M, M)$ ,  $H_{YD}^k(N, N)$ ,  $H^{k-1}(Y^{\bullet,0}(M, N), d_m^{\bullet,0})$ , and  $H^{k-1}(Y^{0,\bullet}(M, N), d_c^{0,\bullet})$  are trivial, then so is  $H_{YD}^k(f)$ .*

### 3. DEFORMATION OF A MORPHISM OF YETTER-DRINFEL'D MODULES

Let  $f: (M, \omega_M, \rho_M) \rightarrow (N, \omega_N, \rho_N)$  be a fixed morphism of Yetter-Drinfel'd modules over a bialgebra  $(A, \mu, \Delta)$ .

For a 1-cochain  $\theta \in Y^1(M, M)$ , we will also write it as  $\theta = (\omega; \rho)$ , where  $\omega = \theta_{1,0} \in Y^{1,0}(M, M)$  and  $\rho = \theta_{0,1} \in Y^{0,1}(M, M)$ .

**3.1. Deformation and infinitesimal.** First we define the absolute case. A *deformation* of  $M$  is a power series

$$\Theta_t = \sum_{i=0}^{\infty} \theta_i t^i$$

in which  $\theta_0 = (\omega_M; \rho_M)$  and each  $\theta_i = (\omega_i; \rho_i) \in Y^1(M, M)$  such that  $(M[[t]], \omega_t, \rho_t)$  is a Yetter-Drinfel'd module over the  $K[[t]]$ -bialgebra  $A[[t]]$ . Here  $\omega_t = \sum_{i=0}^{\infty} \omega_i t^i$  and  $\rho_t = \sum_{i=0}^{\infty} \rho_i t^i$  are extended linearly to the appropriate modules of power series.

By a *deformation of  $f$* , we mean a power series

$$\Gamma_t = \sum_{i=0}^{\infty} \gamma_i t^i$$

in which each  $\gamma_i = (\theta_{M,i}; \theta_{N,i}; f_i) \in Y^1(f)$  such that:

1. For each  $X \in \{M, N\}$ ,  $\Theta_{X,t} = \sum_{i=0}^{\infty} \theta_{X,i} t^i$  is a deformation of  $X$ .
2.  $F_t = \sum_{i=0}^{\infty} f_i t^i: (M[[t]], \omega_{M,t}, \rho_{M,t}) \rightarrow (N[[t]], \omega_{N,t}, \rho_{N,t})$  is a morphism of Yetter-Drinfel'd modules over the  $K[[t]]$ -bialgebra  $A[[t]]$ .

Here  $\theta_{X,i} = (\omega_{X,i}; \rho_{X,i})$ ,  $\omega_{X,t} = \sum_{i=0}^{\infty} \omega_{X,i} t^i$ , and  $\rho_{X,t} = \sum_{i=0}^{\infty} \rho_{X,i} t^i$ . We will also denote a deformation  $\Gamma_t$  as  $(\Theta_{M,t}; \Theta_{N,t}; F_t)$ . The linear coefficient  $\gamma_1 \in Y^1(f)$  is called the *infinitesimal* of  $\Gamma_t$ .

A *formal isomorphism of  $f$*  is a power series

$$\Phi_t = \sum_{i=0}^{\infty} \phi_i t^i$$

in which each  $\phi_i = (\phi_{M,i}; \phi_{N,i}) \in Y^0(f)$  and  $\phi_0 = (\text{Id}_M; \text{Id}_N)$ . Such a formal isomorphism will also be denoted by  $(\Phi_{M,t}; \Phi_{N,t})$ , where  $\Phi_{X,t} = \sum_{i=0}^{\infty} \phi_{X,i} t^i$ .

Two deformations  $\Gamma_t$  and  $\bar{\Gamma}_t$  are said to be *equivalent* if and only if there exists a formal isomorphism  $\Phi_t$  of  $f$  such that the following three statements hold ( $X \in \{M, N\}$ ):

1.  $\bar{\omega}_{X,t} = \Phi_{X,t} \circ \omega_{X,t} \circ (\text{Id}_A \otimes \Phi_{X,t}^{-1})$ .
2.  $\bar{\rho}_{X,t} = (\Phi_{X,t} \otimes \text{Id}_A) \circ \rho_{X,t} \circ \Phi_{X,t}^{-1}$ .
3.  $\bar{F}_t = \Phi_{N,t} \circ F_t \circ \Phi_{M,t}^{-1}$ .

Here  $\text{Id}_A$  is extended linearly to  $A[[t]]$ . Such extensions will be performed wherever they are needed without further comments. In this situation, we write

$$\Phi_t: \Gamma_t \cong \bar{\Gamma}_t.$$

Equivalence defined this way is clearly an equivalence relation. In fact, we have

$$\Phi_t^{-1}: \bar{\Gamma}_t \cong \Gamma_t,$$

where

$$\Phi_t^{-1} = (\Phi_{M,t}^{-1}; \Phi_{N,t}^{-1}).$$

Moreover, given a deformation  $\Gamma_t$  and a formal isomorphism  $\Phi_t$ , one can construct an equivalent deformation  $\bar{\Gamma}_t$  using the above three conditions.

We now show that the infinitesimal can be properly thought of as a cohomology class in  $H_{YD}^1(f)$ .

**Theorem 3.2.** *Let  $\Gamma_t = \sum_{i=0}^{\infty} \gamma_i t^i$  be a deformation of  $f$ . Then the infinitesimal  $\gamma_1$  is a 1-cocycle in  $Y^1(f)$  whose cohomology class is determined by the equivalence class of  $\Gamma_t$ . More generally, if*

$$\gamma_1 = \cdots = \gamma_k = 0,$$

*then  $\gamma_{k+1}$  is a 1-cocycle.*

*Proof.* Since  $Y^\bullet(M, M)$  controls the deformation of  $M$  [9], the linear coefficient  $\theta_{M,1} \in Y^1(M, M)$  is a 1-cocycle. Likewise,  $\theta_{N,1} \in Y^1(N, N)$  is a 1-cocycle. Now  $F_t$  is a morphism of Yetter-Drinfel'd modules, so we have:

$$\omega_{N,t} \circ (\text{Id}_A \otimes F_t) = F_t \circ \omega_{M,t}, \quad (3.2.1a)$$

$$\rho_{N,t} \circ F_t = (F_t \otimes \text{Id}_A) \circ \rho_{M,t}. \quad (3.2.1b)$$

Comparing the coefficients of  $t$  in (3.2.1a), we obtain

$$\omega_N \circ (\text{Id}_A \otimes f_1) + \omega_{N,1} \circ (\text{Id}_A \otimes f) = f \circ \omega_{M,1} + f_1 \circ \omega_M.$$

This can be rewritten as

$$f \circ \omega_{M,1} - \omega_{N,1} \circ (\text{Id}_A \otimes f) - d_m^{0,0} f_1 = 0. \quad (3.2.2)$$

Similarly, the coefficients of  $t$  in (3.2.1b) gives the condition

$$(f \otimes \text{Id}_A) \circ \rho_{M,1} - \rho_{N,1} \circ f - d_c^{0,0} f_1 = 0. \quad (3.2.3)$$

Combining (3.2.2), (3.2.3), and that

$$d^1 \theta_{X,1} = 0$$

for  $X \in \{M, N\}$ , we conclude that

$$\gamma_1 = (\theta_{M,1}; \theta_{N,1}; f_1)$$

is a 1-cocycle in  $Y^1(f)$ . The last assertion is proved similarly.

Now suppose that  $\Phi_t: \Gamma_t \cong \bar{\Gamma}_t$  for some deformation  $\bar{\Gamma}_t$  and formal isomorphism  $\Phi_t$ . Then the coefficients of  $t$  in the first condition of equivalence,

$$\bar{\omega}_{X,t} = \Phi_{X,t} \circ \omega_{X,t} \circ (\text{Id}_A \otimes \Phi_{X,t}^{-1}),$$

imply that

$$\omega_{X,1} - \bar{\omega}_{X,1} = \omega_X \circ (\text{Id}_A \otimes \phi_{X,1}) - \phi_{X,1} \circ \omega_X. \quad (3.2.4)$$

Similarly, the linear terms in the second condition of equivalence,

$$\bar{\rho}_{X,t} = (\Phi_{X,t} \otimes \text{Id}_A) \circ \rho_{X,t} \circ \Phi_{X,t}^{-1},$$

yield

$$\rho_{X,1} - \bar{\rho}_{X,1} = \rho_X \circ \phi_{X,1} - (\phi_{X,1} \otimes \text{Id}_A) \circ \rho_X. \quad (3.2.5)$$

The linear terms in

$$\bar{F}_t = \Phi_{N,t} \circ F_t \circ \Phi_{M,t}^{-1}$$

yield

$$f_1 - \bar{f}_1 = f \circ \phi_{M,1} - \phi_{N,1} \circ f. \quad (3.2.6)$$

Combining (3.2.4), (3.2.5), and (3.2.6), we have

$$\gamma_1 - \bar{\gamma}_1 = d^0(\phi_{M,1}; \phi_{N,1}).$$

This is a 1-coboundary, as desired.  $\square$

In order to obtain the desired rigidity result below, we need the following preliminary observation.

**Proposition 3.3.** *Let*

$$\Gamma_t = \gamma_0 + \gamma_l t^l + (\text{higher terms in } t)$$

be a deformation of  $f$  for some  $l \geq 1$  in which  $\gamma_l$  is a 1-coboundary,  $\gamma_l = d_f^0(\phi_M; \phi_N)$ , in  $Y^1(f)$ . Let  $\Phi_t$  be the formal isomorphism  $(\text{Id}_M + \phi_M t^l; \text{Id}_N + \phi_N t^l)$ . Then the deformation  $\bar{\Gamma}_t = \sum_{i=0}^{\infty} \bar{\gamma}_i t^i$  defined by  $\Phi_t: \Gamma_t \cong \bar{\Gamma}_t$  satisfies  $\bar{\gamma}_j = 0$  for  $1 \leq j \leq l$ .

*Proof.* Since

$$\Phi_t \equiv (\text{Id}_M; \text{Id}_N) \pmod{t^l},$$

it suffices to consider  $\bar{\gamma}_l$ . Now since  $\bar{\Gamma}_t$  is defined by the three conditions in the definition of equivalence, we have:

$$\begin{aligned} \bar{\gamma}_l = & ((\omega_{M,l} + \phi_M \circ \omega_M - \omega_M \circ (\text{Id}_A \otimes \phi_M); \\ & \rho_{M,l} + (\phi_M \otimes \text{Id}_A) \circ \rho_M - \rho_M \circ \phi_M); \\ & (\omega_{N,l} + \phi_N \circ \omega_N - \omega_N \circ (\text{Id}_A \otimes \phi_N); \\ & \rho_{N,l} + (\phi_N \otimes \text{Id}_A) \circ \rho_N - \rho_N \circ \phi_N); \\ & f_l + \phi_N \circ f - f \circ \phi_M). \end{aligned}$$

This is equal to 0 by the hypothesis  $\gamma_l = d_f^0(\phi_M; \phi_N)$ . □

**3.4. Rigidity.** The morphism  $f: M \rightarrow N$  is said to be *rigid* if and only if every deformation of  $f$  is equivalent to the trivial deformation,

$$\Gamma_t = \gamma_0 = ((\omega_M; \rho_M); (\omega_N; \rho_N); f).$$

Combining Theorem 3.2 and Proposition 3.3, we obtain the following cohomological criterion for rigidity.

**Corollary 3.5.** *If  $H_{YD}^1(f)$  is trivial, then  $f$  is rigid.*

#### 4. COHOMOLOGY AND DEFORMATION FOR A MORPHISM OF HOPF MODULES

In final this section, we describe the corresponding results from the last two sections for morphisms of Hopf modules. We will omit the proofs, since they are basically identical to the ones above.

**4.1. Cohomology of Hopf modules.** A *left-right Hopf module*  $(M, \omega, \rho)$  over a bialgebra  $(A, \mu, \Delta)$  is a vector space  $M$  together with a left  $A$ -module action  $\omega: A \otimes M \rightarrow M$  and a right  $A$ -comodule coaction  $\rho: M \rightarrow M \otimes A$  that satisfy the Hopf module condition:

$$\rho \circ \omega = (\omega \otimes \mu) \circ (\text{Id}_A \otimes \tau \otimes \text{Id}_A) \circ (\Delta \otimes \rho).$$

We will simply call  $(M, \omega, \rho)$  a *Hopf module* in what follows.

**4.2. Cohomology for a morphism of Hopf modules.** Let  $(M, \omega_M, \rho_M)$  and  $(N, \omega_N, \rho_N)$  be two Hopf modules over the bialgebra  $A$ . In [9], a bicomplex  $C^{\bullet, \bullet}(M, N)$  is defined such that

$$C^{n,p}(M, N) = \text{Hom}(A^{\otimes n} \otimes M, N \otimes A^{\otimes p}) = Y^{n,p}(M, N)$$

as vector spaces for all  $n, p \geq 0$ . We refer the reader to [9, section 3] for the definitions of the horizontal and vertical differentials in  $C^{\bullet, \bullet}(M, N)$ . The cohomology of the total complex  $C^\bullet(M, N)$  of the bicomplex  $C^{\bullet, \bullet}(M, N)$  is denoted by  $H_{Hopf}^\bullet(M, N)$ .

Fix a morphism  $f: (M, \omega_M, \rho_M) \rightarrow (N, \omega_N, \rho_N)$  of Hopf modules, i.e.  $f$  is compatible with both  $\omega$  and  $\rho$  (exactly as in the Yetter-Drinfel'd case). Define the cochain complex  $C^\bullet(f)$  as in §2, using  $C^\bullet$  and  $C^{\bullet, \bullet}$  in place of  $Y^\bullet$  and  $Y^{\bullet, \bullet}$  everywhere. We call  $C^\bullet(f)$  the *deformation complex of  $f$* . Its cohomology is denoted by  $H_{Hopf}^\bullet(f)$ .

It follows exactly as in §2.4, with  $C$  in place of  $Y$  everywhere, that  $C^\bullet(f)$  is a mapping cylinder  $\text{Cyl}(\varphi: D \rightarrow E)$ . We have the following counterpart of Corollary 2.7.

**Proposition 4.3.** *For  $k \geq 3$ , assume that  $H_{Hopf}^k(M, M)$ ,  $H_{Hopf}^k(N, N)$ ,  $H^{k-1}(C^{\bullet, 0}(M, N), d_m^{\bullet, 0})$ , and  $H^{k-1}(C^{0, \bullet}(M, N), d_c^{0, \bullet})$  are trivial. Then so is  $H_{Hopf}^k(f)$ .*

It is shown in [9, Proposition 9] that

$$H_{Hopf}^k(V \otimes A, W \otimes A) = 0$$

for all  $k \geq 1$  and vector spaces  $V$  and  $W$ . Here the Hopf module structure on  $V \otimes A$  has left  $A$ -module action

$$\omega = (\text{Id}_V \otimes \mu) \circ (\tau \otimes \text{Id}_A),$$

where  $\tau: A \otimes V \cong V \otimes A$  is the twist isomorphism, and right  $A$ -coaction

$$\rho = \text{Id}_V \otimes \Delta.$$

The Hopf module structure on  $W \otimes A$  is defined in the same way. It is also shown there that all the rows and columns of  $C^{\bullet, \bullet}(V \otimes A, W \otimes A)$  are acyclic. Combining this with Proposition 4.3, we obtain the following consequence.

**Corollary 4.4.** *Let  $V$  and  $W$  be arbitrary vector spaces, and let*

$$f: V \otimes A \rightarrow W \otimes A$$

*be a Hopf module morphism. Then*

$$H_{\text{Hopf}}^k(f) = 0$$

*for all  $k \geq 3$ .*

Recall that a *skew antipode* on a bialgebra  $A$  is an endomorphism  $S$  such that

$$\sum S(a_{(2)})a_{(1)} = \sum a_{(2)}S(a_{(1)})$$

for all  $a \in A$ . When  $A$  is a bialgebra with a skew antipode, every Hopf module over  $A$  has the form  $V \otimes A$  for some vector space  $V$  [8, p. 16]. Using Corollary 4.4, this leads to the following vanishing criterion.

**Corollary 4.5.** *If  $f$  is a morphism of Hopf modules over a bialgebra with a skew antipode, then*

$$H_{\text{Hopf}}^k(f) = 0$$

*for all  $k \geq 3$ .*

**4.6. Deformation and rigidity for a morphism of Hopf modules.** The notions of *deformation*, *infinitesimal*, *formal isomorphism*, *equivalence* of deformations, and *rigidity* for a Hopf module morphism  $f$  are defined exactly as in §3, with  $C^\bullet$  and ‘‘Hopf’’ replacing, respectively,  $Y^\bullet$  and ‘‘Yetter-Drinfel'd’’ everywhere.

**Theorem 4.7.** *Let  $\Gamma_t = \sum_{i=0}^{\infty} \gamma_i t^i$  be a deformation of a Hopf module morphism  $f$ . Then the infinitesimal  $\gamma_1$  is a 1-cocycle in  $C^1(f)$  whose cohomology class is determined by the equivalence class of  $\Gamma_t$ . Moreover, if  $H_{\text{Hopf}}^1(f)$  is trivial, then  $f$  is rigid as a Hopf module morphism.*

The proof is essentially the same as the one given in the previous section.

#### REFERENCES

- [1] M. Gerstenhaber, On the deformation of rings and algebras, *Ann. Math.* **79** (1964), 59-103.
- [2] M. Gerstenhaber and S. D. Schack, On the deformation of algebra morphisms and diagrams, *Trans. Amer. Math. Soc.* **279** (1983), 1-50.
- [3] M. Gerstenhaber and S. D. Schack, On the cohomology of an algebra morphism, *J. Algebra* **95** (1985), 245-262.
- [4] M. Gerstenhaber and S. D. Schack, Sometimes  $H^1$  is  $H^2$  and discrete groups deform, *Contemp. Math.* **74** (1988), 149-168.
- [5] M. Gerstenhaber and S. D. Schack, Algebras, bialgebras, quantum groups, and algebraic deformations, *Contemp. Math.* **134** (1992), 51-92.
- [6] M. Gerstenhaber and S. D. Schack, Bialgebra cohomology, deformations, and quantum groups, *Proc. Nat. Acad. Sci. U.S.A.* **87** (1990), 478-481.
- [7] L. Lambe and D. Radford, Algebraic aspects of the quantum Yang-Baxter equation, *J. Algebra* **154** (1992), 228-288.
- [8] S. Montgomery, Hopf algebras and their actions on rings, Amer. Math. Soc. Providence, 1993.
- [9] F. Panaite and D. Ştefan, Deformation cohomology for Yetter-Drinfel'd modules and Hopf (bi)modules, *Comm. Algebra* **30** (2002), 331-345.
- [10] D. Radford, Solutions to the quantum Yang-Baxter equation and the Drinfel'd double, *J. Algebra* **161** (1993), 20-32.
- [11] D. Radford and J. Towber, Yetter-Drinfel'd categories associated to an arbitrary bialgebra, *J. Pure Appl. Algebra* **87** (1993), 259-279.
- [12] P. Schauenburg, Hopf modules and Yetter-Drinfel'd modules, *J. Algebra* **169** (1994), 874-890.
- [13] M. Sweedler, Hopf algebras, Benjamin, New York, 1969.
- [14] D. Yetter, Quantum groups and representations of monoidal categories, *Math. Proc. Cambridge Philos. Soc.* **108** (1990), 261-290.

**Received: June 24, 2007**