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Geometric Structure of Lorentzian Three Dimensional Heisenberg Group

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Abstract

In this paper we study some basic theorems for three dimensional Heisenberg group which is endowed with Lorentz metric and some computations for Killing vector fields.

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1. Introduction:

Let K be a topological group and let M be a closed subgroup. Then the space

$$H = K/M$$

is a Haussdorf space and K operates continously on H by the map

$$\begin{array}{ccccc} K & \times & H & \longrightarrow & H \\ (k & , & xM) & \longrightarrow & (kx)M \end{array}.$$

For each $k \in K$ the map

$$\begin{array}{cccc} \tau(g): & H & \longrightarrow & H \\ & xM & \longrightarrow & kxM \end{array}$$

is a homeomorphism, and using this, we see K act transitively on H. The coset space

$$H = K/M$$

is called a homogenous space and K admits a reductive decomposition $\boldsymbol{k}=\boldsymbol{h}\oplus\boldsymbol{m}$ where $\boldsymbol{h}\subset\boldsymbol{k}$. Here $\boldsymbol{h}$, $\boldsymbol{k}$, $\boldsymbol{m}$ are Lie algebras of H, K, M.

A geodesic $\gamma(t)$ through the origin o of $H = K/M$ is called homogenous if it is an orbit of a one parameter subgroup of K, that is

$$\gamma(t) = \exp(tZ)(o), \quad t \in \mathbb{R}$$

where Z is a nonzero vector of $\boldsymbol{k}$. Let K is the maximal connected group of isometries. If γ is homogenous with respect to some isometry group K', then it is also homogenous with respect to any enlarged group of isometries K.

In [7] , obtained that three dimensional Heisenberg group has three left-invariant Lorentz metric which are

$$g_1 = -dx^2 + dy^2 + (xdy + dz)^2$$

$$g_2 = dx^2 + dy^2 - (xdy + dz)^2$$

$$g_3 = dx^2 + (xdy + dz)^2 - ((1 - x)\,dy - dz)^2 .$$

We know from [3] that three dimensional Lorentzian Heisenberg group is a naturally reductive and a geodesic orbit manifold. So we can write $H_3 = K/M$ and K admits a reductive decomposition $\boldsymbol{k} = \boldsymbol{h}_3 \oplus \boldsymbol{m}$. Here $\boldsymbol{h}_3$, $\boldsymbol{k}$, $\boldsymbol{m}$ are Lie algebras of $Heis_3$, K, M.

2. Preliminaries:

Proposition 2.1: Let $H = K/M$ be a homogenous space and H be a K-manifold. For all $p \in H$ we have the vector field

$$\tilde{X}_p = \frac{d}{dt}\Big|_{t=0} (\exp(tX) .p)$$

,[5] .

Definition 2.2: An analytic vector field X defined on a Lie group G is called invariant if for all $a \in G$

$$[(TL(a))(e)]\,X(e) = X(a) .$$

If X is invariant, then X is $L(a)-$invariant for all $a \in G$; that is, X is actually $G-$invariant or left invariant ,[9] .

Proposition 2.2: Let G be a Lie group, let $X \in T(G,e)$, and let

$$\begin{array}{cccc} \tilde{X}: & G & \longrightarrow & T(G) \ , \\ & p & \longrightarrow & \tilde{X}(p) \end{array}$$

where $T(G)$ is the tangent bundle of G with projection map π and $\tilde{X}(p)$ is given by

$$\left(\tilde{X}f\right)(p) = X(f \circ L(p))$$

where f is any real valued analytic function on G. Then $\tilde{X}$ is a G-invariant analytic vector field on G such that $\tilde{X}(e) = X$. Therefore $\tilde{X}$ is the unique G-invariant vector field on G such that $\tilde{X}(e) = X$. Thus any $G-$invariant vector field is of the form $\tilde{X}$,[9] .

Corollary 2.1: The map

$$\begin{array}{cccc} \phi: & L(G) & \longrightarrow & T(G,e) \\ & \tilde{X} & \longrightarrow & X \end{array}$$

is a Lie group isomorphism ,[9] .

Theorem 2.1: Let G be a Lie group and $\boldsymbol{g}$ be Lie algebra of G. For $\forall$ X, $Y \in \boldsymbol{g}$ we have

$$\left[\tilde{X}, \tilde{Y}\right] = -\widetilde{[X,Y]}$$

,[2] .

Theorem 2.2: Let H homogeneous space be a reductive space and D be Levi-Civita connection of H. For $\forall$ X, Y, $Z \in \boldsymbol{h}$ the map $U : \boldsymbol{h} \times \boldsymbol{h} \longrightarrow \boldsymbol{h}$ defined like follows;

$$2g(U(X,Y), Z) = g([Z,X]_{\boldsymbol{m}}, Y) + g(X, [Z,Y]_{\boldsymbol{m}}) .$$

At each $p \in H$ point and for $\forall$ X, $Y \in \boldsymbol{h}$ we have

$$(D_X Y)_p = -\frac{1}{2}[X,Y]_{\boldsymbol{m}} + U(X,Y)$$

,[2] .

Definition 2.2: (H, g) space is naturally reductive if there exists at least one reductive split $\boldsymbol{k} = \boldsymbol{m} \oplus \boldsymbol{h}$ such that

$$g\left([X, Y]_{\boldsymbol{h}}, Z\right) + g\left([X, Z]_{\boldsymbol{h}}, Y\right) = 0 \tag{2.1}$$

for all X, Y, $Z \in \boldsymbol{h}$,[3] .

Definition 2.3: Heisenberg group defined as a group 3×3 upper triangular matrices

$$\begin{pmatrix} 1 & y & t \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Here x, y, $t \in \mathbb{R}$ and endowed with multiplication

$$(\bar{x}, \bar{y}, \bar{t})(x, y, t) = \left(\bar{x} + x, \bar{y} + y, \bar{t} + t - \frac{1}{2}\bar{x}y + \frac{1}{2}\bar{y}x\right)$$

,[6] .

3. Geometric structure of $Heis_3$.

3.1. Basic theorems

In this section g is the metric one of the Lorentz metric g_1, g_2 and g_3.

Theorem 3.1.1: $(Heis_3, g)$ be 3-dimensional Heisenberg group equipped with Lorentz metric g. $C^\infty(Heis_3, \mathbb{R})$ -linear combination of degenerate Killing vector field is a Killing vector field.

Proof: Let X be a Killing vector field and $f \in C^\infty(Heis_3, \mathbb{R})$. For the proof we must show that fX is a Killing vector field. From properties of Lie derivative, we can write for $\forall\ Y$, $Z \in \chi(Heis_3)$,

$$(L_{fX}g)(Y, Z) = (fX)(g(Y, Z)) - g([fX, Y], Z) - g(Y, [fX, Z]). \tag{3.1.1}$$

If we write equality

$$[fX, Y] = f[X, Y] - Y(f)X$$

in equation (3.1.1), can be obtained

$$(L_{fX}g)(Y, Z) = fX(g(Y, Z)) - f\ g([X, Y], Z) - f\ g(Y, [X, Z]). \tag{3.1.2}$$

Right of (3.1.2) is

$$f(L_X g)(Y, Z). \tag{3.1.3}$$

Since X is a Killing vector field, (3.1.3) equal to zero. So, fX is a Killing vector field.

Theorem 3.1.2 (Kozsul formula for Killing vector fields): $(Heis_3, g)$ be three dimensional Heisenberg group equipped with Lorentz metric g and D be Levi-Civita connection on $Heis_3$. If vector fieds X, Y, Z are Killing, the follow equality is ensured;

$$2g\left(D_X Y, Z\right) = g\left(\left[X, Y\right], Z\right) + g\left(\left[X, Z\right], Y\right) + g\left(X, \left[Y, Z\right]\right).$$

Proof: Since X and Z are Killing, we have

$$g\left(\left[X, Z\right], X\right) = g\left(D_X X, Z\right).$$

From here, since $X + Y$ and Z are Killing, we obtain that

$$g\left(\left[X + Y, Z\right], X + Y\right) = g\left(D_{X+Y} X + Y, Z\right).$$

So we deduce

$$g\left(\left[X, Z\right], Y\right) + g\left(\left[X, Y\right], Z\right) + g\left(\left[X, Z\right], Y\right) + g\left(X, \left[Y, Z\right]\right). \tag{3.1.4}$$

Because of $D_X - D_Y = \left[X, Y\right]$, (3.1.4) is equal to

$$2g\left(D_X Y, Z\right) = g\left(\left[X, Y\right], Z\right) + g\left(\left[X, Z\right], Y\right) + g\left(X, \left[Y, Z\right]\right).$$

Theorem 3.1.3: $(Heis_3, g)$ be 3-dimensional Heisenberg group equipped with Lorentz metric and D be Levi Civita connection on $Heis_3$ and p be a point on $Heis_3$. For all X, Y, $Z \in \boldsymbol{h}_3$

$$g\left(\left(D_{\tilde{X}}\tilde{Y}\right)_p, \tilde{Z}_p\right) = -\frac{1}{2}g\left(\left[X, Y\right]_{\boldsymbol{h}_3}, Z\right) + \frac{1}{2}g\left(\left[Z, X\right]_{\boldsymbol{h}_3}, Y\right) + \frac{1}{2}g\left(X, \left[Z, Y\right]_{\boldsymbol{h}_3}\right).$$

Proof: From Kozsul formula for Killing vector field at the point p we have follow equality;

$$g\left(D_X Y, Z\right) = \frac{1}{2}g\left(\left[\tilde{X}_p, \tilde{Y}_p\right], \tilde{Z}_p\right) + g\left(\left[\tilde{X}_p, \tilde{Z}_p\right], \tilde{Y}_p\right) + g\left(\tilde{X}_p, \left[\tilde{Y}_p, \tilde{Z}_p\right]\right). \tag{3.1.5}$$

Because of

$$\left[X, Y\right]_{\boldsymbol{k}} = -\left[\tilde{X}, \tilde{Y}\right],$$

(3.1.5) is equal to

$$g\left(\left(D_{\tilde{X}}\tilde{Y}\right)_p, \tilde{Z}_p\right) = -\frac{1}{2}g\left([X,Y]_{\boldsymbol{k}}, Z\right) + \frac{1}{2}g\left([Z,X]_{\boldsymbol{k}}, Y\right) + \frac{1}{2}g\left(X, [Z,Y]_{\boldsymbol{k}}\right). \tag{3.1.6}$$

Since for all $X \in \boldsymbol{h}_3$ and $Y \in \boldsymbol{m}$ the follows are ensured

$$g(X,Y) = 0$$

and

$$\boldsymbol{k} = \boldsymbol{h}_3 \oplus \boldsymbol{m},$$

(3.1.6) is equal to

$$g\left(\left(D_{\tilde{X}}\tilde{Y}\right)_p, \tilde{Z}_p\right) = -\frac{1}{2}g\left([X,Y]_{\boldsymbol{h}_3}, Z\right) + \frac{1}{2}g\left([Z,X]_{\boldsymbol{h}_3}, Y\right) + \frac{1}{2}g\left(X, [Z,Y]_{\boldsymbol{h}_3}\right).$$

Theorem 3.1.4: Let K be sectional curvature of the plane of span $\tilde{X}_p$, $\tilde{Y}_p$ at $p \in Heis_3$. The following is true;

$$K\left(\tilde{X}_p, \tilde{Y}_p\right) = \frac{g\left(\left[Y, [X,Y]_{\boldsymbol{m}}\right]_{\boldsymbol{h}_3}, X\right) + \frac{1}{4}g\left([X,Y]_{\boldsymbol{h}_3}, [X,Y]_{\boldsymbol{h}_3}\right)}{g(X,X)\,g(Y,Y) - g(X,Y)^2}.$$

Proof: Because of $Heis_3$ naturally reductive for all X, Y, $Z \in Heis_3$, we obtain

$$g\left(\left(D_{\tilde{X}}\tilde{Y}\right)_p, \tilde{Z}_p\right) = -\frac{1}{2}g\left([X,Y]_{\boldsymbol{h}_3}, Z\right).$$

So for a symmetrical function

$$\Psi : \Gamma(TK) \times \Gamma(TK) \longrightarrow \Gamma\left(TK^{\perp}\right),$$

$$\left(D_{\tilde{X}}\tilde{Y}\right)_p = \frac{1}{2}[X,Y]_{\boldsymbol{h}_3} + \Psi.$$

From here we can deduce

$$\begin{aligned}
g\left(R_{\tilde{X}\tilde{Y}}\tilde{X},\tilde{Y}\right)\underset{p}{|} &= g\left(D_{[X,Y]}X,Y\right)\underset{p}{|} - g\left(D_X D_Y X,Y\right)\underset{p}{|} + g\left(D_Y D_X X,Y\right)\underset{p}{|} \\
&= -g\left(D_Y X,[X,Y]\right)\underset{p}{|} - X_p g\left(D_Y X,Y\right) + g\left(D_Y X,D_X Y\right)\underset{p}{|} \\
&\quad + Y_p g\left(D_X X,Y\right) - g\left(D_X X,D_Y Y\right)\underset{p}{|} \\
&= g\left(D_Y X,D_Y X\right)\underset{p}{|} - g\left(D_X X,D_Y Y\right)\underset{p}{|} + Y_p g\left([X,Y],X\right) \\
&= \frac{1}{4}g\left([X,Y]_{\boldsymbol{h}_3},[X,Y]_{\boldsymbol{h}_3}\right) + g\left([Y,[X,Y]],X\right) + g\left([X,Y],[Y,X]\right) \\
&= \frac{1}{4}g\left([X,Y]_{\boldsymbol{h}_3},[X,Y]_{\boldsymbol{h}_3}\right) + g\left(\left[Y,[X,Y]_{\boldsymbol{h}_3}\right]_{\boldsymbol{h}_3},X\right) \\
&\quad + g\left(\left[Y,[X,Y]_{\boldsymbol{h}_3}\right]_{\boldsymbol{m}},X\right) + g\left([Y,[X,Y]_{\boldsymbol{m}}]_{\boldsymbol{h}_3},X\right) \\
&\quad + g\left([Y,[X,Y]_{\boldsymbol{m}}]_{\boldsymbol{m}},X\right) + g\left([X,Y]_{\boldsymbol{h}_3},[Y,X]_{\boldsymbol{h}_3}\right) \\
&\quad + g\left([X,Y]_{\boldsymbol{h}_3},[Y,X]_{\boldsymbol{m}}\right) + g\left([X,Y]_{\boldsymbol{m}},[Y,X]_{\boldsymbol{h}_3}\right) + g\left([X,Y]_{\boldsymbol{m}},[Y,X]_{\boldsymbol{m}}\right) \\
&= \frac{1}{4}g\left([X,Y]_{\boldsymbol{m}},[Y,X]_{\boldsymbol{m}}\right) + g\left(\left[Y,[X,Y]_{\boldsymbol{h}_3}\right]_{\boldsymbol{h}_3},X\right) \\
&\quad + g\left([Y,[X,Y]_{\boldsymbol{m}}]_{\boldsymbol{h}_3},X\right) + g\left([X,Y]_{\boldsymbol{m}},[Y,X]_{\boldsymbol{m}}\right).
\end{aligned}$$

From naturally reductivetly, we have the equality

$$g\left(\left[Y,[X,Y]_{\boldsymbol{h}_3}\right]_{\boldsymbol{h}_3},X\right) = -g\left([X,Y]_{\boldsymbol{m}},[Y,X]_{\boldsymbol{m}}\right).$$

From here we obtain

$$g\left(R_{\tilde{X}\tilde{Y}}\tilde{X},\tilde{Y}\right) \underset{p}{|}= g\left([Y,[X,Y]_{\boldsymbol{m}}]_{\boldsymbol{h}_3},X\right) + \frac{1}{4}g\left([X,Y]_{\boldsymbol{h}_3},[X,Y]_{\boldsymbol{h}_3}\right).$$

So the sectional curvature is

$$K\left(\tilde{X}_p,\tilde{Y}_p\right) = \frac{g\left([Y,[X,Y]_{\boldsymbol{m}}]_{\boldsymbol{h}_3},X\right) + \frac{1}{4}g\left([X,Y]_{\boldsymbol{h}_3},[X,Y]_{\boldsymbol{h}_3}\right)}{g(X,X)\,g(Y,Y) - g(X,Y)^2}.$$

3. 2. Computations for Killing vector fields

3.2.1. Computations for $(Heis_3, g_1)$

Let obtain some computations for the Lorentz metric

$$g_1 = -dx^2 + dy^2 + (xdy + dz)^2$$

The Lie algebra of $Heis_3$, $\boldsymbol{h}_3$ has an orthonormal basis

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x}. \tag{3.7}$$

Here we can obtain

$$g_1(e_1,e_1)=g_1(e_2,e_2)=1,\ g_1(e_3,e_3)=-1.$$

For this basis Lie products can be obtain as

$$[e_2,e_3]=e_1,\quad [e_3,e_1]=0,\quad [e_2,e_1]=0.$$

We know from [6] that Killing vector fields are for the metric g_1

$$\begin{aligned} X_1 &= y\frac{\partial}{\partial x}+x\left(\frac{\partial}{\partial y}-x\frac{\partial}{\partial z}\right)+\frac{1}{2}\left(x^2-y^2\right)\frac{\partial}{\partial z}, \qquad X_2=\frac{\partial}{\partial x}-y\frac{\partial}{\partial z},\\ X_3 &= \frac{\partial}{\partial y}, \qquad X_4=\frac{\partial}{\partial z}. \end{aligned}$$

Firstly let examine these Killing vector fields are spacelike, timelike or nulllike.

$$\begin{aligned} g_1(X_1,X_1) &= g_1\left(y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}-\frac{1}{2}\left(x^2+y^2\right)\frac{\partial}{\partial z},y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}-\frac{1}{2}\left(x^2+y^2\right)\frac{\partial}{\partial z}\right)\\ &= x^2-y^2+\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2. \end{aligned}$$

So case of the Killing vector field X_1 is change with choosing of the x and y.

$$\begin{aligned} g_1(X_2,X_2) &= g_1\left(\frac{\partial}{\partial x}-y\frac{\partial}{\partial z},\frac{\partial}{\partial x}-y\frac{\partial}{\partial z}\right)\\ &= y^2-1 \end{aligned}$$

We obtain that the case of the Killing vector field X_2 is change with choosing of the y.

$$\begin{aligned} g_1(X_3,X_3) &= g_1\left(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right)\\ &= 1>0 \end{aligned}$$

So the Killing vector field X_3 is spacelike.

$$g_1(X_4,X_4)=1>0$$

We deduce that the Killing vector field X_4 is spacelike. For these Killing vector fields if we find Lie brackets, we obtain that

$$\begin{aligned} &[X_1,X_1]=0,\quad [X_1,X_2]=x\frac{\partial}{\partial z},\quad [X_1,X_3]=-y\frac{\partial}{\partial z},\quad [X_1,X_4]=0,\\ &[X_2,X_1]=-x\frac{\partial}{\partial z},\quad [X_2,X_2]=0,\quad [X_2,X_3]=-\frac{\partial}{\partial z},\quad [X_2,X_4]=0,\\ &[X_3,X_1]=y\frac{\partial}{\partial z},\quad [X_3,X_2]=\frac{\partial}{\partial z},\quad [X_3,X_3]=0,\quad [X_3,X_4]=0,\\ &[X_4,X_1]=[X_4,X_2]=[X_4,X_3]=[X_4,X_4]=0. \end{aligned}$$

If we calculate sectional curvatures for fundamental Killing vector fields with theorem 3.1.4, we obtain that

$$K\left(\tilde{X}_1, \tilde{X}_2\right) = \frac{1/4x^2}{y^2 - x^2 - \frac{x^4}{4}\left(1-y^2\right) - \frac{y^4}{2}\left(\frac{5}{2}+x^2\right) + \frac{3}{2}x^2y^2 + \frac{y^6}{4} + y + \frac{1}{2}x^2y - \frac{1}{2}y^3}$$

$$K\left(X_1, X_3\right) = \frac{1/4y^2}{-y^2 + \frac{9x^4}{4} + \frac{y^4}{4} - \frac{5x^2y^2}{2}}$$

$$K\left(X_1, X_4\right) = 0$$

$$K\left(\tilde{X}_2, \tilde{X}_3\right) = \frac{1/4}{y^2 - x^2y^2 + x^2}, \quad K\left(\tilde{X}_2, \tilde{X}_4\right) = 0, \ \ K\left(\tilde{X}_3, \tilde{X}_4\right) = 0.$$

3.2.3. Computations for $(Heis_3, g_3)$

Lastly we want to find same properties for the Lorentz metric

$$g_3 = dx^2 + \left(xdy + dz\right)^2 - \left(\left(1-x\right)dy - dz\right)^2.$$

The Lie algebra of $Heis_3$, $\boldsymbol{h}_3$ has an orthonormal basis

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + \left(1-x\right)\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}. \tag{3.12}$$

Here we can obtain

$$g_3\left(e_1, e_1\right) = g_3\left(e_2, e_2\right) = 1, \ \ g_3\left(e_3, e_3\right) = -1.$$

For this basis the Lie products are

$$[e_2, e_3] = 0, \quad [e_1, e_2] = e_3 - e_2, \quad [e_1, e_3] = e_3 - e_2.$$

We know from [6] that Killing vector fields are for the metric g_3

$$\begin{aligned} X_1 &= y^2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \left(2x-1\right)y\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \\ X_4 &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \left(x^2 - x\right)\frac{\partial}{\partial z}, \quad X_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad X_6 = \frac{\partial}{\partial z} \end{aligned}$$

Firstly let examine these Killing vector fields are spacelike, timelike or null like.

$$\begin{aligned} g_3\left(X_1, X_1\right) &= g_3\left(y^2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - (2x-1)\,y\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - (2x-1)\,y\frac{\partial}{\partial z}\right) \\ &= y^2\left(y^2+2x-1\right) \end{aligned}$$

So case of the Killing vector field X_1 is change with choosing of the x and y.

$$g_3\left(X_2, X_2\right) = 1 > 0$$

We obtain that the Killing vector field X_2 is spacelike.

$$g_3\left(X_3, X_3\right) = 2x - 1$$

So case of the Killing vector field X_3 is change with choosing of the x.

$$g_3\left(X_4, X_4\right) = z^2 + x^2$$

We obtain that the case of the Killing vector field X_4 is change with choosing of the x and z.

$$g_3\left(X_5, X_5\right) = y^2$$

So case of the Killing vector field X_5 is change with choosing of the y.

$$g_3\left(X_6, X_6\right) = 0$$

So the Killing vector field X_6 is a null like vector field. For these Killing vector fields if we find Lie brackets, we obtain that

$$\begin{aligned} \left[X_1, X_1\right] &= 0, \quad \left[X_1, X_2\right] = y\frac{\partial}{\partial z}, \quad \left[X_1, X_3\right] = -y^2\frac{\partial}{\partial z}, \\ \left[X_1, X_4\right] &= \frac{1}{2}\left(xy^2 + yz\right)\frac{\partial}{\partial z}, \quad \left[X_1, X_5\right] = y^2\frac{\partial}{\partial z}, \quad \left[X_1, X_6\right] = 0 \end{aligned}$$

$$\begin{aligned} \left[X_2, X_1\right] &= -y\frac{\partial}{\partial z}, \quad \left[X_2, X_2\right] = 0, \quad \left[X_2, X_3\right] = -\frac{\partial}{\partial z}, \\ \left[X_2, X_4\right] &= x\frac{\partial}{\partial z}, \quad \left[X_2, X_5\right] = 0, \quad \left[X_2, X_6\right] = 0. \end{aligned}$$

$$\begin{aligned} \left[X_3, X_1\right] &= -y^2\frac{\partial}{\partial z}, \quad \left[X_3, X_2\right] = \frac{\partial}{\partial z}, \quad \left[X_3, X_3\right] = 0, \\ \left[X_3, X_4\right] &= z\frac{\partial}{\partial z}, \quad \left[X_3, X_5\right] = y\frac{\partial}{\partial z}, \quad \left[X_3, X_6\right] = 0. \end{aligned}$$

$$\begin{aligned}
[X_4, X_1] &= -\frac{1}{2}\left(xy^2 + yz\right)\frac{\partial}{\partial z}, \quad [X_4, X_2] = -x\frac{\partial}{\partial z}, \quad [X_4, X_3] = z\frac{\partial}{\partial z}, \\
[X_4, X_4] &= 0, [X_4, X_5] = -xy\frac{\partial}{\partial z}, \quad [X_4, X_2] = 0.
\end{aligned}$$

$$\begin{aligned}
[X_5, X_1] &= -y^2\frac{\partial}{\partial z}, \quad [X_5, X_2] = 0, \quad [X_5, X_3] = -y\frac{\partial}{\partial z}, \\
[X_5, X_4] &= xy\frac{\partial}{\partial z}, \quad [X_5, X_6] = 0.
\end{aligned}$$

$$[X_6, X_i] = 0, \quad 1 \leq i \leq 6$$

If we calculate sectional curvatures for fundamental Killing vector fields

$$K\left(\tilde{X}_i, \tilde{X}_j\right) = 0, \ 1 \leq i, j \leq 6.$$

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