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Arithmetic and Metric Properties of p-Adic Alternating Engel Series Expansions

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Abstract

In this paper, the alternating Engel expansion over the field of p-adic series is studied. Metric properties, such as strong and weak number laws, central limit theorem, and iterated logarithm law, of the digits occurring in this expansion are considered. At the same time, the approximation orders by rational fractions which are the partial sums of the series are investigated.

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1 Introduction

Let $\mathbb{Q}$ be the field of rational numbers, p be a prime number and $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the $p-$adic absolute value $|\ \ |_p$ defined on $\mathbb{Q}$ by([6])

$$|0|_p = 0 \ \ and \ \ |A|_p = p^{-a} \ \ if \ \ A = p^a \frac{r}{s}, \ where\ p\ is\ not\ divisible\ with\ r\ and\ s..$$

The exponent a in this definition is the $p-$adic valuation of A, which denoted by $\nu_p(A)$. Then $\mathbb{Q}_p$ is the field of $p-$adic numbers with valuation $|\ \ |_p$, the extension of the original valuation on $\mathbb{Q}$, which has the properties

$$|A|_p \geq 0 \ \ with |A|_p = 0 \ \ iff \ \ A = 0, |AB|_p = |A|_p| \cdot B|_p,$$
$$and \ \ |A+B|_p \leq \max(|A|_p, |B|_p) \ \ with \ \ equality \ \ when \ \ |A|_p \neq |B|_p.$$

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It is familiar that the above non-Archimedean valuation leads to an ultrametric distance function ρ on $\mathbb{Q}_p$, with $\rho(A,B) = |A-B|_p$, making $\mathbb{Q}_p$ into a complete metric space with respect to ρ. As is common, we define the order $\nu_p(A)$ of A by $|A|_p = p^{-\nu_p(A)}$, and set $\nu_p(0) = +\infty$. Then $\nu_p(AB) = \nu_p(A) + \nu_p(B)$, $\nu_p(\frac{A}{B}) = \nu_p(A) - \nu_p(B)$ if $B \neq 0$, and $\nu_p(A+B) \geq \min(\nu_p(A), \nu_p(B))$ with equality when $\nu_p(A) \neq \nu_p(B)$. It is well known that every $A \in \mathbb{Q}_p$ has a unique series representation([6])

$$A = \sum_{n=\nu_p(A)}^{\infty} c_n p^n, \quad c_n \in \{0, 1, \cdots, p-1\}.$$

In the discussion below we call $\langle A \rangle = \sum_{\nu_p(A) \leq n \leq 0} c_n p^n$ the fractional part of A. Then $\langle A \rangle \in S_p$, where we define $S_p = \{\langle A \rangle : A \in \mathbb{Q}_p\} \subset \mathbb{Q}$. The set S_p is not multiplicatively or additively closed. The function $\langle A \rangle$ and the set S_p have been used in the study of certain types of p-adic continued fractions by Laohakosol([7]) in particular.

The fraction part $\langle A \rangle$ was used by A.Knopfmacher and J.Knopfmacher([1,2]) to derive some new unique series expansions for any element $A \in \mathbb{Q}_p$, including particular of certain "Sylvester", "Engel" and "Lüroth" expansions of arbitrary real numbers into series with rational terms([5], Chap. IV).

In the corresponding case of $p-$adic Lüroth type expansions ergodic and other metric properties have been investigated by A. and J.Knopfmacher[3]. For both the p-adic continued fractions and Lüroth expansions, ergodicity of the corresponding transformations were used to derive the results. However, in the case of Engel, Peter J. Grabner and Arnold Knopfmacher have obtained some similar metric asymptotic results for the p-adic Lüroth type expansions.

For Oppenheim expansions over the field of Laurent series, Ai-hua Fan and Jun Wu([5]) studied the metric properties and the approximation orders by rational fractions which are the partial sums of these series. The aim of this paper is to derive metric properties of the digits $\{a_n(x)\}$ for the alternating Engel expansion expansions of p-adic series.

Let I denote the valuation ideal $p\mathbb{Z}_p$ in the ring of p-adic integers $\mathbb{Z}_p$ and let $\mathbf{P}$ denote probability with respect to the Haar measure on $(\mathbb{Q}_p, +)$ normalized by $\mathbf{P}(I) = 1$. The Haar measure on I is the product measure on $\{0, \cdots, p-1\}^{\mathbb{N}}$ defined by $\mathbf{P}(\{x\}) = p^{-1}$ for each factor and any element $x \in \{0, \cdots, p-1\}$.

Considering $\{a_n(x)\}_{n\geq 1}$ as a sequence of random variables defined on the probability space $(I, \mathbf{P})$, we find the finite dimensional joint distributions of this sequence. It turns out that both $\{a_n(x)\}$ and $\nu_p(a_n(x))$ are Markov chains and that their transition probabilities are by explicit formulas. These allow us to prove that $\{\triangle_n(x) = \nu_p(a_n(x)) - \nu_p(a_{n+1}(x))\}_{n\geq 0}$ is a sequence of i.i.d. geomitrical random variable. As a consequence of the main result and the classical law of large numbers and central limit theorem, we deduce the metric properties of $\sum_{j=0}^{n-1} \triangle_j$ and the approximation order of a typical p-adic series by rational functions.

2 Metric theory

We are now in the position to introduce the algorithm of alternating Engel expansion expansion over the field of p-adic series.

Given any $A \in \mathbb{Q}_p$, note that if $\langle A \rangle = a_0 \in S_p$, then $\nu_p(A - a_0) \geq 1$. Then define $A_1 = A - a_0$. Suppose $A_n (n \geq 1)$ is defined. If $A_n \neq 0$, then let $a_n = \langle \frac{1}{A_n} \rangle$, and define $A_{n+1} = 1 - A_n a_n$. If $A_n = 0$, then this recursive stops. We call $\{a_n\}$ the digits of A.

Lemma 2.1 *If $A_n \neq 0$, then $\nu(a_{n+1}) \leq \nu_p(a_n) - 1$.*

Theorem 2.2 *Every $x \in \mathbb{Q}_p$ has a finite or an infinite convergent (relative to ρ) expansion of the form*

$$x = a_0 + \frac{1}{a_1} + \sum_{n=2}^{+\infty} (-1)^{n+1} \frac{1}{a_1 \cdots a_n}. \tag{1}$$

where $a_n \in S_p$, $a_0 = \langle A \rangle$, and $\nu_p(a_1) \leq -1$, $\nu_p(a_{n+1}) \leq \nu_p(a_n) - 1$. Furthermore, the expansion is unique.

The proofs of the Lemma2.1 and the Theorem2.2 are very similar to that of Oppenheim series of p-adic([2]), we will omit the process.

In this section, we are concentrated in deriving the metric properties of digits occurring in the expansion. Actually, we prove that the sequence of digits $\{a_n(x)\}_{n \geq 1}$ is a Markov chain and so is $\{\nu_p(a_n(x))\}_{n \geq 1}$. We also obtain the finite dimensional joint distributions for both Markov chains. Most importantly, we find for any $n \geq 1$, $\triangle_0(x) := -\nu_p(a_1(x))$, $\triangle_n(x) := \nu_p(a_n(x)) - \nu_p(a_{n+1}(x))$ are independent and identically distributed as a sequence of random variables.

Definition 2.3 *A finite sequence $\{k_1, \cdots, k_n\} \subset S_p$ is said to be admissible, if it satisfies the admissibility condition $\nu_p(a_{n+1}) \leq 2\nu_p(a_n) - 1$.*

Lemma 2.4 *Let $x \in I$ whose alternating Engel expansion expansion of p-adic series is*

$$x = \sum_{n=1}^{\infty} c_n(x), \qquad c_n(x) = (-1)^{n+1} \frac{1}{a_1(x) \cdots a_n(x)}.$$

We have $|c_{n+1}(x)| \leq |c_n(x)|$ for all $n \geq 1$.

PROOF.Notice that $\nu_p(c_n(x)) = -\sum_{i=1}^{n} \nu_p(a_i(x))$. The difference $\nu_p(c_{n+1}(x)) - \nu_p(c_n(x))$ is equal to $-\nu_p(a_{n+1}(x))$, which is strictly positive by the admissible condition.

Lemma 2.5 *Suppose* $\{a_1, a_2, \cdots, a_n\} \subset S_p$ $n \geq 1$ *is admissible, then*

$$\{x \in I : a_1(x) = a_1, \cdots, a_n(x) = a_n\} := B(a_1, \cdots, a_n)$$

called n*-digits cylinder is equal to the disc* $B(C_n, D_n)$ *with center*

$$C_n := \frac{1}{a_1} + \sum_{j=2}^{n} (-1)^{j-1} \frac{1}{a_1 \cdots a_n}$$

and diameter

$$D_n = p^{\sum_{j=1}^{n-1} \nu_p(a_j) + 2\nu_p(a_n) - 1}. \tag{2}$$

PROOF. For any $x \in B(a_1, \cdots, a_n)$, by Lemma 2.1,

$$|x - C_n| = |c_{n+1}| = p^{\sum_{j=1}^{n} \nu_p(a_j) + \nu_p(a_{n+1}(x))} \leq p^{\sum_{j=1}^{n-1} \nu_p(a_j) + 2\nu_p(a_n) - 1}.$$

It follow that $x \in B(C_n, B_n)$. Thus we get

$$\{x \in I : a_1(x) = a_1, \cdots, a_n(x) = a_n\} \subset B(C_n, B_n).$$

Conversely, for any $y \in B(C_n, D_n)$, then $y = \sum_{j=1}^{\infty} c_j(y)$. We are going to show that $a_j(y) = a_j$ for all $1 \leq j \leq n$. Firstly, we prove $a_1(x) = a_1$ by contradiction. Suppose $a_1(y) \neq a_1$.There are two cases.

Case I: $\nu_p(a_1(y) \neq \nu_p(a_1)$. In this case, we have

$$|\frac{1}{a_1(y)} - \frac{1}{a_1}| = \max(|\frac{1}{a_1(y)}|, |\frac{1}{a_1}|).$$

By Lemma2.2, we have

$$|\frac{1}{a_1(y)}| \geq |c_j(y)|(j \geq 2); |\frac{1}{a_1}| > |c_j|, (2 \leq j \leq n.)$$

Therefore

$$\max(|\frac{1}{a_1(y)}|, |\frac{1}{a_1}|) > \max(\max_{2 \leq j \leq n} \max(|c_j(y)|, |c_j|), \sup_{j \geq n+1} |c_j(y)|)$$

It follows that

$$|y - C_n| = |\frac{1}{a_1(y)} - \frac{1}{a_1}| = \max(|\frac{1}{a_1(y)}|, |\frac{1}{a_1}|) \geq |\frac{1}{a_1}| = p^{\nu_p(a_1)}$$

By the admissibility condition, we may check that $\log_p D_m$ is decreasing, so that

$$D_n \leq D_1 = p^{2\nu_p(a_1)-1} < p^{\nu_p(a_1)} \leq |y - C_n|,$$

which contradicts the fact $y \in B(C_n, D_n)$.

Case II: $\nu_p(a_1(y)) = \nu_p(a_1)$. In this case, using (4), we can prove by induction on j that

$$|\frac{1}{a_1(y)}|^2 \geq |c_j(y)|, (j \geq 2;) \qquad |\frac{1}{a_1}|^2 \geq |c_j(y)|, (2 \leq j \leq n).$$

Thus

$$\begin{aligned}&|\frac{1}{a_1(y)} - \frac{1}{a_1}| \geq |\frac{1}{a_1}|^2 = |\frac{1}{a_1(y)}|^2 \\ &> \max(\max_{2\leq j\leq n} \max(|c_j(y)|, |c_j|), \sup_{j\geq n+1} |c_j(y)|).\end{aligned}$$

Hence

$$|y - C_n| = |\frac{1}{a_1(y)} - \frac{1}{a_1}| \geq |\frac{1}{a_1}|^2 = p^{2\nu_p(a_1)} > D_1 \geq D_n$$

Hence we have proved $a_1(y) = a_1$. In the same way, $a_j(x) = a_j$, for all $1 \leq j \leq n$.

Proposition 2.6 *Suppose* $\{a_1, , \cdots, a_n, a_{n+1}\} \subset S_p$ *is admissible, then*

$$\mathbf{P}\{B(a_1, \cdots, a_n)\} = p^{\sum_{j=1}^{n-1} \nu_p(a_j) + 2\nu_p(a_n)} \tag{3}$$

Proof. *By Lemma2.5,* $\{x \in I : a_1(x) = a_1, \cdots, a_n(x) = a_n\}$ *is the disc* $B(C_n, D_n)$*, thus by the definition of probability* $\mathbf{P}$*, we have*

$$\mathbf{P}\{B(C_n, D_n)\} = pD_n = p^{\sum_{j=1}^{n-1} \nu_p(a_j) + 2\nu_p(a_n)}.$$

Proposition 2.7 *Suppose that* $\{x \in I_0 : a_1(x) = a_1, \cdots, a_n(x) = a_n\}$ *is an admissible sequence, then we have* $\mathbf{P}\{a_{n+1}(x) = a_{n+1}|a_n(x) = a_n\} = \frac{|a_n|_p}{|a_{n+1}|_p^2}$.

Proof. *By Proposition2.6,*

$$\begin{aligned}&\mathbf{P}\{a_{n+1}(x) = a_{n+1}|a_n(x) = a_n, \cdots, a_1(x) = a_1\} \\ = \;& \frac{\mathbf{P}\{a_1(x) = a_1, \cdots, a_{n+1}(x) = a_{n+1}\}}{\mathbf{P}\{a_1(x) = a_1, \cdots, a_n(x) = a_n\}} \\ = \;& \frac{p^{2\nu_p(a_{n+1})}}{p^{\nu_p(a_n)}} = \frac{|a_n|_p}{|a_{n+1}|_p^2}.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& \mathbf{P}\{a_{n+1}(x)=a_{n+1}|a_n(x)=a_n\} \\
= & \frac{\mathbf{P}\{a_n(x)=a_n, a_{n+1}(x)=a_{n+1}\}}{\mathbf{P}\{a_n(x)=a_n\}} \\
= & \frac{\sum \mathbf{P}\{a_j(x)=l_j, 1\le j\le n-1, a_n(x)=a_n, a_{n+1}(x)=a_{n+1}\}}{\sum \mathbf{P}\{a_j(x)=l_j, 1\le j\le n-1, a_n(x)=a_n\}} \\
= & \frac{\sum p^{\sum_{j=1}^{n}\nu_p(a_j)+2\nu_p(a_{n+1})}}{\sum p^{\sum_{j=1}^{n-1}\nu_p(a_j)+2\nu_p(a_n)}} = \frac{|a_n|_p}{|a_{n+1}|_p^2}.
\end{aligned}
$$

where both the summation in the numerators are taken over all sequences $\{l_1,\cdots,l_{n-1}\}\subset S_p$, *such that* $\{l_1,\cdots,l_{n-1},k_n\}$ *is admissible. Consequently, the sequence* $\{a_j(x)\}_{j\ge 1}$ *forms a Markov chain with the transition probability*

$$\mathbf{P}\{a_{n+1}(x)=a_{n+1}\big|a_n(x)=a_n\} = \frac{|a_n|_p}{|a_{n+1}|_p^2}.$$

Proposition 2.8 *Suppose that* $\{k_1,\cdots,k_n\}$ *is an admissible sequence, then we have*

$$
\begin{aligned}
& \mathbf{P}\{x\in I:\nu_p(a_1(x))=\nu_p(k_1),\cdots,\nu_p(a_n(x))=\nu_p(k_n)\} \\
= & (p-1)^n p^{\nu_p(k_n)},
\end{aligned}
$$

and consequently, the sequence $\{\nu_p(a_j(x))\}_{j\ge1}$ *forms a Markov chain with the transition probability*

$$\mathbf{P}\{\nu_p(a_{n+1}(x))=\nu_p(a_{n+1})|\nu_p(a_n(x))=\nu_p(a_n)\} = (p-1)|\frac{|a_n|_p}{|a_{n+1}|_p}.$$

Proof.First remark that for any $n\ge 1$ and any $a,l\in S_p$ such that $\nu_p(k)=\nu_p(l)\le -1$, we have $\nu_p(k)=\nu_p(l)$, by Proposition2.6, we have

$$
\begin{aligned}
& \mathbf{P}\{x\in I:\nu_p(a_1(x))=\nu_p(k_1),\cdots,\nu_p(a_n(x))=\nu_p(k_n)\} \\
= & \sum \mathbf{P}\{a_1(x)=l_1,\cdots,a_n(x)=l_n\} \\
= & \sum p^{\sum_{j=1}^{n-1}\nu_p(l_j)+2\nu_p(l_n)} \\
= & (p-1)^n p^{\nu_p(k_n)}
\end{aligned}
$$

where both the summation in the numerators are taken over all sequences $\{l_1,\cdots,l_n\}\subset S_p$, such that $\nu_p(l_j)=\nu_p(k_j),\ 1\le j\le n$. It is trivial that

$$\mathbf{P}\{\nu_p(a_{n+1}(x))=\nu_p(a_{n+1})|\nu_p(a_n(x))=\nu_p(a_n)\} = (p-1)\frac{|a_n|_p^2}{|a_{n+1}|_p}.$$

Theorem 2.9 $\{\triangle_n\}_{n\geq 0}$ *is a sequence of independent and identically distributed random variables. Furthermore,* $\deg a_1(x)$ *is geometrical, i.e. for any* $m \geq 1$,

$$\mathbf{P}\{x \in I : \triangle_1(x) = m\} = \frac{p-1}{p^m}. \tag{4}$$

Proof.

$$\begin{aligned}
\mathbf{P}\{x \in I : \triangle_{n+1} = m\} &= \sum_{a_1,\cdots,a_{n+1}} \mathbf{P}(B(a_1,\cdots,a_{n+1})) \\
&= \sum_{a_1,\cdots,a_n} \mathbf{P}(B(a_1,\cdots,a_n)) \sum_{a_{n+1}} p^{2\nu_p(a_{n+1})} \\
&= \sum_{a_1,\cdots,a_n} \mathbf{P}(B(d_1,\cdots,a_n))(p-1)p^{-m} = \frac{p-1}{p^m}
\end{aligned}$$

where the summation on $\{a_1,\cdots,a_n,a_{n+1}\}$ is over all the admissible sequence with $2\nu_p(a_n(x)) - \nu_p(a_{n+1}(x) = m$.
At the same time, it is easy to see, for any $m \geq 1$,

$$\mathbf{P}\{x \in I : \triangle_1(x) = m\} = \frac{p-1}{p^m}.$$

For any positive integer $m_1, m_2, \cdots, m_{n+1}$,

$$\begin{aligned}
& \mathbf{P}\{x \in I : \triangle_j(x) = m_j, 1 \leq j \leq n+1\} \\
= & \sum_{a_1,\cdots,a_{n+1}} \mathbf{P}\{x \in I : a_1(x) = a_1,\cdots,a_{n+1}(x) = a_{n+1}\} \\
= & \sum_{a_1,\cdots,a_n} \mathbf{P}(a_1,\cdots,a_n) \sum_{a_{n+1}} p^{2\nu_p(a_{n+1})-2\nu_p(a_n)} \\
= & \sum_{a_1,\cdots,a_n} \mathbf{P}(a_1,\cdots,a_n)\frac{p-1}{p^{m_{n+1}}} \\
= & \mathbf{P}\{x : \triangle_j(x) = m_j, 1 \leq j \leq n\}\frac{p-1}{p^{m_{n+1}}} \\
= & \prod_{j=0}^{n} \mathbf{P}\{\triangle_j(x) = m_{j+1}\}
\end{aligned}$$

where all the summations are taken over all admissible sequence with $2\nu_p(a_{j-1}) - \nu_p(a_j) = m_j$, for all $1 \leq j \leq n+1$.
By this theorem, it can be easily testified the following proposition.

Proposition 2.10

$$\mathbf{E}(\triangle_n(x)) = \frac{p}{p-1}, \qquad \mathbf{Var}(\triangle_n(x)) = \frac{p}{(p-1)^2}. \tag{5}$$

Proof.

$$\mathbf{E}(\triangle_n(x)) = \sum_{l=1}^{\infty} l\mathbf{P}(\triangle_n(x) = l) = (p-1)\sum_{l=1}^{\infty} lp^{-l} = \frac{p}{p-1},$$

$$\mathbf{E}(\triangle_n^2(x)) = \sum_{l=1}^{\infty} l^2\mathbf{P}(\triangle_n(x) = l) = \frac{p}{p-1} + 2\frac{p}{(p-1)^2},$$

from which the formula for $Var(\triangle_n(x))$ is immediate.As consequence of Theorem 2.9, and the classical limit theorems on $i.i.d.$ random variables, we get immediately the following metric properties:

Theorem 2.11 *For the alternating Engel expansion expansions over the field of p-adic series, we have:*

(i) *For* $\mathbf{P}$*-almost all* $x \in I$, $\triangle_n(x) > \log_p \phi(n)$ *i.o. iff* $\sum_{n=1}^{+\infty} \frac{1}{\phi(n)} = +\infty.$

(ii) $\lim_{n\to+\infty} \mathbf{P}\{x \in I : \frac{\sum_{j=1}^{n} \nu_p(a_j(x)) - \nu_p(a_{n+1}(x))}{\sqrt{np}/(p-1)} < t\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-u^2/2}du.$

(iii) *For* $\mathbf{P}$*-almost all* $x \in I$,

$$\lim_{n\to+\infty} \frac{1}{n}\Big(\sum_{j=1}^{n} \nu_p(a_j(x)) - \nu_p(a_{n+1}(x))\Big) = \frac{p}{p-1}. \tag{6}$$

(iv) *For* $\mathbf{P}$*-almost all* $x \in I_0$,

$$\limsup_{n\to+\infty} \frac{\sum_{j=1}^{n} \nu_p(a_j(x)) - \nu_p(a_{n+1}(x)) - \frac{p}{p-1}n}{\sqrt{2n\log\log n}} = \frac{\sqrt{p}}{p-1} \tag{7}$$

and

$$\liminf_{n\to+\infty} \frac{\sum_{j=1}^{n} \nu_p(a_j(x)) - \nu_p(a_{n+1}(x)) - \frac{p}{p-1}n}{\sqrt{2n\log\log n}} = -\frac{\sqrt{p}}{p-1}. \tag{8}$$

An direct application of (i) yields:

Corollary 2.12 *For* $\mathbf{P}$*-almost all* $x \in I$,

$$\limsup_{n\to+\infty} \frac{2\nu_p(a_n(x)) - \nu_p(a_{n+1}(x)) - \log_p n}{\log_p \log_p n} = 1. \tag{9}$$

Theorem 2.13 $\frac{1}{n\log_p n}\sum_{j=1}^{n} p^{\triangle_j(x)}$ *converges in probability to* $p-1$. *That is to say, for any fixed* $\epsilon > 0$,

$$\lim_{n\to+\infty} \mathbf{P}\{x\in I : |\frac{1}{n\log_p n}\sum_{j=1}^{n} p^{\triangle_j(x)} - (p-1)| > \epsilon\} = 0. \tag{10}$$

Proof. Fix $n \geq 1$. For any $1 \leq k \leq n$, define

$$U_k(x) = \left\{ \begin{array}{ccc} p^{\triangle_k(x)}, & \text{if } p^{\triangle_k(x)} \leq n\log_p n, & 0 \\ 0, & \text{otherwise} & p^{\triangle_k(x)} \end{array} \right\} = V_k(x).$$

$$\begin{aligned} &\mathbf{P}\{x\in I : |\frac{1}{n\log_p n}\sum_{k=1}^{n} p^{\triangle_k(x)} - (p-1)| > \epsilon\} \\ &\leq \mathbf{P}\{x\in I : |\frac{\sum_{k=1}^{n} U_k(x)}{n\log_p n} - (p-1)| > \epsilon\} + \mathbf{P}\{x\in I : \sum_{k=1}^{n} V_k(x) \neq 0\} \\ &=: \mathbf{P}(A_n) + \mathbf{P}(B_n). \end{aligned}$$

Since

$$\begin{aligned} \mathbf{P}(B_n) \leq & \sum_{k=1}^{n} \mathbf{P}\{x\in I : V_k(x)\neq 0\} = \sum_{k=1}^{n} \mathbf{P}\{x\in I : p^{\triangle_k(x)} > \log_p n\} \\ = & \sum_{k=1}^{n} \sum_{l:p^l > n\log_p n} \frac{p-1}{p^l} = O(\frac{1}{\log n}). \end{aligned}$$

Notice that

$$\begin{aligned} &\mathbf{E}(U_k(x)) = \int_{p^{\triangle_k(x)}\leq n\log_p n} p^{\triangle_k(x)} d\mathbf{P} = \sum_{l:p^l\leq n\log_p n} p^l \mathbf{P}(\triangle_k(x) = l) \\ &= \sum_{l:p^l\leq n\log_p n} p^l \frac{p-1}{p^l} \sim (p-1)\log_p n. \end{aligned} \tag{11}$$

$$\mathbf{Var}(U_k(x)) \leq \mathbf{E}(U_k^2(x)) = \sum_{l:p^l\leq n\log_p n} (p-1)p^l \leq cn\log n.$$

Then by Chebyshev's inequality, we get

$$\begin{aligned} &\mathbf{P}\{x\in I_0 : |\sum_{k=1}^{n} U_k(x) - \sum_{k=1}^{n}\mathbf{E}(U_k(x))| > \epsilon\sum_{k=1}^{n}\mathbf{E}(U_k(x))\} \\ &\leq \frac{\mathbf{Var}(\sum_{k=1}^{n} U_k(x))}{(\epsilon\sum_{k=1}^{n}\mathbf{E}(U_k(x)))^2} = O(\frac{n^2\log n}{n^2\log_p^2 n}) = O(\frac{1}{\log n}). \end{aligned}$$

That is to say, $\frac{1}{\sum_{k=1}^{n} \mathbf{E}(U_k(x))} \sum_{k=1}^{n} U_k(x)$ converges in probability to 1. Since $E(U_k) \sim (p-1)\log_p n$ as $n \to +\infty$, we get

$$\lim_{n\to+\infty} \frac{1}{n\log_p n} \sum_{k=1}^{n} E(U_k(x)) = p-1.$$

Thus $\mathbf{P}(A_n) \to 0$, as $n \to +\infty$.

3 The speed of convergence

In this section, we consider the approximation of p-adic series by its convergent. For any $x \in I$, we define its partial sum

$$C_n(x) = \frac{1}{a_1(x)} + \sum_{j=2}^{n} \frac{1}{a_n(x)}.$$

Theorem 3.1 *For alternating Engel expansion expansions of p-adic series,*

$$\lim_{n\to+\infty} \frac{1}{n}(\log_p |x - C_n(x)| - \sum_{j=1}^{n} \nu_p(a_j(x))) = -\frac{p}{p-1}. \tag{12}$$

Proof.As we can see,$|C_{n+1}(x)| \leq |C_n(x)|$, then

$$\begin{aligned} \log_p |x - C_n(x)| &= \nu_p(a_{n+1}(x)) \\ &= \sum_{j=1}^{n} \nu_p(a_j(x)) - \sum_{j=1}^{n} \triangle_j(x) + \nu_p(a_1(x)), \end{aligned}$$

then by theorem2.11, for $\mathbf{P}$-almost all $x \in I$,

$$\lim_{n\to+\infty} \frac{1}{n}(\log_p |x - C_n(x)| - \sum_{j=1}^{n} \nu_p(a_j(x))) = -\frac{p}{p-1}.$$

Theorem 3.2 *For $\mathbf{P}$-almost all $x \in I_0$,*

$$\lim_{n\to+\infty} \frac{1}{2^n} \log_p |x - C_n| = -G(x)$$

where $G(x) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \triangle_n(x)$.

Proof.Since $\triangle_0(x) = -\nu_p(a_1(x)), \triangle_n(x) = 2\nu_p(a_n(x)) - \nu_p(a_{n+1}(x)),\ \forall n \geq 1,$ By Theorem2.9, Theorem2.11 and Toeplitz Lemma, for $\mathbf{P}$-almost all $x \in I_0$,

$$\lim_{n\to+\infty} \frac{1}{2^n}(\log_p |x - C_n(x)| = \lim_{n\to+\infty} \frac{\nu_p(a_{n+1}(x))}{2^{n+1} - 2^n}.$$

Notice that

$$\frac{-\nu_p(a_{n+1}(x))}{2^n} = \sum_{j=0}^{n} \frac{\triangle_j(x)}{2^j}$$

and

$$\int \sum_{j=0}^{\infty} \frac{\triangle_k(x)}{(2-L)^j} d\mathbf{P} = \sum_{j=0}^{\infty} \frac{1}{(2-L)^j} \frac{p}{p-1} < +\infty,$$

Hence by the Beppo-levi theorem, $G(x)$ exists, for $\mathbf{P}$-almost all $x \in I$.

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