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A Note on Just-Non-$\mathfrak{X}$ Groups

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Abstract

Let $\mathfrak{X}$ be a class of groups. A group which belongs to $\mathfrak{X}$ is said to be an $\mathfrak{X}$-group. A group is said to be a Just-Non-$\mathfrak{X}$ group, if it is not an $\mathfrak{X}$-group but all of its proper quotients are $\mathfrak{X}$-groups. We study the role of the generalized Fitting subgroup in Just-Non-$\mathfrak{X}$ groups. Finally, we may apply some recent results on groups having few normalizer subgroups, whenever $\mathfrak{X}$ gives information of this kind.

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1. Introduction and terminology

Let $\mathfrak{X}$ be a class of groups. A group which belongs to $\mathfrak{X}$ is said to be an $\mathfrak{X}$-group. A group is said to be a *Just-Non-*$\mathfrak{X}$ group, or briefly a JNX-group, if it is not an $\mathfrak{X}$-group but all of its proper quotients are $\mathfrak{X}$-groups. The structure of Just-Non-$\mathfrak{X}$ groups has already been studied for several choices of the class $\mathfrak{X}$, so there is a well developed theory about the topic (see [16]). JNX-groups have been investigated both in Theory of Finite Groups and in Theory of Infinite Groups (see [[2],§2.3], [14], [16]). H.Schunk was interested in studying Just-Non-$\mathfrak{X}$ groups with respect to some problems of Local Theory of Finite Groups as [[2], Theorem 2.3.7] and [[10], §3] testify. He called such groups the *groups of boundary* $\mathfrak{X}$ and obtained deep results of factorization as in [[2], Theorems 2.3.15, Propositions 2.3.16, 2.3.17,Theorems 2.3.20, 2.3.24, 2.4.12, Statements 6.5.10-6.5.19, Theorem 6.5.21, Corollary 6.5.22]. Most of the times, the description of JNX-groups overlaps either the results of H.Schunk or a well-known splitting's Theorem of I.Schur and H.Zassenhaus [[15], 18.1, 18.2]. In locally finite groups we may find a generalization of the results of H.Schunk in [9]. Recently, also a compact group which is not a Lie group but all of its proper topological quotients are Lie groups has been classified (see [27]). Therefore, many techniques and methods have general application.

On the other hand, there is a long standing line of research (see [2], [10]) which investigates the structure of a group which is not an $\mathfrak{X}$-group but all of its proper subgroups are $\mathfrak{X}$-groups. Such groups are called $\mathfrak{X}$-*critical* groups (see [2], [10]), or *minimal non*-$\mathfrak{X}$-groups (see [18], [24]). In the present paper we follow the terminology of [24], using respectively the terms Just-Non-$\mathfrak{X}$ group, or briefly JNX-group, and minimal non-$\mathfrak{X}$-group. The literature shows that the terminology and the notations are not uniform and some results can be found independently by means of different approaches (compare for instance [[2], Theorem 6.4.4] and [[24], Theorem 3.44]).

The reason why Just-Non-$\mathfrak{X}$ groups and minimal non-$\mathfrak{X}$-groups are correlated is due to a unexpected symmetry in their structure. Roughly speaking, a Just-Non-$\mathfrak{X}$ group means a group which is dual with respect to a minimal non-$\mathfrak{X}$-group. This is afforded by looking at the main results in [16]. For instance, if $\mathfrak{A}$ is the class of the abelian groups, Just-Non-$\mathfrak{A}$ groups have been completely described by M.F.Newman (see [[16], §11] and [[16], Theorems 11.1, 11.2]). He proved that a Just-Non-$\mathfrak{A}$ group is characterized to be a homomorphic image of a direct product of an extra-special group by a locally cyclic group. Minimal non-$\mathfrak{A}$-groups have been completely described by O.Yu.Schmidt (see [[2], p.268, l.4] or [28]). It is interesting to point out the great symmetry which pervades the result of M.F.Newman and that of O.Yu.Schmidt.

Easy remarks show that the Fitting subgroup $F(G)$ of a group G plays in the structure of a Just-Non-$\mathfrak{A}$ group the same role which is played by $G/Frat(G)$ in the structure of a minimal non-$\mathfrak{A}$-group, where $Frat(G)$ denotes the Frattini subgroup of G (see [[16], §11] and [28]). Many other analogies can be found if one analyzes deeply the results in literature.

The importance of Just-Non-$\mathfrak{X}$ groups and minimal non-$\mathfrak{X}$-groups becomes more relevant when we think situations as in [[18], Theorem 7.4.1] or [[2], l.2-18]. For instance, [[18], Theorem 7.4.1] states that a finitely generated group G which is not polycyclic has a section S which is not polycyclic but all of its proper quotients S/N are polycyclic groups, where N is a normal subgroup of S (see [18]). This property happens for many choices of the class $\mathfrak{X}$ and not only when $\mathfrak{X}$ is the class of the polycyclic groups. Actually, many problems remain unsolved both in Theory of Finite Groups and in Theory of Infinite Groups also for easy choices of the class $\mathfrak{X}$ as [[19], Problem 9.74] and [[16], open questions] show.

In Section 2 of the present paper, we recall the notion of generalized Fitting subgroup $F^*(G)$ of a group G. This notion has been introduced in the last thirty years (see [[2], l.97-98] and [[17], §6.5]) and largely used in working with simple groups and finitary linear groups (see [20], [[21], §6], [[22], Theorem A]). It seems that a systematic discussion of Just-Non-$\mathfrak{X}$ groups, using the notion of the generalized Fitting subgroup, has been not completely given. Here we will extend some classical results of W.Gaschütz, H.Schunk and H.Wielandt [[2], §3 and 6] both in finite and infinite case [[9],§6 and 7], involving $F^*(G)$.

Sections 3 is devoted to Just-Non-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is related to have few normalizer subgroups. Here we will use recent results of [3], [4], [5], [6], [7], [8].

Most of our notation is standard and follows [2], [17], [18], [24].

2. Some auxiliary results

Most of the following statements has been showed to convenience of the reader and follows from [1], [[2], 2.2], [[17], §6.5], [22].

Definitions 2.1 *(see [[2], 2.2] or [[17], §6.5]).* –

(i) *A group G is said to be quasi-simple if $[G, G] = G$ and $G/Z(G)$ is a non-abelian simple group.*

(ii) *A subgroup H of G is said to be a component of G if H is a quasi-simple subnormal subgroup of G.*

(iii) *The layer $E(G)$ of a group G is the subgroup $E(G)$ generated by all components of G, i.e. the product of all components of G.*

Lemma 2.2 *(see [[2], 2.2] or [[17], §6.5]).* –

(i) *Assume that H and K are subnormal subgroups of a finite group G. If H is the unique maximal normal subgroup of G, $H = [H, H]$ and H is not contained in K, then $K \leq N_G(H)$.*

(ii) *Let G be a group. If H and K are components of G, then either $H = K$ or $[H, K] = 1$.*

(iii) *$F^*(G) = F(G)E(G)$ and $[F(G), E(G)] = 1$ hold in each group G.*

(iv) *$C_G(F^*(G)) \leq F^*(G)$, where G is either a finite group or a finitary linear group.*

(v) *Let G be a group. $E(G)$ is the central product of all components of G, but not the product of any proper subset of them.*

Now we recall [[1],§1]. Let $\mathcal{D}$ be a class of pairs (B, A) such that B is a group acting faithfully on the group A. Let G be a group acting on a group

N. A G-invariant normal series $\mathcal{A}$ of N is called a $\mathcal{D}$-series for G on N if $(G/C_G(A), A) \in \mathcal{D}$ for all factors A of $\mathcal{A}$. An ascending $\mathcal{D}$-series for G on N is called a hyper-$\mathcal{D}$ series. If such a series exists we say that G acts hyper-$\mathcal{D}$ on N. G is hyper-$\mathcal{D}$ means that G acts hyper-$\mathcal{D}$ on G. If $\mathcal{G}_1$, $\mathcal{G}_2$ are classes of groups, then $(\mathcal{G}_1, \mathcal{G}_2)$ denotes the class of pairs (B, A) with $B \in \mathcal{G}_1$, $A \in \mathcal{G}_2$ and B acting faithfully on A. We denote the class of all groups with $*$. So $(*, *)$ denotes the class of all pairs of groups (B, A) with B acting faithfully on A. Consider the case $N = G$. Observe that hyper-($*$,abelian) groups are the hyperabelian groups and hyper-$(1, *)$ groups are the hypercentral groups. We say that G is hypersolvable if G is hyper-(solvable,$*$). This notation might be slightly misleading since one probably would be tempted to define a hypersolvable group to be a hyper-($*$,solvable) group. But as the hyper-($*$,solvable) groups are just the hyperabelian groups such a definition would not be of much use. Similarly we define a hypernilpotent group to be a hyper-(nilpotent,$*$)-group. Unwinding the definitions we see that a group G is hypersolvable if and only if G has a normal ascending series $\mathcal{A}$ such that $G/C_G(A)$ is solvable for all factors A of $\mathcal{A}$.

We recall some notions in group varieties as in [1]. From now the symbol $\mathbb{N}$ will denote the set of the positive integers. For $n \in \mathbb{N}$, $F(n)$ denotes the free group on n-generators $x_1, x_2, \ldots, x_n$. Let G be a group and $m \in \mathbb{N} \cup \{\infty\}$ with $m \geq n$ and $g = (g_i)_{i=1}^m \in G^m$. Then there exists a unique homomorphism $\phi_g : F(n) \to G$ with $x_i \to g_i$ for all $1 \leq i \leq n$. Given a word $w \in F(n)$, we write $w(g)$ for $\phi_g(w)$. So if $w = x_{i_1} x_{i_2} \ldots x_{i_m}$ with $1 \leq i_k \leq n$, then $w(g) = g_{i_1} g_{i_2} \ldots g_{i_m}$. If $m \leq n$, we view $F(m)$ as subgroup of $F(n)$. Let $m = m(w) \in \mathbb{N}$ be minimal with $m \in F(m)$. Let $F = \bigcup_{n=1}^{\infty} F(n)$ and $\mathcal{W}$ be the set of subsets of F. So the elements of $\mathcal{W}$ are sets of words. Put $G^w = \langle w(g) : g \in G^n \rangle$ and note that G^w is a normal subgroup of G. For a set $W \in \mathcal{W}$ let $G^W = \langle G^w : w \in W \rangle$. Let $\mathcal{G}(W)$ be the class of groups with $G^W = 1$, that is, $\mathcal{G}(W)$ is the variety defined by W.

The following list of results is referred to [1].

Proposition 2.3 *(See [[1], Proposition 3.1]).* – *Let $W \in \mathcal{W}$ and let G be a group. Then G is hyper-$(\mathcal{G}(W), *)$ if and only if G^W is hypercentral.*

Definitions 2.4 *(See [[1], Definition 3.2]).* –
Let $W = (W_i)_{i=1}^{\infty} \in \mathcal{W}^{\infty}$ be a sequence of sets of words.

(a) *W is decreasing if $F^{W_{i+1}} \leq F^{W_i}$ for all i.*

(b) *W is almost decreasing if for all nonzero integers i, j there exists an integer $k \leq j$ such that $F^{W_k} \leq F^{W_i}$.*

(c) *$\mathcal{G}(W) = \bigcup_{i=1}^{\infty} \mathcal{G}(W_i)$.*

Definitions 2.5 *(See [[1], Definition 3.4]). – Let G be a group acting on a group N, $W \in \mathcal{W}^\infty$ and α an ordinal.*

(a) *Define $H_\alpha = Hyp_\alpha^W(G, N)$ inductively as follows:*

$$H_\alpha = 1, if\ 0 = \alpha,$$

$$H_\alpha = \bigcup_{\beta<\alpha} H_\beta, if\ 0 \neq \alpha\ is\ a\ limit\ ordinal,$$

$$H_\alpha/H_{\alpha-1} = C_{N/H_{\alpha-1}}([N, G^{W_k}]G^{W_k}),$$

if $\alpha = \beta + k$ with β a limit ordinal and k nonzero integer.

(b) *$\delta = \delta^W(G, N)$ is the least ordinal such that $H_\delta = H_\beta$ for all $\beta \geq \delta$. Moreover, $Hyp^W(G, N) = H_\delta$.*

(c) *A hyper-W series is a hyper-($\mathcal{G}(W)$,$*$) and a hyper-W group is a hyper-($\mathcal{G}(W)$,$*$) group.*

Proposition 2.6 *(See [[1], Proposition 3.5]). –*
Let G be a group and $W \in \mathcal{W}^\infty$.

(a) $(H_\alpha)_\alpha$ is a hyper-W series for G on $Hyp_\alpha^W(G)$.

(b) Let $A \lhd G$ and $(A_\alpha)_\alpha$ be a a hyper-W series for G on A.

  (b.1) For every ordinal α there exists an ordinal α^* with $A_\alpha \leq H_{\alpha^*}$. In particular, $A \leq Hyp^W(G)$.

  (b.2) If W is almost decreasing we can choose α^* such that $\alpha^* = \alpha + n_\alpha$ for some $n_\alpha \in \mathbb{N}$ and $n_\alpha = 0$ if α is a limit ordinal.

(c) G is a hyper-W if and only if $G = Hyp^W(G)$.

Definitions 2.7 *(See [[1], Definition 3.6]). –*

(a) For $i = 1, 2$ let w_i be a word and $m_i = m(w_i)$. Put

$$[w_1, w_2] = [w_1((x_i)_{i=1}^{m_1}), w_2((x_{m_1+i}))_{i=1}^{m_2}] \in F(m_1 + m_2),$$

$[w_1, w_2]$ is called the outer commutator of w_1 and w_2.

(b) Following [[1], Definition 3.6], outer commutators are inductively defined as follows:

  (b.1) $w = x_1$ is the only commutator word with $m(w) = 1$.

(b.2) If $m(w) > 1$, then w is an outer commutator word provided that there exist outer commutator words w_1, w_2 with $m(w_i) < m(w)$ and $w = [w_1, w_2]$.

(b.3) Let $w \in F^n$, $n \in \mathbb{N} \cup \{\infty\}$. Then $\breve{w}$ is inductively defined as follows: $\breve{w}_1 = x_1$ and $\breve{w}_{i+1} = [\breve{w}_i, w_i]$.

(b.4) Let $W \in \mathcal{W}^n$, $n \in \mathbb{N} \cup \{\infty\}$. Then $\check{W} \in \mathcal{W}^{n+1}$ is inductively defined as follows: $\check{W}_1 = \{x_1\}$ and $\check{W}_{i+1} = \{[v, w] : v \in \check{W}_i, w \in W_i\}$.

Definitions 2.8 *(See [[1], Definition 3.8]).* –

(a) *Let $W \in \mathcal{W}^\infty$. Then $\mathcal{H}(W)$ is the class of groups G such that for all $g \in G^\infty$ and all $w \in Dr_{i=1}^{\infty} W_i$ there exists a nonzero integer n with $w_n(g) = 1$.*

(b) *Let $\mathcal{D} \subset (*, *)$. Then $\mathcal{HD}$ is the class of hyper-$\mathcal{D}$-groups. $\mathcal{FD}$ is the class of finitely hyper-$\mathcal{D}$-groups.*

Theorem 2.9 *(See [[1], Theorem 3.9]).* – *Let $W \in \mathcal{W}^\infty$. Then*

(a) *$\mathcal{G}(\check{W}) \subseteq \mathcal{F}(\mathcal{G}(W), *)$ with equality if W is almost decreasing.*

(b) *$\mathcal{H}(\check{W}) \subseteq \mathcal{H}(\mathcal{G}(W), *)$ with equality if W is almost decreasing.*

The main result of this Section is the following.

Theorem 2.10. – *Let G be a hyper-$\mathcal{H}(\mathcal{G}(W),*)$ group with $F(G) = 1$ and $C_G(F^*(G)) \leq F^*(G)$. G is a Just-Non-(hyper-$\mathcal{H}(\mathcal{G}(W),*)$) group if and only if the following conditions are satisfied:*

(i) *$E(G) = H_1 \times H_2 \times \ldots$, where H_i is a component of G for each $i = 1, 2, \ldots$;*

(ii) *$E(G)$ is the unique minimal normal subgroup of G;*

(iii) *$G/E(G)$ is a hyper-$\mathcal{H}(\mathcal{G}(W),*)$ group;*

(iv) *G acts transitively on the set $\{H_1, H_2, \ldots\}$ of the components of G;*

(v) *H_i is simple for each $i = 1, 2, \ldots$;*

(vi) *$N_G(H_i)/C_G(H_i)$ is a hyper-$\mathcal{H}(\mathcal{G}(W),*)$ group for each $i = 1, 2, \ldots$.*

Proof. – Let G be a Just-Non-(hyper-$\mathcal{H}(\mathcal{G}(W),*)$) group. We will see that the statements (i)-(vii) are satisfied. If G is trivial then the result is obviously true. Since G is a nontrivial group and $F(G) = 1$, Definition 2.1 and Lemma 2.2 imply that $F^*(G) = E(G)$ is a nontrivial normal subgroup of G. Then the statement (iii) follows.

We will show that a component H_i of G is simple for each $i = 1, 2, \ldots$. By Definition 2.1, H_i is a subnormal subgroup of G so that its center $Z(H_i)$ is a subnormal subgroup of G. Then $Z(H_i) \leq F(G)$ so that $Z(H_i) = 1$. This means that H_i is a simple group so the statement (v) follows.

By Definition 2.1 and $F(G) = 1$, $E(G) = \langle H_1, H_2, \ldots \rangle$. (ii) Lemma 2.1 implies $[H_i, H_j] = 1$ for each $i, j \in \{1, 2, \ldots\}$. This allows us to say that H_i is a normal subgroup of G. But,

$$H_i \cap \langle H_j : i \neq j \rangle \leq Z(H_i) = 1$$

for each $i, j \in \{1, 2, \ldots\}$. Therefore, the statement (i) follows.

We will prove the statement (iv). We may deduce easily from Definitions 2.1 and Lemma 2.2 that G acts transitively on $E(G)$. But, $E(G)$ is the direct product of the components of G, as noted before. The statement (iv) follows.

It is enough to check that each nontrivial normal subgroup N of G contains $E(G)$ in order to prove that the statement (ii) is true. Assume that $[N, E(G)] = 1$. $C_G(E(G)) \leq E(G)$ so we have

$$N \leq C_G(E(G)) \leq E(G).$$

Then $N \leq Z(E(G)) = 1$ because of (i) and this gives a contradiction. Now assume that $M = N \cap E(G)$. The same argument shows that $[M, E(G)] \neq 1$. Then there exists an integer $i = 1, 2, \ldots$ such that $[H_i, M] \neq 1$. But, $1 \neq [M, H_i]$ is a normal subgroup of H_i and H_i is a simple group thanks to the statement (v). From this, $H_i = [M, H_i]$ is contained in M. Now the statement (iv) implies that

$$E(G) = \langle H_i \rangle^G \leq M \leq N,$$

as claimed.

It remains to see that (vi) is satisfied. Let $\overline{N_G(H_i)} = N_G(H_i)/C_G(H_i)$ and $\overline{H_i} = H_i/C_G(H_i)$, where $i \in \{1, 2, \ldots n\}$. Firstly, we prove that $\overline{H_i}$ is the unique minimal normal subgroup of $\overline{N_G(H_i)}$. Let $\overline{N}$ be a nontrivial normal subgroup of $\overline{N_G(H_i)}$ and N be the inverse image of $\overline{N_G(H_i)}$. If N is not contained in $C_G(H_i)$, then $1 \neq [N, H_i]$ is a normal subgroup of the simple group H_i. Then $[N, H_i] = H_i$ is contained in N and we have that $\overline{H_i}$ is contained $\overline{N}$. This shows that $\overline{H_i}$ is the unique minimal normal subgroup of $\overline{N_G(H_i)}$.

Secondly, we prove that $\overline{N_G(H_i)}$ is a hyper-$\mathcal{H}(\mathcal{G}(W),*)$ group. Of course,

$$\overline{N_G(H_i)}/\overline{H_i} \simeq N_G(H_i)/H_iC_G(H_i),$$

then

$$F^*(G) \leq H_iC_G(H_i) \leq N_G(H_i) \leq G.$$

This and the statement (iii) are enough to state that (vi) is true.

Conversely, assume that the statements (i)-(vi) are true. The statements (ii) and (iii) imply that G is a Just-Non-(hyper-$\mathcal{H}(\mathcal{G}(W),*)$) group. Then our result has been completely proved.□

Corollary 2.11. – *Let G be a group which satisfies the conditions (i)-(vi) of Theorem 2.10. Then $F^*(G) \leq N_G(H_i) \triangleleft G$ for each component H_i of G and $N_G(H_i) = N_G(H_j)$ for each $i, j \in \{1, 2, \dots\}$.*

Corollary 2.12. – *Let G be a group which satisfies the conditions (i)-(vi) of Theorem 2.10. Then G is isomorphic to a subgroup of $Aut(F^*(G))$. Furthermore,*

$$Aut(F^*(G)) = Aut(H_1 \times H_2 \times \dots) wr Sym(|I|),$$

where $|I|$ denotes the number of the components $H_1, H_2, \dots$ of G.

Corollary 2.13. – *Let G be a group which satisfies the conditions (i)-(vi) of Theorem 2.10. Then $G/F^*(G))$ is isomorphic to a subgroup of*

$$Out(H_1 \times H_2 \times \dots) wr Sym(|I|).$$

Example 2.14. – Let $G = PSL(2, q^2)\langle d\sigma\rangle = AB$, where $A = PSL(2, q^2)$, $B = \langle d\sigma\rangle$,

$$\sigma : 1 \in K \to \sigma(1) \in K^q$$

is an automorphism of the field K of q^2 elements, ζ is an element of order the second part of $q^2 - 1$, and

$$d : \zeta \in K \longmapsto d(\zeta) = d = \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, K).$$

In this situation, we have $F(G) = Z(G) = 1$ and

$$A = E(G) = F^*(G) = PSL(2, q^2).$$

Furthermore, B acts transitively on A. If $q = 3$, then $G = M_{10}$, that is, G is the Mathieu group of type 10. G is not an abelin group and A is the unique minimal normal subgroup of G, so each proper quotient of G is abelian. This construction satisfies the conditions of Theorem 2.10 when the class of abelian groups is involved.

Example 2.15. – Assume that $G = HwrA$, where H is a simple group and A is an abelian group which acts transitively on H. We have that

$$F^*(G) = E(G) = \underbrace{H \times H \times \ldots \times H}_{|A|-times},$$

$F(G) = Z(G) = 1$. Again $E(G)$ is the unique minimal normal subgroup of G and each proper quotient of G is abelian. Of course, G is not abelian. Again, G satisfies Theorem 2.10 and we note that it can be constructed both finite and infinite, provided the order $|A|$.

Example 2.16. – Assume that $G = (H \times H)wrA$, where H is a simple group and A is an abelian group which acts transitively on $H \times H$. We have that

$$F^*(G) = E(G) = \underbrace{H \times H \times \ldots \times H}_{2|A|-times},$$

$F(G) = Z(G) = 1$. Again $E(G)$ is the unique minimal normal subgroup of G and each proper quotient of G is abelian. Of course, G is not abelian. This example shows that we may extend Example 2.15.

3. JNX_i-groups for $i = 0, 1, 2, 3, 4$

A result of Ya.D.Polovicky [23] of 1980s states that a group G with finitely many normalizers of abelian subgroups has the center $Z(G)$ of finite index in G. In literature the groups which have the center of finite index in the whole group are called *central-by-finite* groups. Since central-by-finite groups have been described by B.H.Neumann in a famous paper of 1955, it is clear that restrictions on the number of the normalizer subgroups in a group has a strong influence on the structure of the group.

If $\mathfrak{X}_0$ is the class of groups having finitely many normalizer of abelian subgroups, then we may introduce the notion of JNX_0-*group*. We say that a group

G is an $\mathfrak{X}_0$-group, if G has finitely many normalizers of abelian subgroups. A group G is said to be a *Just-Non-*$\mathfrak{X}_0$ group, or briefly a JNX_0-group, if G is not a $\mathfrak{X}_0$-group but all of its proper quotient groups are $\mathfrak{X}_0$-groups. The knowledge of $\mathfrak{X}_0$-groups has been recently concluded in [3]. We may consider further classes as we have done for $\mathfrak{X}_0$-groups, thanks to [4], [5], [6], [7], [8].

We say that a group G is an $\mathfrak{X}_1$-group, if G has finitely many normalizers of non-abelian subgroups. We say a group G is an $\mathfrak{X}_2$-group, if G has finitely many normalizers of non-(locally nilpotent) subgroups. We say that a group G is an $\mathfrak{X}_3$-group, if G has finitely many normalizers of subnormal subgroups. We say that a group G is an $\mathfrak{X}_4$-group, if G has finitely many normalizers of non-subnormal subgroups. A corresponding definition of JNX_i-group for $i = 1, 2, 3, 4$ can be given.

We are interested in studying JNX_i-groups for $i = 0, 1, 2, 3, 4$ as in [16]. In order to do this, the following concepts have to be considered. As in [18], a group G is called *finite-by-abelian* if there is a finite normal subgroup F of G such that the quotient group G/F is abelian. In [26] the following notions have been introduced. A group G which is not finite-by-abelian but all of its proper quotients are finite by-abelian is called *Just-Non-(Finite-by-Abelian)* group, or briefly $JNFA$-group. A group G which is not central-by finite but all of whose proper quotients are central-by-finite is called *Just-Non-(Central-by-Finite)* group, or briefly $JNCF$-group. $JNFA$-groups and $JNCF$-groups have been classified in [26]. The notions of *Just-Non-Nilpotent* group, *Just-Non-Finite* group and JNT-group can be found in [16]. It is easy to see that finite-by-abelian groups, nilpotent groups, T-groups and central-by-finite groups are hyper-$\mathcal{H}(\mathcal{G}(W),*)$ groups, for suitable choices of $\mathcal{G}(W)$.

Proposition 3.1. – *Let G be a group with $F(G) \neq 1$. Then the following statements are true.*

(i) *If G is a JNX_0-group, then G is a $JNCF$-group.*

(ii) *If G is a JNX_1-group and each proper quotient of G is locally graded, then G is a $JNFA$-group.*

(iii) *If G is a JNX_2-group, then either G has each proper quotient which is a locally nilpotent group or G is a $JNFA$-group.*

(iv) *Assume that G is a soluble JNX_3-group which locally satisfies the maximal condition on subgroups. If each proper quotient of G has proper Wielandt subgroup, then either G is a JNT-group or G is a Just-Non-Finite group.*

(v) *Assume that G is a soluble group which is either torsion-free or of finite exponent. If G is a JNX_4-group, then G is either a Just-Non-Nilpotent group or a $JNCF$-group.*

Proof. – (i). Assume that G is a JNX_0-group. Each proper quotient Q of G satisfies [[3], Theorem 2.2], then Q is central-by-finite. Assume that G is central-by-finite. Neumann's Lemma (see [24]) implies that G is covered by finitely many abelian subgroups so that G has finitely many normalizers of abelian subgroups. Then G should be an $\mathfrak{X}_0$-group and we get to a contradiction. Therefore G is a $JNCF$-group.

(ii). Assume that G is a JNX_1-group. Each proper quotient Q of G satisfies [[3], Theorem 3.5], then Q has finite derived subgroups. In particular, Q is finite-by-abelian. Since a finite-by-abelian group has finitely many normalizers of non-abelian subgroups, G is not an $\mathfrak{X}_1$-group. Therefore G is a $JNFA$-group.

(iii). Assume that G is a JNX_2-group. Each proper quotient Q of G satisfies [[3], Theorem 3.7], then either Q is locally nilpotent or Q has finite derived subgroup. In particular, either Q is locally nilpotent or Q is finite-by-abelian. Since both locally nilpotent and finite-by-abelian groups have finitely many normalizers of non-locally nilpotent subgroups, G is not an $\mathfrak{X}_3$-group. Therefore the result follows.

(iv). Assume that G is a soluble JNX_3-group. We recall that *max-locally* is closed with respect to quotients. Therefore each proper quotient Q of G satisfies [[3], Theorem 4.1]. If $\omega(Q)$ denotes the Wielandt subgroup of G (see [24]) and $\omega(Q) = 1$, then Q is finite from [[3], Theorem 4.1]. If $\omega(Q) = Q$, then Q is a soluble T-group again from [[3], Theorem 4.1]. Since both finite and soluble T-groups have finitely many normalizers of subnormal subgroups, G is not an $\mathfrak{X}_3$-group. Therefore the result follows.

(v). Assume that G is a JNX_4-group. Each proper quotient Q of G satisfies [[3], Theorem 5.2], then either Q has each subgroup which is subnormal or Q has each subgroup which is almost normal. In the first case [[18],Theorems 12.2.5 and 12.5.1] imply that Q is nilpotent. In the second case a famous result of B.H.Neumann implies that Q is central-by-finite. Since both nilpotent and central-by-finite groups have finitely many normalizers of non-subnormal subgroups, G is not an $\mathfrak{X}_4$-group. Therefore the result follows. □

Corollary 3.2. – *Let G be a locally nilpotent group with $F(G) \neq 1$.*

(i) *If G is a JNX_1-group, then G is a $JNFA$-group.*

(ii) *If G is a soluble JNX_3-group whose proper quotients have proper Wielandt subgroup, then either G is a JNT-group or G is a Just-Non-Finite group.*

Proof. – It follows from Proposition 3.1. □

We note that Proposition 3.1 can be reformulated without using the results of [3], [4], [5], [6], [7], [8]. The main passages of this standard procedure can be found in [[13], §1,2] or more generally in [16].

Corollary 3.3.– *Let* G *be a locally nilpotent* JNX_i*-group for* $i = 0, 1, 2$ *with* $1 = F(G)$*. Then* G *satisfies the conditions of Theorem 2.10.*

Proof. – This follows from Theorem 2.10 for a suitable choice of $\mathcal{G}(W)$. □

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