International Journal of Algebra, Vol. 2, 2008, no. 7, 301 - 313

A Note on the Quasicenter of a Group

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Abstract

The quasicenter $Q(G)$ of a group G is the subgroup generated by the elements x of G such that the subgroup $\langle x \rangle$ is permutable in G. For the infinite groups the knowledge of $Q(G)$ has been recently concluded by J.Beidleman and H.Heineken. The present paper studies the groups in which the quasicenter has finite index in the whole group. Incidentally, if G is a finite group, then we find that the section $G/C_G(Q^*(G))$ is supersoluble, where $Q^*(G)$ denotes the last term of a series of generalized quasicenters.

Mathematics Subject Classification: Primary 20F19, 20E07, 20F14; Secondary 20D25, 20D10, 20D40

Keywords: quasinormal subgroups, hyperquasicenter, hypercyclically embedded subgroups

1 Introduction

Two subgroups H and K of a group G are said to permute if $HK = KH$. It is easily seen that H and K permute if and only if the set HK is a subgroup of G. A subgroup of G is *permutable* (or *quasinormal*) if it permutes with every subgroup of G. An element x of G is called *quasicentral (q.c.) element* if $\langle x \rangle$ is a permutable subgroup of G. The *quasicenter* of G, denoted by $Q(G)$ or briefly by Q, is the subgroup generated by all q.c. elements of G. The *hyperquasicenter* of G, denoted by $Q^*(G)$ or briefly by Q^*, is the largest term of the chain of normal subgroups

$$1 = Q_0(G) \leq Q_1(G) = Q(G) \leq \ldots \leq Q_\alpha(G) \leq Q_{\alpha+1}(G) \ldots$$

where α is an ordinal,

$$Q_{\alpha+1}(G)/Q_\alpha(G) = Q(G/Q_\alpha(G)),$$

λ is a limit ordinal and

$$Q_\lambda = \bigcup_{\alpha<\lambda} Q_\alpha(G).$$

[11] and [12] describe the properties of $Q(G)$ in the class of finite groups, while [1] describes the properties of $Q(G)$ in the class of infinite groups. Since the structure of $Q(G)$ is well-known (see [[1], Theorem 3]), it is possible to investigate those groups which are finite extension of the quasicenter. A group G such that the index $|G : Q|$ is finite is called *quasicentral-by-finite*.

Our paper is devoted to the knowledge of the quasicentral-by-finite groups. By definitions and by Corollary 3 of [1], some classes of groups are easily seen to belong to the class of quasicentral-by-finite groups: the class of central-by-finite groups, i.e. groups which are finite extension of the center; the class of hypercentral-by-finite groups, i.e. groups which are finite extension of the hypercenter; the class of Dedekind groups, i.e. groups in which each subgroup is normal; the class of Hamiltonian groups, i.e. nonabelian Dedekind groups; the class of quasihamiltonian groups, i.e. groups in which each subgroup is permutable. [13] and [15] contain famous results about these classes of groups. [4], [5], [6], [7], [8], [9], [10], [17] give a wider description of these classes of groups.

We recall briefly some notations of [1] in order to introduce our Main Theorems. Let G be a group and p be a prime. L will denote the subgroup of G generated by the q.c. elements of finite order. L_p will denote the maximal p-subgroup of L. $O^p(G)$ will denote the minimal normal subgroup of G with factor group a p-group. The relations between the subgroups L, L_p, $O^p(G)$ and Q of a group G have been described in [[1], Theorem 2].

Our Main Theorems are listed below. The first two results describe quasicentral-by-finite groups.

Theorem A. *Let G be a quasicentral-by-finite group such that the product of two q.c. elements of p-power order is a q.c. element of G (p is a prime).*

(i) *If $[L_p, L_p] \leq O^p(G)$ and q is a prime distinct from p, then G contains a normal series*

$$1 \triangleleft H \triangleleft K \triangleleft L \triangleleft Q \triangleleft G,$$

where H and K/H are periodic abelian groups generated by q.c. elements of p-power order, L/K is the direct product of L_q, Q/L is isomorphic to the subgroup of G generated by all the q.c. elements of G of infinite order, G/Q is finite.

(ii) *Furthermore if Q is either a p-group or a torsion-free group, then G is metabelian-by-finite.*

Theorem B. *Let G be a group and Q be the quasicenter of G. If the index $|G : Q|$ is finite, then there is a finitely generated subgroup K of G such that the index $|\langle y\rangle^K : \langle y\rangle|$ is finite, for each q.c. element y of G.*

Finally, a third result which is not new describes an interesting property of the centralizer $C_G(Q^*(G))$ of $Q^*(G)$ in the group G, when G is finite.

Theorem C. *Let G be a finite group. Then $G/C_G(Q^*(G))$ is supersoluble.*

Theorem C can be found for instance in [15]. However it seems to be instructive to give a proof by means of the introduction of the notion of quasicenter of a group.

Our group-theoretic notation and terminology are standard and they follow [15]. The proofs of Theorem A and Theorem B are contained in Section 2. Section 3 regards some special classes of quasicentral-by-finite groups which are close to the class of central-by-finite groups. In Section 4 we furnish examples of quasicentral-by-finite groups.

2 Proofs of main results

In general the product of q.c. elements need not be a q.c. element as [[1], Preliminaries, line 3] testifies. This means that if we consider an element of Q, then it could not be a q.c. element of G, but, it could be written as product of finitely many q.c. elements of G.

If the product of two q.c. elements of G is a q.c. element of G, then we may assume that each element of Q is a q.c. element of G.

Proof of Theorem A. From [[1], Theorem 2, (5)], either L_p is abelian or $[L_p, O^p(G)] = 1$.

Assume that L_p is abelian. [[1], Theorem 2, (1)] implies that L is periodic hypercentral, so that L has a decomposition in the direct product of its primary components (see for instance [13]). From [[1], Theorem 2] we discover that each primary component of L is of the form L_q for $q \in \mathbb{P}$ so that $L = Dr_{q\in\mathbb{P}} L_q$, where L_q denotes the maximal q-subgroup of L generated by those q.c. elements of G of q-power order. We claim that the statement (i) holds by choosing $H = K = L_p$. As testified in [[1], Lemma 3, (2)], L_p is characteristic in G and [[1], Theorem 2, (2)] implies that L_p is generated by q.c. elements of p-power order. Clearly $L/L_p = Dr_{q\in\mathbb{P}\setminus\{p\}} L_q$. From [[1], Theorem 3, (1)] and [[1], Proof of Theorem 3, lines 1-2], $Q = MN$ where M is the normal subgroup of G generated by the q.c. elements of G of finite order and N is the normal subgroup of G generated by the q.c. elements of G of infinite order. In particular, $Q = LN$, L is normal in G and Q/L is naturally isomorphic to

N. Since Q is normal in G and $|G:Q|$ is finite, (i) follows.

Assume that $[L_p, O^p(G)] = 1$. We consider $L_p \cap O^p(G)$ and note that

$$[L_p \cap O^p(G), L_p \cap O^p(G)] \leq L_p \cap [O^p(G), L_p \cap O^p(G)] \leq$$

$$\leq L_p \cap [O^p(G), L_p] \cap O^p(G) = 1.$$

Then $L_p \cap O^p(G)$ is abelian. Moreover

$$[L_p/(O^p(G) \cap L_p), L_p/(O^p(G) \cap L_p)] = [L_p, L_p](O^p(G) \cap L_p)/(O^p(G) \cap L_p) \leq$$

$$\leq (O^p(G) \cap L_p)/(O^p(G) \cap L_p) = 1.$$

Then $L_p/(O^p(G) \cap L_p)$ is abelian. Define $H = O^p(G) \cap L_p$ and $K = L_p$. K is periodic by [[1], Theorem 2], then K/H is periodic abelian. Since a homomorphic image of a permutable subgroup of p-power order is again a permutable subgroup of p-power order, $(L_pH)/H = K/H$ lies in the maximal p-subgroup of $L(G/H)$, where $L(G/H)$ denotes the subgroup of G/H generated by the q.c. elements of G/H of finite order. In particular, K/H is generated by q.c. elements of p-power order. Recalling that $O^p(G)$ is normal in G we may conclude as in the previous case that H, K, L, Q are normal subgroups of G, L/K is the direct product of L_q, Q/L is isomorphic to the subgroup of G generated by all the q.c. elements of G of infinite order, G/Q is finite. Then (i) follows.

Assume that Q is a p-group. Then $Q = L_p$. As before [[1], Theorem 2, (5)] implies that either L_p is abelian or $[L_p, O^p(G)] = 1$. In the first case G is clearly abelian-by-finite In the second case, $[[L_p, L_p], L_p] \leq [O^p(G), L_p] = 1$, then L_p is metabelian and G is metabelian-by-finite.

Assume that Q is torsion-free. Then $Q = N$ where N has been previously defined (see [[1], Theorem 3]). In particular, N' is abelian so that $N'' = 1$ and G is metabelian-by-finite.□

Note that for a soluble group in which each subgroup is subnormal, a result similar to Theorem A is given by [[7], Corollary 3.4].

If X, Y are subgroups of a group G and $Y \leq X$, the interval $[X/Y]$ is defined to be the set of subgroups H of G such that $Y \leq H \leq X$. This notion can be found in [15].

Proof of Theorem B. The statement is obviously true if Q is trivial. Let y be a nontrivial element of Q. The result is true if we take $K = \langle y \rangle$. If $K = \langle x, y \rangle$, where x is any element of G, the result is also true. In fact, $\langle x, y \rangle = \langle x \rangle \langle y \rangle$, since y is a q.c. element of G. If $\langle x \rangle$ is finite, then $|\langle x, y \rangle : \langle x \rangle \cap \langle y \rangle|$ is finite and the result is true. Similarly, if $\langle x \rangle$ is infinite and the intersection of $\langle x \rangle$ and $\langle y \rangle$ is nontrivial, the result is still true, arguing in the group $\langle x, y \rangle/(\langle x \rangle \cap \langle y \rangle)$. Finally, if $\langle x \rangle$ is infinite and the intersection of $\langle x \rangle$ and $\langle y \rangle$ is trivial, then $\langle y \rangle$ is normal in $\langle x, y \rangle$ by [[14], 13.2.3] and the result is still true. Then our theorem is completely proved.□

Proof of Theorem C. $Q^*(G)$ is the maximal supersolubly embedded normal subgroup of G (see [1]). Let N be a normal subgroup of G. We prove by induction on $|N|$.

A first step consists to verify that

(*) if N is supersolubly embedded in G, then $G/C_G(N)$ is supersoluble.

(*) is surely true for $|N| = p$, where p is a prime. Let n be a positive integer and assume that (*) is true for all N with $|N| < n$. We want to show (*) for N with $|N| = n$.

Let L be a minimal G-invariant subgroup contained in N and $K \geq L$ be a maximal G-invariant subgroup contained in N. We have by the induction hypothesis that $G/C_G(K)$ is supersoluble. Then $(G/L)/C_{G/L}(N/L)$ is supersoluble. Let $C_{G/L}(N/L) = D/L$. We have that $G/(C_G(K) \cap D)$ is supersoluble. Obviously $C_G(N) \leq C_G(K) \cap D$. Now if $x \in C_G(K) \cap D$ and $y \in N \setminus K$, then $[y, x] \in L$, so $x^{-1}yx = yt$ for some $t \in L$. Since $|L|$ and $|N/K|$ are primes, there exists a prime q such that $t^q = 1$. But $x \in C_G(K)$, so $x^{-1}tx = t$ and $x^{-1}yx = y$. We obtain $|C_G(K) \cap D : C_G(N)| = q$ if $t \neq 1$ or $|C_G(K) \cap D : C_G(N)| = 1$ if $t = 1$. So $(C_G(K) \cap D)/C_G(N)$ is cyclic and $G/C_G(N)$ is supersoluble. In particular $G/C_G(Q^*(G))$ is supersoluble.□

Theorem B adapts Neumann's Theorem (see [[13], p.127]) and Maier-Schmid's Theorem (see [[15], Theorem 5.2.3]).

3 Some classes of quasicentral-by-finite groups

We recall that, given a group G, and a normal subgroup H of G, H is said to be hypercyclically embedded in G if it contains a G-invariant series whose factors are cyclic. This notion arouses interest in the investigation of the quasicenter of a group as testified in [[1], Theorem 1].

[[1], Corollary 3] reduces the study of the quasicentral-by-finite groups to the study of the hypercentral-by-finite groups. A result of Baer (see [[13], Theorem 4.18]) characterizes these groups only as the groups which can be covered by finitely many hypercentral subgroups. In particular a quasicentral-by-finite group is always covered by finitely many hypercentral subgroups. At this point it could be interesting to see if a quasicentral-by-finite group can be covered by finitely many hypercentrally embedded subgroups. Unfortunately, a finite simple group as $\mathbb{A}_5$, the alternating group on 5 elements, has trivial quasicenter, is obviously quasicentral-by-finite, but, can not be covered by finitely many hypercentrally embedded subgroups. Then the two notions can not be related. However the following result holds.

Proposition 3.1. *Let G be a group. The following conditions are equivalent:*

(i) *G is covered by finitely many hypercyclically embedded subgroups;*

(ii) *G is hypercyclic;*

(iii) *$G = Q^*(G)$.*

Proof: (i)⇒(ii). If H and K are hypercyclically embedded subgroups of G, then HK is hypercyclically embedded in G by [[15], Lemma 5.2.1]. This fact implies that if G is covered by finitely many hypercyclically embedded subgroups, then G coincides with $\sigma(G)$, the largest hypercyclically embedded subgroup of G. In particular G is hypercyclic.

(ii)⇒(iii). $G = \sigma(G)$, then we finish by [[1], Theorem 1].

(iii)⇒(i). By [[1], Theorem 1] $G = \sigma(G)$ and it is obvious to end.□

The conditions to be covered by normal subgroups are discussed in [1], in particular if the covering of Proposition 3.1 satisfies Engel-conditions, then [[5],Proposition 3.3] or [[6], Theorem 1] reduce the study of quasicentral-by-finite groups to the well-known theory of central-by-finite groups.

In order to introduce the next result, we recall that a *norm* of a group G is the subgroup $N(G)$ of G defined by

$$N(G) = \bigcap_{H \leq G} N_G(H),$$

that is, the intersection of the normalizers of the subgroups of G. In [3] and [15] properties of $N(G)$ have been described.

Proposition 3.2. *Let G be a quasicentral-by-finite group, $Q(G)$ be the quasicenter of G, $N(G)$ be the norm of G.*

(i) *if $Q(G)$ contains only elements of prime or infinite order and $Q(G) = N'$, where N is the subgroup generated by the q.c. elements of infinite order, then G is finite;*

(ii) *if there is an element $x \in N(G)$ such that the index $|Q(G) : \langle x \rangle_G|$ is finite, then G is central-by-finite;*

(iii) *G is central-by-finite if and only if the index $|Q(G) : N(G)|$ is finite.*

Proof: (i). $Q(G) \geq N$ and $N' = Q(G)$, then $N' \geq N$. Following the Proof of [[1], Theorem 3] we note that $N' \leq Z(N)$. Then $N \leq Z(N)$, so $N' = 1$ and $Q(G) = 1$. This implies that G is finite.

(ii). Let x be an element of $N(G)$ such that the index $|Q(G) : \langle x\rangle_G|$ is finite. For each subgroup H of G, $\langle x\rangle \leq N_G(H)$, then $\langle x\rangle_G \leq core_G(N_G(H))$ and the index $|Q(G) : core_G(N_G(H))|$ is finite. The group G has each quotient group $G/core_G(N_G(H))$ which is finite and this implies that G has each index $|G : N_G(H)|$ which is finite. Thus G is central-by-finite (see [[13], p.127]).

(iii). Let G be a central-by-finite group. Since $Z(G) \leq N(G) \leq Q(G)$, obviously the index $|Q(G) : N(G)|$ is finite. Conversely if $Z(G) = 1$, then $N(G) = 1$ by [[15], Theorem 1.4.3], so G is finite. If $N(G)$ is finite, then G is again finite. Without loss of generality we may suppose that G has nontrivial center and infinite nontrivial norm. For each subgroup H of G,

$$N(G) = \bigcap_{K\leq G} N_G(K) \leq \bigcap_{g\in G} N_G(H^g) = \bigcap_{g\in G} N_G(H)^g = core_G(N_G(H))$$

so that G has each quotient $G/core_G(N_G(H))$ which is finite. Again G has each index $|G : N_G(H)|$ which is finite, then G is central-by-finite.□

In many cases we may distinguish certain finite extensions of the quasicenter, thanks to the order of the elements of $Q(G)$. We recall that for a group G the symbol $\pi(G)$ denotes the set of prime divisors of the orders of the elements of G. Moreover if H and K are two subgroups of a group G, H and K are said to be *totally permutable* in G if every subgroup of H permutes with every subgroup of K (see [2]).

Proposition 3.3. *Let G be a quasicentral-by-finite group with quasicenter Q. If one of the following conditions holds*

(i) *Q is a p-group and the index $|G : Q|$ is a q-power for some distinct primes p , q such that $(p-1, q) = 1$;*

(ii) *Q is torsion-free and the product of two q.c. elements is a q.c. element;*

(iii) *Q is periodic and $(p-1, q) = 1$ for any distinct primes $p \in \pi(Q)$ and $q \in \pi(G/Q)$ with $\pi(Q) \cap \pi(G/Q) = \varnothing$;*

then $G = QF$ where F is a finite subgroup of G such that each subgroup of F is permutable with each finitely generated subgroup of Q.

Proof: (i). If $|G : Q| = n$ where n is a q-power, then a transversal of Q in G is given by elements $g_1, \ldots , g_n \in G \setminus Q$ such that $G/Q = \{g_1Q, \ldots , g_nQ\}$. This implies that there exists a q-subgroup F of order $|G : Q|$ such that $G = FQ$. Now, if H is a subgroup of Q and K a subgroup of F, [[1], Lemma 4, (2)] implies that $[h, k] = 1$ for each element h of H and for each element k of K. Thus we have that $[H, K] = 1$ and the arbitrary choice of H and K implies that F and Q are totally permutable in G. In particular $G = QF$ and each subgroup of F is permutable with each subgroup of Q (see also [2]).

(ii). We may write again $G = FQ$, following the notation of the previous step (i). Let x be an element of infinite order in Q and y be an element of F. Since the product of two q.c. elements of G is a q.c. element of G, we may assume that each element of Q is a q.c. element of G. Thus x is a q.c. element of G and $\langle x \rangle$ is permutable in G. [[15], Lemma 5.2.7, Theorem 6.2.10] imply $y \in N_G(x)$. Thus $F \leq N_G(x)$, $\langle x \rangle^F = \langle x \rangle [\langle x \rangle, F] = \langle x \rangle$, then $[\langle x \rangle, F] \leq \langle x \rangle$, but x has infinite order, so $[\langle x \rangle, F] = 1$. If $H = \langle h_1, h_2 \rangle$ is a 2-generated subgroup of Q, then

$$[F, \langle h_1, h_2 \rangle] = [F, \langle h_1 \rangle \langle h_2 \rangle] = [F, \langle h_1 \rangle]^{\langle h_2 \rangle} [F, \langle h_2 \rangle] = 1.$$

It is clear how to generalize this method to the case of an m-generated subgroup of Q, where m is a fixed positive integer. It follows that each subgroup of F is permutable with each finitely generated subgroup of Q.

(iii). [[1], Corollary 3] implies that Q is $(\omega + 1)$-hypercentral, then it has a primary decomposition in the direct product of p_j-Sylow subgroups P_j, where p_j is a prime and j is a positive integer. Applying the argument in (i), the result follows.□

In the proofs of the statements (i) and (iii) of Proposition 3.3, it is shown that a group G which is an extension of its quasicenter Q by a finite group F is the product of the totally permutable subgroups Q and F. [[2],Theorems 1, 2, 3] describe in details this situation.

Moreover the consideration of the group

$$G = \mathbb{D}_\infty \times \mathbb{D}_\infty = \langle a, b : a^b = a^{-1}, b^2 = 1 \rangle \times \langle c, d : c^d = c^{-1}, d^2 = 1 \rangle,$$

where $\mathbb{D}_\infty$ denotes the infinite dihedral group, shows that the condition that the product of two q.c. elements is a q.c. elements can not be omitted from (ii) of Proposition 3.3. Here a and c are q.c. elements, but ac is not a q.c. element of G, since $F = \langle b, d \rangle$ is not permutable with $\langle ac \rangle$.

Other information on the structure of a quasicentral-by-finite group can be obtained by a famous Hall's Theorem (see [[13], Theorem 9.57]) and by [[1], Lemma 3].

Corollary 3.4. *If G is a quasicentral-by-finite group, then every chief factor of G is finite and every maximal subgroup of G has finite index.*

Proof: This follows by [[1], Corollary 3] and [[13], Theorem 9.57].□

4 Examples

In this Section a list of examples has been furnished in order to point out the relations between the subgroups $Z(G), N(G), Q(G), Q^*(G)$ of a quasicentral-by-finite group G.

Example 4.1.
1. The infinite dihedral group

$$H = \mathbb{D}_\infty = \langle a, x : a^x = a^{-1}, x^2 = 1\rangle$$

and the infinite locally dihedral 2-group

$$K = \mathbb{D}_{2^\infty} = Y \ltimes B,$$

where B is a quasicyclic 2-group, $Y = \langle y\rangle$ has order 2 and $b^y = b^{-1}$ for each $b \in B$, are both quasicentral-by-finite groups.

H has $Z(H) = N(H) = 1$, where $N(H)$ denotes the norm of H; $Q(H)$ coincides with $\langle a\rangle$ so $Q(H)$ is infinite cyclic and $|G : Q(H)| = 2$; $Q^*(H) = H$ since $Q^*(H)$ is the largest hypercyclically embedded subgroup of H. K has $Z(K) = N(K)$ which is cyclic of order 2; $Q(K) = B$ and $|G : Q(K)| = 2$; $Q^*(K) = K$ since $Q^*(K)$ is the largest hypercyclically embedded subgroup of K. The sections $Q(H)/N(H)$ and $Q(K)/N(K)$ are both infinite and H and K are not central-by-finite according with (iii) of Proposition 3.2.

2. Following the notations of the previous step 1 let G be the direct product of K by the alternating group on 5 elements $\mathbb{A}_5$. It is well known that $\mathbb{A}_5$ is a finite simple non-abelian group. G has $Q^*(G) = K$, $Q(G) = B$ and $N(G) = Z(G)$ is cyclic of order 2. G is a quasicentral-by-finite group with the chain $1 < Z(G) = N(G) < Q(G) < Q^*(G) < G$.

Example 4.2.
1. Let G be the group presented by

$$\langle x, y, a | x^y = x^{-1}, y^8 = a^2 = 1, y^a = y^5, x^a = x\rangle.$$

Put $A = \langle a\rangle$, $X = \langle x\rangle$, $Y = \langle y\rangle$, $K = \langle x, y | x^y = x^{-1}, y^8 = 1\rangle$, where A is cyclic of order 2, X is infinite cyclic and Y cyclic of order 8, we have $K = Y \ltimes X$ and $G = A \ltimes K$. In particular G is isomorphic to $(C_2 \ltimes C_8) \ltimes C_\infty$. Following [[16], Example 3], $x \notin Z(G) \neq 1$, since $x^{y^{-1}} = x^{-1}$, but $x^2 \in Z(G)$, and A is not normal in G. We have $A \leq N(G)$, where $N(G)$ denotes the norm of G. But y^2 has order 4 and $(y^2)^a = (y^a)^2 = (y^5)^2 = y^{10} = y^2$, $(y^4)^a = (y^a)^4 = (y^5)^4 = y^{20} = y^4$, $(y^6)^a = (y^a)^6 = (y^5)^6 = y^{30} = y^6$. This implies that $y^2 \in Z(G)$. Now $[a, y] \neq 1$, thus $y \notin Z(G)$ and $Z(G) = \langle x^2, y^2\rangle$. Since x is torsion-free, y is periodic, both $\langle x^2\rangle$ and $\langle y^2\rangle$ are normal subgroups of G, $Z(G) = \langle x^2\rangle \times \langle y^2\rangle$ and it is isomorphic to $C_\infty \times C_4$.

In order to describe $N(G)$ it is enough to calculate the normalizers in G of the subgroups $\langle a\rangle, \langle x\rangle, \langle y\rangle$.

$$N_G(a) = N_G(\langle a\rangle) = \langle a, y^2, x\rangle,$$

$$N_G(x) = N_G(\langle x\rangle) = G,$$

$$N_G(y) = N_G(\langle y\rangle) = \langle a, y, x\rangle = G$$

then $N(G) = \langle a, y^2, x\rangle$.

On the other hand G is not nilpotent, since it contains the infinite dihedral group. But, G is supersoluble by construction so that $Q^*(G) = G$.

$N(G) \leq Q(G)$, $\langle y\rangle$ is permutable in G, so we conclude that $Q(G) = Q^*(G) = G$. Also here the section $Q(G)/N(G)$ is finite and G is central-by-finite by (iii) of Proposition 3.2.

G has the chain $1 < Z(G) < N(G) < Q(G) = Q^*(G) = G$.

2. Following [[16], Example 4], let p be an odd prime and Y be a cyclic group of order p^2 generated by y. Put $Y_1 = Y/Y^p$ and $y_1 = yY^p$. Let $R = \mathbb{Z}Y_1$ be the integral group ring of Y_1 and let $I = R(y_1 - 1)$ be the augmentation ideal of R. Then I becomes an Y-module via the natural action $u^y = uy_1$ for all $u \in I$. We form the split extension $K = Y \ltimes I$, a free abelian group of rank $p-1$ extended by C_{p^2}.

Let A be a cyclic group of order p generated by a and define an action of A on K by

$$u^a = u, \quad y^a = y^{1+p}$$

for all $u \in I$. Then form the split extension $G = A \ltimes K$. Thus A is not normal in G and we claim that

$$(*) \qquad A \leq N(G),$$

where $N(G)$ denotes the norm of G.

For, A centralizes $IY^pA = L$, say. A typical cyclic subgroup of G, not contained L, is generated by an element of the form $g = y^iua^j$, where $i \equiv 1 (mod p)$ and $u \in I$. Then $g^a = y^{i(1+p)}ua^j$. But we have

$$(**) \qquad g^{1+p} = y^{i(1+p)}ua^j.$$

For, $\langle y^iu, a\rangle$ is nilpotent of class 2, since its derived subgroup is Y^p, which lies in $Z(G)$. Therefore

$$g^{1+p} = (y^iu)^{1+p}a^j.$$

Also

$$(yu)^{1+p} = yu(yu)^p = yu(y^pu(1 + y_1 + y_1^2 + \ldots + y_1^{p-1})) = y^{1+p}u.$$

Thus $(y^i u)^{1+p} = y^{i(1+p)}u$ and $(**)$ is true. Therefore $(*)$ is true.

Following the considerations of the previous step 1 we discover that $Z(G)$ is isomorphic to $I \times C_{p^2}$, $N(G) = \langle a, I, y^p \rangle$, $Q^*(G) = Q(G) = G$, G is supersoluble.

Also here G has the chain $1 < Z(G) < N(G) < Q(G) = Q^*(G) = G$.

3. Let G be the direct product of the group H described in the previous step 1. or 2. by the alternating group on 5 elements $\mathbb{A}_5$. G has $Q^*(G) = H$, $Q(G) = Q(H)$, $N(G) = N(H)$ and $Z(G) = Z(H)$. G is a quasicentral-by-finite group with the chain $1 < Z(G) < N(G) < Q(G) = Q^*(G) < G$.

Example 4.3.

1. We consider the split extension $A \ltimes_\phi P$, where $P = \langle u \rangle \times \langle v \rangle$ is an elementary abelian p-group of order p^2 and A is infinite cyclic. Given $H \triangleleft A$ of index $|A : H| = p$ and σ the automorphism of P defined on the basis $\{u, v\}$ by $u^\sigma = u$ and $v^\sigma = uv$, we have that A acts on P via $\phi = \pi\psi$, where π is the natural epimorphism from A to A/H and ψ is an isomorphism between A/H and P. This gives the presentation

$$G = \langle a, b, c : a^p = b^p = [a, b] = 1, a^c = a, b^c = ab \rangle.$$

G is nilpotent of class 2 with $G' = Z(G)$ and $Z(G) \neq 1$, because $a \in Z(G)$ and $c^p \in Z(G)$. Moreover, $Z(G) = \langle a \rangle \times \langle c^p \rangle$. Here $Q^*(G) = G$, $Q(G) = \langle a \rangle = N(G)$ since $b \notin Q(G)$, and G has the chain $1 < Z(G) = N(G) = Q(G) < Q^*(G) = G$.

2. Let G be the direct product of the group H described in the previous step 1. by the alternating group on 5 elements $\mathbb{A}_5$. G has $Q^*(G) = H$, $Q(G) = Q(H) = N(G) = N(H) = Z(G) = Z(H)$. G is a quasicentral-by-finite group with the chain $1 < Z(G) = N(G) = Q(G) < Q^*(G) < G$.

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Received: November 7, 2007