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Some Groups Whose Proper Quotients Have Minimax Conjugacy Classes

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Abstract

A group G is said to be an MC-group if $G/C_G(< x >^G)$ is a finite extension of a soluble minimax group, for each element x of G. The class of MC-groups is a generalization of the class of FC-groups, which have been studied by several authors. A group which is not an MC-group, but whose proper quotients are MC-groups, is called a $JNMC$-group. The present paper is devoted to the description of periodic $JNMC$-groups, $JNMC$-groups with a unique minimal normal subgroup and $JNMC$-groups, subject to certain restrictions on their abelian rank.

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1. Introduction.

A *minimax group* is a group which has a series of finite length whose factors satisfy either the maximal condition or the minimal condition on subgroups. The maximal condition on subgroups is often denoted with *max* and the minimal condition on subgroups is often denoted with *min*. Thus *minimax* is a finiteness property which generalizes both *max* and *min*. It is easy to verify that the class of minimax groups is closed with respect to homomorphic images, subgroups and extensions. A group G is said to be *soluble minimax* if it has a characteristic series of finite length whose factors are abelian minimax groups. Abelian minimax groups are well-known: an abelian group is minimax if and only if it is an extension of a group with *max* by a group with *min* [9, Lemma 10.31] and consequently soluble minimax groups are well-known [9, Sections 10.3 and 10.4].

Recalling that an FC-group is a group whose elements have only finitely many conjugates, the notion of soluble minimax group allows us to define a wider class of groups, including the class of FC-groups.

A group G is said to be an *MC-group*, or group with (soluble minimax)-by-finite conjugacy classes, if $G/C_G(< x >^G)$ is a finite extension of a soluble minimax group, for each element x of G. Following [9], a finite extension of a soluble minimax group is called a (*soluble minimax*)-*by-finite* group. In a group G, an element x of G is said to be an *MC-element* of G if $G/C_G(< x >^G)$ is a finite extension of a soluble minimax group. Obviously, a group G is an MC-group if and only if its elements are all MC-elements and it is clear that each FC-group is an MC-group. The set of all MC-elements of a group G forms a characteristic subgroup $MC(G)$ of G as noted in [7]. $MC(G)$ is called the *MC-center* of G and contains the center $Z(G)$ of G.

The class of MC-groups was introduced in [4], after the study of the groups with polycyclic-by-finite conjugacy classes and Chernikov conjugacy classes in [2], [3] and [8].

Let $\mathfrak{X}$ be a class of groups. A group G is said to be *just-non-*$\mathfrak{X}$, or $JN\mathfrak{X}$-group, if it does not belong to $\mathfrak{X}$ but all its proper quotients belong to $\mathfrak{X}$. A group which belongs to $\mathfrak{X}$ is often called an $\mathfrak{X}$-group. Of course, every simple group which is not an $\mathfrak{X}$-group is a $JN\mathfrak{X}$-group, so that in the investigation concerning $JN\mathfrak{X}$-groups it is natural to require that they have a nontrivial Fitting subgroup, i.e. that they contain a nontrivial abelian normal subgroup. The structure of $JN\mathfrak{X}$-groups has already been studied for several choices of the class $\mathfrak{X}$, so there is a well-developed theory about this topic. In particular [5] represents a work of reference. We are interested in studying $JN\mathfrak{X}$-groups, where $\mathfrak{X}$ is the class of MC-groups. These groups are called *JNMC-groups*.

In studying $JNMC$-groups, it will be useful to investigate $JN\mathfrak{X}$-groups, where $\mathfrak{X}$ is the class of (soluble minimax)-by-finite groups. These groups are called *just-non-*((*soluble minimax*)-*by-finite*) groups (see [1]).

At the moment the knowledge of $JNMC$-groups is an open problem, but many subclasses of $JNMC$-groups such as $JNCC$-groups and $JNFC$-groups (see [5]) are known. The present paper studies $JNMC$-groups which have a unique minimal normal subgroup, $JNMC$-groups which are periodic and $JNMC$-groups which are subject to certain restrictions on their abelian rank. Here the general problem to giving structural information about $JNMC$-groups without minimal normal subgroups is not treated.

In Section 2 we discuss some general properties of $JNMC$-groups and we investigate $JNMC$-groups which are subject to certain restrictions on their abelian rank. Our results on $JNMC$-groups, which have a unique minimal normal subgroup, are shown in Section 3. Section 4 describes $JNMC$-groups which are periodic, overlapping a result in [5].

Most of our notation is standard and can be found in [9]. The general properties of MC-groups and related classes of generalized FC-groups are referred to [2],[3],[4], [7] and [11].

2. General properties of $JNMC$-groups.

The following two lemmas recall properties of MC-groups which are described in [4] and [7], so the proofs are omitted.

LEMMA 2.1. *Let G be a group. If X is a finite subset of MC-elements of G, then $< X >^G$ is a (soluble minimax)-by-finite subgroup of G.*

If G is an MC-group, then Lemma 2.1 can be also expressed by saying that G is locally (normal and (soluble minimax)-by-finite), or that G can be covered by normal (soluble minimax)-by-finite subgroups.

LEMMA 2.2. *The class of MC-groups is closed with respect to homomorphic images, subgroups and direct products.*

In the following statement, we mention the notion of finite abelian section rank which can be found in [9].

PROPOSTION 2.3. *Let G be a $JNMC$-group. If G has finite abelian section rank, then G is a just-non-((soluble minimax)-by-finite) group and $MC(G) = 1$. If G has not finite abelian section rank, then G is covered by (soluble minimax)-by-finite subgroups and $MC(G) \neq 1$.*

PROOF. Assume that G has finite abelian section rank. From [4], an MC-group with finite abelian section rank is a (soluble minimax)-by-finite group. Then each proper quotient of G is a (soluble minimax)-by-finite group, from which G is a just-non-((soluble minimax)-by-finite) group. Now, let $MC(G)$ be nontrivial and $x \in MC(G)$. From Lemma 2.1, $G/ < x >^G$ can be covered by (soluble minimax)-by-finite normal subgroups. Then there exists a normal series

$$1 \triangleleft < x >^G = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G$$

with (soluble-by-finite) normal factors. If G has finite abelian section rank, then there exists an integer $i \geq 0$ such that $G_i = G_{i+1} = \ldots = G$ and then G would be a (soluble minimax)-by-finite group. This gives a contradiction. Then $MC(G)$ is nontrivial. The remaining part of the statement follows, whenever G has not finite abelian section rank. $\square$

Another interesting fact is that a $JNMC$-group is subdirectly indecomposable.

Lemma 2.4. *Let G be a $JNMC$-group, then every intersection of two nontrivial normal subgroups of G is nontrivial.*

Proof. Let H and K be two nontrivial normal subgroups of G. Suppose that $H \cap K$ is trivial. G is isomorphic to a subgroup of the direct product of G/H and G/K. But G/H and G/K are MC-groups, so Lemma 2.2 implies that G is an MC-group and this contradicts the fact that G is a $JNMC$-group.□

Lemma 2.5. *If G is a $JNMC$-group with $MC(G) = 1$, then G has trivial center.*

Proof. We have $MC(G) \geq Z(G)$.□

Lemma 2.6. *Let G be a $JNMC$-group with $MC(G) = 1$. If N is a normal nilpotent subgroup of G, then N is abelian.*

Proof. Let N be a nontrivial normal nilpotent subgroup of G such that $N' \neq 1$. Then G/N' is an MC-group. Now we apply [6, Lemma 3.1] which is a generalization of the famous Hall's criterion of nilpotence (see for instance [9]). Then G is an MC-nilpotent group, that is, G has a characteristic series

$$1 = M_0 \triangleleft M_1 \triangleleft M_2 \triangleleft \ldots \triangleleft M_c = G,$$

where c is a positive integer,

$$M_1 = MC(G),\ M_2/M_1 = MC(G/M_1),\ \ldots,\ M_c/M_{c-1} = MC(G/M_{c-1}).$$

Further details can be found for instance in [6] and [7]. Since G is an MC-nilpotent group, $MC(G) \neq 1$ which is a contradiction. Then the result follows. □

Proposition 2.7. *Let G be a $JNMC$-group with $MC(G) = 1$. If $A = Fit\, G \neq 1$, then either A is torsion-free abelian or A is an elementary abelian p-group for some prime p.*

Proof. Let x and y be two nontrivial distinct elements of A. Then there are normal nilpotent subgroups L_x and L_y of G such that $x \in L_x$ and $y \in L_y$. According to Fitting's Theorem the subgroup L_xL_y is nilpotent, Lemma 2.6 implies that it is abelian, thus $xy = yx$ and A is abelian.

Let $T = T(A)$ be the torsion subgroup of A and let T be nontrivial. Without loss of generality we may suppose that T is a p-group for some prime p.

Assume that $T^p = \{a^p : a \in T\} \neq 1$. Then $T_1 = T/T^p = \{b \in T : b^p = 1\} \neq T$. Since G/T_1 is an MC-group, T/T_1 is a periodic abelian MC-group thanks to Lemma 2.2. From Lemma 2.1 it follows that an MC-group has an ascending series of normal subgroups, every factor of which either is a finite group or is an abelian minimax group. Hence T/T_1 is a periodic abelian minimax group, then it is a Chernikov group by [9, Section 10.3, p.166]. Then T/T_1 includes a finite nontrivial subgroup P/T_1, which is normal in G/T_1.

We define $H/T_1 = C_{G/T_1}(P/T_1)$ so that H is a normal subgroup of G with finite index in G. If $h \in H$ and $c \in P \setminus T_1$, then

$$[c, h]T_1 = [cT_1, hT_1] \in [P/T_1, H/T_1] = 1$$

and $[c, h] \in T_1$. This allows us to say that $1 = [c, h]^p = [c^p, h]$, $c \notin T_1$, $c^p \neq 1$, then $c \in Z(H)$, $Z(H) \neq 1$. Now $< c^p >$ is a finite nontrivial normal subgroup of G, and so it should belong to $MC(G) = 1$. This contradiction implies that T is elementary abelian and coincides with T_1.

Let $A \neq T$. The Prüfer-decomposition of A implies that $A = T \times B$, being B a torsion-free abelian subgroup of A isomorphic to A/T. Hence $B \geq A^p$ and $B^p \cap T = 1$. Since $A \neq T$, $A^p \neq 1$. This contradicts Lemma 2.4. Thus $A = T$ as claimed.□

PROPOSITION 2.8. *Let G be a $JNMC$-group with $MC(G) = 1$. If $1 \neq A = FitG$, then $A = C_G(A)$ and A is maximal abelian in G.*

PROOF. A is abelian thanks to Proposition 2.7, then $C = C_G(A)$ includes A. If $C' = 1$, then C is normal abelian in G and $C \leq A$, so that $C = A$. If C is nilpotent, then $C \leq A$ and $C = A$. Let C be not nilpotent and assume that $C \neq A$. The quotient C/A is a nontrivial MC-group and by Lemma 2.1 there exists an element $xA \neq A$ of C/A such that $< xA >^{G/A} = X/A$ is a nontrivial (soluble minimax)-by-finite group. If X/A is central-by-finite, then we conclude following the argument of the last part of [5,Theorem 10.5].

Let X/A be not central-by-finite and let it be (abelian minimax)-by-finite. From [9, Lemma 10.31], X/A contains a G-invariant abelian finitely generated subgroup Y/A. Then Y is a normal nilpotent subgroup of G so that Y lies in A. It follows that X/A must be periodic, then it is a Chernikov group. Hence A must be a Chernikov group, but this contradicts $MC(G) = 1$ and Proposition 2.7.

[9, Theorem 10.35] implies that each abelian subgroup of X/A must be abelian minimax and the previous argument holds also in the general case of X/A which is (soluble minimax)-by-finite. We conclude that $C = A$.

Put $C_G(C) = L$ and $x \in L$. $L \geq A$, but also $[A, x] = 1$, thus $L = C$. This is sufficient to prove that C is maximal abelian in G.□

3. Monolithic $JNMC$-groups.

In a $JN\mathfrak{X}$-group, where $\mathfrak{X}$ is a prescribed class of groups (see [5]), the action of the Fitting subgroup is fundamental to obtain structural conditions on the whole group. There is a natural dichotomy of $JNMC$-groups into those without minimal normal subgroups and those with a unique minimal normal subgroup. A group with a unique minimal normal subgroup is called *monolithic* and in this section we describe monolithic $JNMC$-groups.

We will use the next result, due to D.J.Robinson.

Theorem 3.1. *Let G be a group with an abelian subgroup A satisfying the minimal condition on its G-invariant subgroups and let K be a normal subgroup of G satisfying the following conditions:*

(i) *$K \geq A$ and K/A is locally nilpotent;*

(ii) *the FC-hypercenter of $G/C_K(A)$ includes $K/C_K(A)$;*

(iii) *$A \cap Z(K) = 1$.*

Then G contains a free abelian subgroup X such that the index $|G : XA|$ is finite and $X \cap A = 1$ (nearly splitting of G on A). Moreover the complements of A in G fall into finitely many conjugacy classes.

Theorem 3.1 is frequently used for giving structural information on $JN\mathfrak{X}$-groups, where $\mathfrak{X}$ is a class of generalized FC-groups. The proof of Theorem 3.1 involves homological machineries and the requirement of FC-hypercentrality plays an important role in the economy of the proof. For this reason, it is not possible to substitute the requirement of FC-hypercentrality to the requirement of weaker conditions of generalized hypercentrality, hoping to get an improvement. Further details can be found in [10] .

Theorem 3.2. *Let G be a monolithic group with $1 \neq A = FitG$, $MC(G) = 1$ and p be a prime. If G is a $JNMC$-group and $H = G/A$ is locally nilpotent, then*

(j) *A is torsion-free abelian or p-elementary abelian;*

(jj) *$A = C_G(A)$ is the unique minimal normal subgroup of G;*

(jjj) *G contains a free abelian subgroup X such that $|G : XA|$ is finite and $X \cap A = 1$ (nearly splitting of G on A). If G splits over A, the complements of A fall into finitely many conjugacy classes.*

Proof. (j). This follows by Proposition 2.7.

(jj). By Proposition 2.8, $A = C_G(A)$ so it is sufficient to prove that $A = M$, where M is the unique minimal normal subgroup of G. Certainly $A \geq M$, so M is abelian. Conversely we suppose that M is nontrivial abelian and $M > A$. Then $M > C_G(A)$ and $C_G(A)$ is not maximal abelian in G. This is against Proposition 2.8.

(jjj). By the previous steps (j) and (jj), A is an abelian subgroup of G which satisfies the minimal condition on its G-invariant subgroups. G/A is a locally nilpotent MC-group and it is hypercentral [4, Theorem 4]. At the same time $G/C_G(A) = G/A$ is FC-hypercentral by a classical McLain's Theorem [9, Theorem 4.38]. Now Lemma 2.5 implies that $Z(G) \cap A = 1$. We may apply Theorem 3.1 so that (jjj) is proved. □

Let $\mathfrak{X}$ be a class of groups. Examples of monolithic $JN\mathfrak{X}$-groups are usually given by means of the wreath product of two suitable groups.

We may construct a wreath product $G = HwrK$ of the groups H and K, where K is nontrivial, such that the base subgroup B is p-elementary abelian, or torsion-free abelian, and the quotient G/B is an $\mathfrak{X}$-group. D.J.Robinson and J.Wilson introduced this method, by choosing $\mathfrak{X}$ as the class of polycyclic groups (see [5]). In this construction $B = C_G(B)$ is the unique minimal normal subgroup of G and it coincides with the Fitting subgroup of G. Moreover each proper quotient of G is an $\mathfrak{X}$-group. To convenience of the reader an adapted example is shown.

Example 3.3. Let p be a prime, $\mathbb{Q}$ be the additive group of the rational numbers and $\mathbb{Q}_p$ be the additive group of the rational numbers whose denominators are p-numbers. Clearly the direct product D of countabily many copies of $\mathbb{Q}_p$ is an MC-group. Put $\mathbb{Q} = Q$, the group $G = QwrD$ can be written as $G = D \ltimes B$, where B is the base subgroup of the wreath product, then B is the direct product of countabily many copies of $\mathbb{Q}$. Here $C_G(B) = B$, G/B is an MC-group, B is the unique minimal normal subgroup of G. If N is a nontrivial normal subgroup of G, then $N \geq B$ and G/N is an MC-group. Given a nontrivial element g of G, $< g >^G$ must contain B. Then $C_G(< g >^G) \leq C_G(B) = B$ and $G/C_G(< g >^G)$ has not finite abelian rank. It follows that $G/C_G(< g >^G)$ is not (soluble minimax)-by-finite. We have shown that G is a $JNMC$-group. Moreover the construction of G implies $Z(G) = 1$.

4. Periodic $JNMC$-groups and soluble groups with many min-by-max quotients.

This section contains results on the structure of periodic $JNMC$-groups. It is useful to recall that a group G is said to be a CC-group or a group with Chernikov conjugacy classes if $G/C_G(< x >^G)$ is a Chernikov group, for each element x of G. A group which is not a CC-group, but whose proper quotients are CC-groups, is said to be a $JNCC$-group. The structure of $JNCC$-groups is well-known and the corresponding results are referred to [5].

LEMMA 4.1. *If G is a periodic $JNMC$-group, then it is a $JNCC$-group.*

PROOF. This follows by definitions and [9, Section 10.3, p.166].□

Introducing the following statement, it is useful to recall that, if G is a group, A is an abelian subgroup of G and $H = G/A$ is a quotient, then A can be naturally regarded as $\mathbb{Z}H$-module, where $\mathbb{Z}H$ denotes the group ring with integer coefficients over H. In this situation A is said to be a *just infinite $\mathbb{Z}H$-module* if for every submodule B of A the module A/B is finite and the intersection of the family of all nontrivial submodules of A is trivial.

THEOREM 4.2. *Let G be a non-monolithic periodic $JNMC$-group and $1 \neq A = FitG$. If G is locally soluble, then every proper quotient of G is central-by-finite. Moreover*

(i) *A is a just infinite $\mathbb{Z}H$-module, where $H = G/A$ is central-by-finite and $C_H(A) = 1$.*

(ii) *Conversely, let A be a $\mathbb{Z}H$-module such that $C_H(A) = 1$, where $H = G/A$ is a nontrivial central-by-finite group. Then every extension of A by H including the given module structure is a group, which is not central-by-finite, but whose proper quotients are central-by-finite. Such group has trivial center and Fitting subgroup A.*

PROOF. This follows by Lemma 4.1 and [5, Corollary 16.26, Theorem 16.28].□

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