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K-Forms of 2-Step Splitting Trivectors

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Abstract. In this paper, we describe the K-forms of 2-step splitting trivectors of rank 8 where K is any field of characteristic other than 2 and 3 .

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1. Introduction

First we recall some definitions. Let E be an n-dimensional vector space over a field K and let $\Lambda^m E$ denote the exterior power of degree m over E, the classification of trivectors is the study of the action of General linear group $GL(E)$ on the space $\Lambda^3 E$ defined by $f \cdot \omega = (\Lambda^3 f)(\omega)$. The equivalence classes are the $GL(E)$-orbits under this action. As $\Lambda^3 E^* \simeq (\Lambda^3 E\ )^*$, there is no difference between trilinear alternating forms and trivectors. *The support* of the trivector ω is the least subspace F of E such that $\omega \in \Lambda^3 F$. Denote this subspace by S_ω, its dimension is *the rank* of ω and will be denoted by $r(\omega)$. This classification could help to solve some open problems in the theory of self dual codes, see[10]. The relationship between trivectors and nilpotent lie algebras can be seen in [6].

A general classification of trivectors exists in dimension 7 and less [2,7,12]. For n=8 Gurevitch [4], D. Djokovic [3] and L.Noui [9] give an answer to the classification problem with $K = \mathbb{C}$, $K = \mathbb{R}$ and for K algebraically closed field of arbitrary characteristic respectively.

If ω is a trivector defined over the field K, a K-form of ω is another trivector, of the same type as that of ω, defined over K which is isomorphic to ω over $\overline{K}$, the algebraic closure of K. In [6], the K-forms of a trivector of rank 8 in the open dense orbit are determined when K has characteristic zero. In this paper, K is a field of characteristic other than 2 and 3, we are interested in describing the K-forms of 2-step splitting trivectors of rank 8.

1.1. **Invariants and trivectors.** First we talk about arithmetical invariant $d_1(\omega)$ that one may assign to a trivector ω. For every subspace α of E let $\overline{\omega}_\alpha$ the canonical image of ω in $\Lambda^3(E/\alpha)$. We put $d_k(\omega) = \inf_\alpha r(\overline{\omega}_\alpha)$ where α runs over all k-dimensional subspaces of E. The numbers $d_k(\omega)$ form a chain

$$d_0(\omega) = r(\omega) > d_1(\omega) > d_2(\omega) > \cdots > d_r(\omega) = 0$$

where r is less than or equal to $\dim E$. Note that $d_1(\omega) = 0$ if and only if ω is divisible by a vector [8].

The element ω of $\Lambda^3 E$ is called *splitting* if there exists a decomposition $E = E_1 \oplus E_2$ such that $\omega \in E_1 \otimes \Lambda^2 E_2$.

If $\dim E_1 = r$, ω is called *$r-$step splitting.* In particular ω is 1-step splitting means that ω is divisible: $\omega = e_1 u$ where $e_1 \in E_1$ and $u \in \Lambda^2 E_2$. We refer to $\mathrm{Aut}\,(\omega)$ the automorphisms group of ω as the stabilizer (in $GL(E)$) of ω, other invariants as Derivation and Commutant are defined in [7, 8, 13].

Theorem 1. [11] *There exists a system of divided powers in the ideal* $\underset{k\geqslant 0}{\oplus} \wedge^{2k} E$ *of the exterior power of* E .

This result is well known in the case of fields and it has a generalization in the ring case. If $u \in \wedge^2 M^*$, u is nondegenerate if and only if $\gamma_m(u)$ generates the A-module $\wedge^{2m} M^*$. If $u = \sum_{i=1}^{r} x_{2i-1}x_{2i}$ where $\{x_1, \ldots, x_{2r}, \ldots, x_n\}$ is a basis of M,

$$\gamma_k(u) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq r} x_{2i_1-1}x_{2i_1} \cdots x_{2i_k}$$

such that $\mathrm{rank}\,(u) = 2r$ is equivalent to $\gamma_k(u) \neq 0$ and $\gamma_{k+1}\,(u) = 0$.

1.2. **Trivectors of rank 8.**

Theorem 2. [9] *Let E be a vector space of dimension 8 over an algebraically closed field K of arbitrary characteristic. Then any trivector of rank 8 in $\Lambda^3 E$ is equivalent to one of the trivectors $\omega_{8,i}$ of Table 1.*

Table 1

$\omega_{8,i}$	Notation of Gurevitch	Trivector	$d_1(\omega_{8,i})$
$\omega_{8,1}$	XVI	$e_1(e_2e_3+e_4e_5)+e_6e_7e_8$	3
$\omega_{8,2}$	XI	$e_1(e_2e_3+e_4e_5+e_6e_7)+e_5e_6e_8$	3
$\omega_{8,3}$	$XVII$	$e_1(e_3e_4+e_5e_6)+e_2(e_3e_5+e_7e_8)$	5
$\omega_{8,4}$	$XIII$	$e_1(e_2e_3+e_4e_5)+e_6(e_2e_7+e_4e_8)$	5
$\omega_{8,5}$	XIX	$e_1(e_2e_3+e_4e_5)+e_6(e_2e_3+e_7e_8)$	5
$\omega_{8,6}$	XIV	$e_1(e_2e_3+e_4e_5+e_6e_7)+e_8(e_4e_3+e_5e_6)$	5
$\omega_{8,7}$	XII	$e_1(e_2e_3+e_4e_5+e_6e_7)+e_2(e_5e_6+e_7e_8)$	5
$\omega_{8,8}$	$XVIII$	$e_1(e_2e_8+e_3e_6+e_4e_7)+e_6e_7e_8+e_3e_4e_5$	6
$\omega_{8,9}$	XXI	$e_1[e_2(e_3+e_4)+e_5e_6]+e_3e_5e_7+e_4e_6e_8$	6
$\omega_{8,10}$	XX	$e_1(e_2e_8+e_6e_7)+e_2e_3e_5+e_3e_4e_6+e_4e_5e_7$	6
$\omega_{8,11}$	XV	$e_1(e_3e_7+e_5e_4+e_8e_2)+e_8(e_4e_3+e_6e_7)+e_2e_4e_6$	6
$\omega_{8,12}$	$XXII$	$e_1[(e_4-e_7)(e_3-e_8)+e_5e_7]+e_2(e_3e_4+e_5e_6)+e_6e_7e_8$	7
$\omega_{8,13}$	$XXIII$	$e_1[e_5(e_3+e_7)+e_8e_4]+e_2(e_3e_4+e_5e_6)+e_6e_7e_8$	7

Remark 1. *The trivectors* $\omega_{8,i}$, $i=1,\dots,6$. *are 2 step-splitting.*

2. Automorphisms group and Galois Cohomology.

In this section, K is an arbitrary field of characteristic other than 2 and 3, we will compute the automorphisms group A_i of each trivector $\omega_{8,i}$, $1\leqslant i\leqslant 6$. If $\overline{K}$ is the algebraic closure of K and G is $Gal(\overline{K}/K)$, the classification of trivectors using Galois descent will involve the cohomology $H^1(G,A_i)$ for not necessarily abelian coefficients A_i, $H^1(G,A_i)$ classifies trivectors over K that become isomorphic over $\overline{K}$ to $\omega_{8,i}$. Indeed if $A_i=\mathrm{Aut}_{\overline{K}}(\omega_{8,i})$, there is a bijection from $H^1(G,A_i)$ to the set of K-isomorphisms classes of trivectors ω such that $\omega_{\overline{K}}\backsimeq\omega_{8,i}$ ([12, 14]).

To establish the main result for K-forms we need to introduce a few more concepts.

2.1. **Etale algebras.** Following [5] by etale algebra we mean a finite dimensional commutative K-algebra L such that $L\simeq K_1\times\dots\times K_r$, with the K_i's finite separable field extentions of K. In other words, if we denote by K_s a separable closure of K then a finite-dimensional K-algebra L is called etale if $L\bigotimes_K K_s$ is isomorphic to a split K_s-algebra $K_s\times\dots\times K_s$. For the set $X=\{1,2,...,n\}$, let S_n be the symmetric group of X, i.e the group of all permutations of X. It is well-known that the Galois cohomology set $H^1(G,S_n)$ is in canonical one -to-one correspondence with the isomorphism classes of etale K-algebras of dimension n, see[5], that is, the pointed set $H^1(G,S_n)$ classifies etale K-algebras of degree n. In particular, for $n=3$ we obtain the cubic etale algebras as commutative associative K-algebras L of dimension 3 over K such that $L \simeq K_s \times K_s \times K_s$. They are of four possible types:

- $L_1 \simeq K \times K \times K$, the split algebra.
- $L_2 \simeq K \times K'$, where K' is a Galois quadratic field extension of K.
- L_3 a cyclic cubic field extension of K, and
- L_4 a general separable field extension of degree 3 over K.

2.2. **Invariant domains.** Let E be a vector space of dimension n over the field K and if ω is a trivector in $\Lambda^3 E$, we define two domains which are invariant under the group of automorphisms of the trivector ω.
First, the domain $\overline{R}_i(\omega) = \{x \in E \ / \ \overline{\omega}_{\langle x \rangle}$ is of type $\omega_i\}$ is invariant under $\mathrm{Aut}\,(\omega)$, indeed, let $x \in R_i$ and put $E = Kx \oplus E'$, then $\omega = xu + \omega'$ where ω' is a trivector in $\Lambda^3 E'$. Since $x \notin E'$, $\overline{\omega}_{\langle x \rangle} = \overline{\omega'}_{\langle x \rangle} = \omega'$ which is of type ω_i. If $f \in \mathrm{Aut}\,(\omega)$ then

$$\Lambda^3 f(\omega) = \omega = f(x)\Lambda^2 f(u) + \Lambda^3 f(\omega'),$$

this yields $\overline{\omega}_{\langle f(x) \rangle} = \overline{\Lambda^3 f(\omega')}_{\langle f(x) \rangle}$, as f is a bijective linear map, $\Lambda^3 f(\omega')$ is equivalent to ω'. $f(x) \notin f(E')$ imply that $\overline{\omega}_{\langle f(x) \rangle} = \Lambda^3 f(\omega')$ which is of type ω_i, so $f(x) \in R_i$, that is, $f(R_i) \subset R_i$.

For the second domain, suppose $x \in E$ and let f be a nonzero trilinear alternating form on E. Consider the bilinear alternating form f^x on E defined by

$$f^x(y, z) = f(x, y, z) \qquad (y, z \in E).$$

Its rank $rk(f^x)$ is an even number, then the domain $R_i(f) = \{x \in E / \, rk(f^x) = 2i\}$ $(0 \leq 2i \leq n)$ is invariant under $\mathrm{Aut}\,(f)$ [2].

In order to compute the automorphisms group A_i we use the previous invariant domains.

2.3. **Automorphisms groups.**

Proposition 1. *The automorphisms group* $A_1 = \mathrm{Aut}\,(\omega_{8,1})$ *verify the following exact sequences :*

$$\begin{aligned} 1 \ &\longrightarrow \ SL_3(K) \longrightarrow A_1 \longrightarrow \mathrm{Aut}\,(\omega_5) \longrightarrow 1 \\ 1 \ &\longrightarrow \ A_0 \longrightarrow \mathrm{Aut}\,(\omega_5) \longrightarrow K^* \longrightarrow 1 \\ 1 \ &\longrightarrow \ K^4 \longrightarrow A_0 \longrightarrow SP_4(K) \longrightarrow 1 \end{aligned}$$

where $\omega_5 = e_1(e_2e_3 + e_4e_5)$

Proof. Let $\omega_{8,1} = e_1\,(e_2e_3 + e_4e_5) + e_6e_7e_8$, first we determine the domain $R_{\leqslant 1}(\omega_{8,1}) = R_0(\omega_{8,1}) \cup R_1(\omega_{8,1})$. Direct computation shows that

$$R_{\leqslant 1}(\omega_{8,1}) = < e_2, e_3, e_4, e_5 > \cup < e_6, e_7, e_8 > = V_1 \cup V_2.$$

As $V_1 \cap V_2 = \{0\}$ then if $f \in A_1 = \mathrm{Aut}\,(\omega_{8,1})$ we have $f(R_{\leqslant 1}(\omega_{8,1})) \subset R_{\leqslant 1}(\omega_{8,1})$ so f either stabilizes each of the two linear spaces V_1, V_2 or interchanges them, that is,

$$\{f(V_1) \subset V_1 \text{ and } f(V_2) \subset V_2\} \quad \text{or} \quad \{f(V_1) \subset V_2 \text{ and } f(V_2) \subset V_1\}.$$

The second case is impossible and the matrix of f is of the shape:

$$\begin{pmatrix} \alpha_1 & o_{1\times 4} & o_{1\times 3} \\ \vdots & A & o_{4\times 3} \\ \alpha_8 & o_{3\times 4} & B \end{pmatrix}$$

Let $f \in A_1$, $\Lambda^3 f(\omega_{8,1}) = \omega_{8,1}$ implies $B \in SL_3(K)$ and $\alpha_2 = \cdots = \alpha_8 = 0$, in this case $f(e_1) = \alpha_1 e_1$ and $\wedge^3 f\,[e_1(e_2e_3 + e_4e_5)] = e_1(e_2e_3 + e_4e_5) = \omega_5$, consequently $f \mid < e_1, \ldots, e_5 > \in \mathrm{Aut}\,(\omega_5)$, this allows us to define a homomorphism of groups

$$A_1 \ni f \longmapsto \varphi(f) = f|_{<e_1,\ldots,e_5>} \in \mathrm{Aut}\,(\omega_5)$$

φ is obviously surjective, we have $\ker\varphi = \{f \ / \ f \in A_1 \text{ and } \ \alpha_1 = 1 \ , \ A = id_4\}$. $\Lambda^3 f(\omega_{8,1}) = \omega_{8,1}$ implies that $\ker\varphi \backsimeq SL_3(K)$, so the sequence

$$1 \longrightarrow SL_3(K) \longrightarrow A_1 \longrightarrow (\omega_5) \longrightarrow 1$$

is exact, as $\mathrm{Aut}\,(\omega_5)$ verify the following exact sequences [8]

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_0 & \longrightarrow & (\omega_5) & \longrightarrow & K^* & \longrightarrow & 1 \\ 1 & \longrightarrow & K^* & \longrightarrow & A_0 & \longrightarrow & SP_4(K) & \longrightarrow & 1 \end{array}$$

so we obtain the desired result. □

Proposition 2. *The automorphisms group* $A_2 = \mathrm{Aut}\,(\omega_{8,2})$ *verify the following exact sequences :*

$$\begin{array}{l} 1 \ \longrightarrow \ A_2^{'} \longrightarrow A_2 \longrightarrow K^* \longrightarrow 1 \\ 1 \ \longrightarrow \ A_2^{''} \longrightarrow A_2^{'} \longrightarrow SL_2(K) \longrightarrow 1 \\ 1 \ \longrightarrow \ A_2^{'''} \longrightarrow A_2^{''} \longrightarrow K^* \longrightarrow 1 \\ 1 \ \longrightarrow \ U(16) \longrightarrow A_2^{'''} \longrightarrow SL_2(K) \longrightarrow 1 \end{array}$$

Proof. Choosing a suitable basis of E, we can write

$$\omega_{8,2} = e_1\,(e_5e_6 + e_8e_2 + e_3e_7) + e_2e_3e_4.$$

First determine the domain $\overline{R}_3(\omega_{8,2}) = \Big\{x \in E / \ \overline{\omega_{8,2}}_{\langle x\rangle} \text{ has rank } 3\Big\}$. Notice that $e_1 \in \overline{R}_3(\omega_{8,2})$. Direct computation yields $\overline{R}_3(\omega_{8,2}) =< e_1 > -\{0\}$, then we can define a groups homomorphism $\varphi : A_2 \longrightarrow K^*$ by $\varphi(f\ ) = \alpha$ where $f(e_1) = \alpha e_1$, $A_2^{'} = \ker\varphi$. φ is surjective so the first sequence is exact:

$$1 \longrightarrow A_2^{'} \longrightarrow A_2 \longrightarrow K^* \longrightarrow 1.$$

Let $f \in A_2^{'}$, hence $\alpha = 1$ and $\Lambda^3 f(\omega_{8,2}) = \omega_{8,2}$ imply that f stabilizes the linear space $< e_1, e_2, e_3, e_4 >$. In this case the matrix of f has the form :

$$\begin{pmatrix} \begin{matrix} 1 & a \; b \; c \\ 0 & \\ 0 & A \\ 0 & \end{matrix} & B \\ O_{4\times 4} & \begin{matrix} C & D \\ O_{2\times 2} & E \end{matrix} \end{pmatrix}, \quad A \in GL_3(K), \quad C, E \in GL_2(K).$$

By simple computation we deduce that $C \in SL_2(K)$. This allows us to define a groups homomorphism $\pi : A_2^{'} \longrightarrow SL_2(K)$. π is surjective, therefore the following sequence is exact

$$1 \longrightarrow A_2^{''} \longrightarrow A_2^{'} \longrightarrow SL_2(K) \longrightarrow 1$$

where $A_2^{''} = \ker \pi$. As $\Lambda^3 f(\omega_{8,2}) = \omega_{8,2}$, then by simple computation, the matrix A gives the last sequences with $U(16)$ is unipotent group of dimension 16. □

Proposition 3. *The automorphisms group* $A_3 = \mathrm{Aut}\,(\omega_{8,3})$ *verify the following exact sequences :*

$$\begin{array}{lll} 1 & \longrightarrow & A_3^{'} \longrightarrow A_3 \longrightarrow K^* \times K^* \longrightarrow 1 \\ 1 & \longrightarrow & A_3^{''} \longrightarrow A_3^{'} \longrightarrow SL_2(K) \longrightarrow 1 \\ 1 & \longrightarrow & K^9 \longrightarrow A_3^{''} \longrightarrow SL_2(K) \longrightarrow 1 \end{array}$$

Proof. Let $\omega_{8,3} = e_1\,(e_3e_4 + e_5e_6) + e_2\,(e_3e_5 + e_7e_8)$, the invariant domain $\overline{R}_5(\omega_{8,3}) = \left\{ x \in E /\ \overline{\omega_{8,3}}_{\langle x\rangle} \text{ has rank } 5 \right\}$ is equal to

$$\{V_1 \cup V_2\} - \{0\} = \{< e_1 > \cup < e_2 >\} - \{0\}.$$

Let $f \in A_3$, then $f(\overline{R}_5(\omega_{8,3})) \subset \overline{R}_5(\omega_{8,3})$ so f either stabilizes each of the two linear spaces V_1, V_2 or interchanges them, the case $f(V_1) \subset V_2$ and $f(V_2) \subset V_1$ is impossible because if $f(e_1) = \lambda e_2$ and $f(e_2) = \beta e_1$, a computation of the coefficient of $e_2e_7e_8$ in the equality $\Lambda^3 f(\omega_{8,3}) = \omega_{8,3}$ leads to $1 = 0$, a contradiction. So $f(V_1) \subset V_1$ and $f(V_2) \subset V_2$, hence $f(e_1) = \alpha e_1$ and $f(e_2) = \beta e_2$, then we can define a groups homomorphism $\varphi : A_3 \longrightarrow K^* \times K^*$ by $\varphi(f) = (\alpha, \beta)$, this allows us to write the first exact sequence

$$1 \longrightarrow A_3^{'} \longrightarrow A_3 \longrightarrow K^* \times K^* \longrightarrow 1$$

where $A_3^{'} = \ker \varphi$. Let $f \in A_3^{'}$, then $\alpha = \beta = 1$, from the equality $\Lambda^3 f(\omega_{8,3}) = \omega_{8,3}$, we get:

$$M(f) = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \\ \multicolumn{2}{c}{0_{4\times 2}} \end{matrix} & \begin{matrix} A \\ B \end{matrix} & D \\ O_{2\times 2} & O_{2\times 4} & C \end{pmatrix} \quad \begin{array}{l} B \in GL_4(K) \\ C \in GL_2(K) \end{array}$$

By a computation using the equality $\det M(f\ \ ) = \det B \times \det C \neq 0$, we deduce that $C \in SL_2(K)$, then we can define a groups homomorphism $\Psi : A_3^{'} \longrightarrow SL_2(K)$ by $\varphi(f) = C$, consequently we obtain the second exact sequence. Let $f \in A_3^{''} = ker\Psi$ then $C = Id_2$. The requirement $\wedge^3 f(\omega_{8,3}) = \omega_{8,3}$ gives the last sequence. □

Proposition 4. *The automorphisms group* $A_4 = \mathrm{Aut}\,(\omega_{8,4})$ *verify the following exact sequences ::*

$$1 \longrightarrow A_4^{'} \longrightarrow A_4 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$
$$1 \longrightarrow A_4^{''} \longrightarrow A_4^{'} \longrightarrow K^* \times K^* \longrightarrow 1$$
$$1 \longrightarrow K^{12} \longrightarrow A_4^{''} \longrightarrow SL_2(K) \times SL_2(K) \longrightarrow 1$$

Proof. Choosing a suitable basis of E, we can write $\omega_{8,4} = e_1(e_3e_6 + e_4e_7) + e_2(e_3e_5 + e_4e_8)$. The invariant domain $\overline{R}_5(\omega_{8,4})$ is equal to

$$\{< e_1, e_2 > \cup < e_3, e_4 >\} - \{0\} = \{V_1 \cup V_2\} - \{0\},$$

obviously $V_1 \cap V_2 = \{0\}$, then if $f \in A_4,\ f(\overline{R}_5(\omega_{8,4})) \subset \overline{R}_5(\omega_{8,4})$ consequently f either stabilizes each of the two linear spaces V_1, V_2 or interchanges them, this allows us to define a groups homomorphism

$$\varphi : A_4 \longrightarrow \mathbb{Z}/2\mathbb{Z}, \quad f \longmapsto \varphi\,(f) = \begin{cases} 0 & \text{if } f(V_1) \subset V_1 \text{ and } f(V_2) \subset V_2 \\ 1 & \text{otherwise} \end{cases}$$

φ is surjective, indeed $\varphi\,(id_E) = 0$ and $\varphi\,(f_0) = 1$ with f_0 is defined by $f_0(e_1) = e_3,\ f_0(e_2) = e_4,\ f_0(e_3) = e_1,\ f_0(e_4) = e_2,\ f_0(e_5) = -e_7,\ f_0(e_6) = -e_6,\ f_0(e_7) = -e_5,\ f_0(e_8) = -e_8$. Thus we obtain the following sequence :

$$1 \longrightarrow A_4^{'} \longrightarrow A_4 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

where $A_4^{'} = \ker\varphi$.

Let $f \in A_4^{'}$, then $f(V_1) \subset V_1$ and $f(V_2) \subset V_2$, consequently the matrix of f has the form:

$$M(f) = \begin{pmatrix} A & O_{2\times 2} & \\ O_{2\times 2} & B & C \\ & O_{4\times 4} & \end{pmatrix} \quad A, B \in GL_2(K)$$

this allows us to define a groups homomorphism

$$\psi : A_4^{'} \longrightarrow K^* \times K^*,\ \psi(f) = (\det A, \det B).$$

ψ is surjective, hence we obtain the exact sequence

$$1 \longrightarrow A_4^{''} \longrightarrow A_4^{'} \longrightarrow K^* \times K^* \longrightarrow 1$$

where $A_4^{''} = \ker\psi$. Let $f \in A_4^{''}$, then $\det A = \det B = 1$, so $A\ , B \in SL_2(K)$, thus we can define a groups homomorphism $\pi : A_4^{''} \longrightarrow SL_2(K) \times SL_2(K)$ by $\pi(f) = (A, B)$, π is surjective, determine its kernel $\ker\pi$. Let $f \in \ker\pi$, then $A = B = id_2$ and $\wedge^3 f(\omega) = \omega$ imply $\ker\pi \backsimeq K^{12}$, as wanted. □

Proposition 5. *The automorphisms group* $A_5 = \mathrm{Aut}\,(\omega_{8,5})$ *verify the following exact sequences :*

$$\begin{array}{lllll} 1 & \longrightarrow & A_5^{'} \longrightarrow A_5 \longrightarrow S_3 \longrightarrow 1 \\ 1 & \longrightarrow & A_5^{''} \longrightarrow A_5^{'} \longrightarrow K^* \longrightarrow 1 \\ 1 & \longrightarrow & A_5^{'''} \longrightarrow A_5^{''} \longrightarrow SL_2(K) \longrightarrow 1 \\ 1 & \longrightarrow & K^6 \longrightarrow A_5^{'''} \longrightarrow SL_2(K) \times SL_2(K) \longrightarrow 1 \end{array}$$

Proof. Choosing a suitable basis of E, we can write $\omega_{8,5} = e_1(e_3e_4 + e_5e_6) + e_2(e_3e_4 + e_7e_8)$. The invariant domain $\overline{R}_5(\omega_{8,5})$ is equal to

$$\{< e_1 > \cup < e_2 > \cup < e_1 + e_2 >\} - \{0\} = \{V_1 \cup V_2 \cup V_3\} - \{0\}.$$

If $f \in A_5$, $f(\overline{R}_5(\omega_{8,5})) \subset \overline{R}_5(\omega_{8,5})$ so that $f(V_i) \subset V_{\sigma(i)}$ with $\sigma \in S_3$, this allows us to define a groups homomorphism

$$\varphi : A_5 \longrightarrow S_3, \quad \varphi(f) = \sigma \quad \text{where} \quad \sigma = \begin{pmatrix} V_1 & V_2 & V_3 \\ V_{\sigma(1)} & V_{\sigma(2)} & V_{\sigma(3)} \end{pmatrix}.$$

It readily follows that φ is surjective, so that the first sequence with $A_5^{'} = \ker\varphi$ is exact. Let $f \in A_5^{'}$ then $f(V_1) \subset V_1$ and $f(V_2) \subset V_2$, $f(V_3) \subset V_3$, consequently $f(e_1) = \alpha e_1$, $f(e_2) = \beta e_2$, $f(e_1 + e_2) = \delta(e_1 + e_2)$, hence $\alpha = \beta$. This allows us to define a groups homomorphism $\psi : A_5^{'} \longrightarrow K^*$ by $\psi(f\ ) = \alpha$. We readily find that ψ is surjective, so the second sequence with $A_5^{''} = \ker\psi$ is exact. Let $f \in A_5^{''}$, then $\alpha = 1$ and from the equality $\Lambda^3 f(\omega_{8,5}) = \omega_{8,5}$, the matrix of f must have the form

$$M(f\ ) = \left(\begin{array}{cccc} 1 & 0 & A & \\ 0 & 1 & B & D \\ \multicolumn{2}{c}{O_{6\times 2}} & O_{2\times 4} & C \end{array} \right) \qquad \begin{array}{l} C \in GL_2(K) \\ B \in GL_4(K) \end{array}$$

Since $\det M(f\ ) = \det B \times \det C \neq 0$, $\det C \neq 0$. Moreover $\wedge^3 f(\omega) = \omega$ implies $\det C = 1$ so $C \in SL_2(K)$, hence we can define a groups homomorphism $\pi : A_5^{''} \longrightarrow SL_2(K)$ by $\pi(f) = C$, π is surjective then the third sequence is exact

$$1 \longrightarrow A_5^{'''} \longrightarrow A_5^{''} \longrightarrow SL_2(K) \longrightarrow 1$$

with $A_5^{'''} = \ker\pi$. Let $f \in \ker\pi$, we get $C = id_2$. But $\Lambda^3 f(\omega) = \omega$ shows that the matrix B has the form : $\begin{pmatrix} X & O_{2\times 2} \\ O_{2\times 2} & Y \end{pmatrix}$ with $\det X = \det Y = 1$, then $X, Y \in SL_2(K)$. This allows us to define a groups homomorphism $h : A_5^{'''} \longrightarrow SL_2(K) \times SL_2(K)$ by $h(f) = (X, Y)$, it readily follows that h is surjective and its kernel ker h is isomorphic to K^6. □

Proposition 6. *The automorphisms group* $A_6 = \mathrm{Aut}\,(\omega_{8,6})$ *verify the following exact sequences :*

$$\begin{array}{lcl} 1 & \longrightarrow & A'_6 \longrightarrow A_6 \longrightarrow K^* \times K^* \longrightarrow 1 \\ 1 & \longrightarrow & U(13) \longrightarrow A'_6 \longrightarrow SL_2(K) \longrightarrow 1 \end{array}$$

Proof. Choosing a suitable basis of E, we can write $\omega_{8,6} = e_1(e_8e_3 + e_6e_5 + e_4e_7) + e_2(e_6e_3 + e_5e_4)$. A direct computation leads to the relations

$$\overline{R}_5(\omega_{8,6}) = < e_1 > - \{0\}$$

and

$$\begin{array}{rcl} \overline{R}_{7,1}(\omega_{8,6}) & = & \{x \in E \ / \ \overline{\omega_{8,6}}\langle x \rangle) \ \text{ is of type } \ \omega_{7,1}[8,13]\} \\ & = & \{x \in E \ / \ x = \alpha_1 e_1 + \alpha_2 e_2 \ \text{ and } \ \alpha_2 \neq 0\}\,. \end{array}$$

Let $f \in A_6$, from previous invariant domains the matrix of f must have the form

$$\left(\begin{array}{ccc} \beta & \alpha_1 & \\ 0 & \alpha_2 & A \\ \multicolumn{2}{c}{O_{6\times 2}} & \end{array} \right),$$

thus we can define a groups homomorphism $\varphi : A_6 \longrightarrow K^* \times K^*$ by $\varphi(f) = (\beta, \alpha_2)$. Put $A'_6 = \ker \varphi$, it readily follows that φ is surjective and the first sequence is exact.

$$1 \longrightarrow A'_6 \longrightarrow A_6 \longrightarrow K^* \times K^* \longrightarrow 1$$

Let $f \in A'_6$, $f(e_1) = e_2 + \alpha_1 e_1$ and $\Lambda^3 f(\omega_{8,6}) = \omega_{8,6}$ implies $\det B = 1$ where B is a 2×2 submatrix of the matrix A such that $\det M(f) = \det B \times \det C$; $C \in GL_2(K)$. Thus we can construct a homomorphism of groups $\psi : A'_6 \longrightarrow SL_2(K)$ by $\psi(f\ ) = B$, ψ is surjective and its kernel is isomorphic to $U(13)$. Hence we obtain the exact sequence

$$1 \longrightarrow U(13) \longrightarrow A'_6 \longrightarrow SL_2(K) \longrightarrow 1.$$

This ends the proof of the proposition. □

3. K-forms of 2-step splitting trivectors of rank 8.

We are now ready to formulate the main results. The following theorem gives us a description of K-forms of 2-step splitting trivectors of rank 8.

Theorem 3. *Let E be a vector space of dimension 8 over a field K of characteristic other than 2 and 3. Then any K-form of 2-step splitting trivector of rank 8 in $\Lambda^3 E$ is equivalent to one of the trivectors of Table 2.*

Table 2

$\omega_{8,i}$	**Trivector**	$\mathbf{d}_1\,(\omega_{8,i})$
$\omega_{8,1}$	$e_1\,(e_2e_3 + e_4e_5) + e_6e_7e_8$	3
$\omega_{8,2}$	$e_1\,(e_2e_3 + e_4e_5 + e_6e_7) + e_5e_6e_8$	3
$\omega_{8,3}$	$e_1\,(e_3e_4 + e_5e_6) + e_2\,(e_3e_5 + e_7e_8)$	5
$\omega_{8,4}$	$e_1\,(e_2e_3 + e_4e_5) + e_6\,(e_2e_7 + e_4e_8)$	5
$\omega_{8,4,d}$	$e_5\,(e_1e_2 + e_3e_4) + e_6\,(e_1e_3 + de_2e_4) + e_7\,(e_1e_4) + e_8\,(e_2e_3)$	6
$\omega_{8,5}$	$e_1\,(e_2e_3 + e_4e_5) + e_6\,(e_2e_3 + e_7e_8)$	5
$\omega_{8,5,d}$	$e_2\,(e_5e_3 - de_4e_6) + e_1\,(e_5e_4 + e_7e_8 + e_6e_3)$	5
$\omega_{8,5,a}$	$e_1\,(ae_3e_4 + ae_5e_6 + e_7e_8) + e_2\,(e_3e_5 + e_4e_7 + e_6e_8)$	7
$\omega_{8,5,b,c}$	$e_7\,(e_1e_2 + e_3e_4 + e_5e_6) + e_8[e_1(e_4 + be_5) + e_2e_6 + ce_3e_5]$	7
$\omega_{8,6}$	$e_1\,(e_2e_3 + e_4e_5 + e_6e_7) + e_8\,(e_4e_3 + e_5e_6)$	5

Here, $d \in K_* - K_*^2$, $a \in K_* - K_*^3$, and $bc \neq 0$.

Denote by C_i the set of K-isomorphism classes of the K-forms of $\omega_{8,i}, 1 \leq i \leq 6$. In order to prove Theorem 3, the following lemmas are necessary.

Lemma 1. *The trivector $\omega_{8,4}$ has two K-forms*

- $\omega_{8,4}$
- $\omega_{8,4,d} = e_5\,(e_1e_2 + e_3e_4) + e_6\,(e_1e_3 + de_2e_4) + e_7\,(e_1e_4) + e_8\,(e_2e_3)$ *where* $d \in K_* - K_*^2$.

Proof. According to [14] $H^1(G,X) = 0$ whenever X is one of the groups $\overline{K}^m$, $\overline{K}_*^m$, $SL_m(\overline{K})$, by proposition 4 and using the exact sequences of Galois cohomology sets, we get $|C_4| = |H^1(G, \frac{\mathbb{Z}}{2\mathbb{Z}})|$. Since $H^1(G, \frac{\mathbb{Z}}{n\mathbb{Z}})$ classifies cyclic K-algebras of degree n, then $|C_4| = |H^1(G, \frac{\mathbb{Z}}{2\mathbb{Z}})| = |\frac{K_*}{K_*^2}|$. We deduce that if $L = K(\sqrt{d})$ is quadratic extension of K, $d \notin K_*^2$, there exists a trivector $\omega_L \in \wedge^3 E$ such that $\omega_L \not\simeq \omega_{8,4}$ and $\omega_L \otimes L \in \wedge^3(E \otimes_K L)$ is L-isomorphic to $\omega_{8,4}$.

We construct ω_L as follows. $\omega_{8,4} = e_1(e_2e_5 + e_3e_6) + e_4(e_7e_2 + e_8e_3)$ is 4-step splitting because

$$\omega_{8,4} = e_5u_1 + e_6u_2 + e_7u_3 + e_8u_4$$

where $u_1 = e_1e_2,\ u_2 = e_1e_3,\ u_3 = e_2e_4$ and $u_4 = e_3e_4$, thus $V = \text{vect}\,\{u_1, u_2, u_3, u_4\}$ is a subspace of dimension 4 of $\wedge^2 K^4$. As in [12] we define the quadratic form γ_2 on V and we put $\omega_L = \omega_{8,4,d} = e_5v_1 + e_6v_2 + e_7v_3 + e_8v_4$ with

$$v_1 = e_1e_2 + e_3e_4,\ v_2 = e_1e_3 + de_2e_4,\ v_3 = e_1e_4,\ v_4 = e_2e_3, \text{where } d \notin K_*^2.$$

To each of the forms $\omega_{8,4}$, $\omega_{8,4,d}$ we associate a quadratic form $\gamma_2(xu_1 + yu_2 + zu_3 + tu_4)$, then we get

$$\gamma_2(xu_1 + yu_2 + zu_3 + tu_4) = -yz,\ \gamma_2(xv_1 + yv_2 + zv_3 + tv_4) = x^2 - dy^2$$

respectively. The two forms are not equivalent over K but they may become equivalent over the algebraic closure $\overline{K}$.
We can also prove that $\omega_{8,4}$ is not equivalent to $\omega_{8,4,d}$ by using the arithmetical invariant $d_1(\omega)$. $\square$

For the K-forms of $\omega_{8,5}$ we have the following

Lemma 2. *$\omega_{8,5}$ has four K-forms.*

- $\omega_{8,5}$;
- $\omega_{8,5,d} = e_2(e_5e_3 - de_4e_6) + e_1(e_7e_8 + e_5e_4 + e_6e_3),\ d \in K - K_*^2$
- $\omega_{8,5,a} = e_1(ae_3e_4 + ae_5e_6 + e_7e_8) + e_2(e_3e_5 + e_4e_7 + e_6e_8)\ ,\ \ a \notin (F_{q*})^3$.
- $\omega_{8,5,b,c} = e_7(e_1e_2 + e_3e_4 + e_5e_6) + e_8(e_1[e_4 + be_5) + e_2e_6 + ce_3e_5)]$ *where* $bc \neq 0$.

Proof. For the K-forms of $\omega_{8,5}$, using the exact sequences of proposition 5 we deduce that $|C_5| = |H^1(G, S_3)|$, on the other hand we know that there exists a bijective correspondence from $H^1(G, S_3)$ to the isomorphism classes of cubic etale algebras. By according 2-1 we get four types of cubic algebras so $\omega_{8,5}$ has four K-forms.
Let L be a cubic extension of K and F a $L-$ space of dimension 3. Choose a basis $\{e_1, e_2, e_3\}$ of F and let the determinant form $\varphi : \wedge^3 F \longrightarrow L$ such that $\varphi(e_1, e_2, e_3) = 1$. and $Tr : L \longrightarrow K$ the trace forme hence $\omega_L = Tr_L \circ \varphi : F \times F \times F \longrightarrow K$: is an alternating trilinear form of rank 9 on $E = F$ viewed as vector space over K.

If $cara(L) \neq 3$ we take $L = K(t)$ with $t^3 = a$ in this case $\{e_1, e_2, e_3, e_4 = te_1, e_5 = te_2, e_6 = te_3, e_7 = t^2e_1, e_8 = t^2e_2, e_9 = t^2e_3\}$ is a basis of E. Let (e_i^*), $1 \leqslant i \leqslant 9$; its dual basis; Hence it readily follows that:
$\omega_L(e_1, e_2, e_3) = 3$, $\omega_L(e_4, e_5, e_6) = 3a$, $\omega_L(e_7, e_8, e_9) = 3a^2$, $\omega_L(e_1, e_5, e_9) = 3a$, $\omega_L(e_1, e_6, e_8) = -3a$, $\omega_L(e_2, e_4, e_9) = -3a$, $\omega_L(e_2, e_6, e_7) = 3a$, $\omega_L(e_9, e_4, e_8) = 3a$, $\omega_L(e_3, e_5, e_7) = -3a$ and $\omega_L(e_i, e_j, e_k) = 0$ otherwise. Hence

$$\begin{aligned}\tfrac{1}{3}\omega_L \ &= \ e_1^*e_2^*e_3^* + ae_4^*e_5^*e_6^* + a^2e_7^*e_8^*e_9^* + ae_1^*e_5^*e_9^* - ae_1^*e_6^*e_8^* \\ &\qquad + ae_2^*e_4^*e_9^* - ae_2^*e_6^*e_7^* + ae_3^*e_4^*e_8^* - ae_3^*e_5^*e_7^*\end{aligned}$$

and

$$\frac{1}{3}\overline{\omega}_{\langle e_9^* \rangle)} = e_3^*(e_1^*e_2^* + ae_4^*e_8^* + ae_7^*e_5^*) + ae_6^*(e_4^*e_5^* + e_1^*e_8^* + e_2^*e_7^*),$$

then the trivector $e_3(e_1e_2 + ae_4e_8 + ae_7e_5) + ae_6(e_4e_5 + e_1e_8 + e_2e_7)$ is equivalent to trivector $\omega_{8,5,a}$ (it suffices to consider linear map f defined by $f(e_1) = e_3$,$f(e_2) = ae_6$, $f(e_3) = e_4$, $f(e_4) = e_8$, $f(e_5) = e_5$,$f(e_6) = -e_7$, $f(e_7) = -e_1$, $f(e_8) = -e_2$).

If we take $L = K^{`} \times K$ with $K^{`} = K(\alpha)$ and $\alpha^2 = d$ $(d \notin K^{*^2})$, in this case, $\{e_1, e_2, e_3, e_4 = \alpha e_1, e_5 = \alpha e_2, e_6 = \alpha e_3, e_7, e_8, e_9\}$ is a basis of E. Let (e_i^*) , $1 \leqslant i \leqslant 9$; its dual basis then

$$\omega_L(e_1, e_2, e_3) = 3,\ \omega_L(e_7, e_8, e_9) = 3,\ \omega_L(e_1, e_5, e_6) = 3d,$$
$$\omega_L(e_2, e_4, e_6) = -3d,\ \omega_L(e_3, e_4, e_5) = 3d,\ \text{and}$$

$$\omega_L(e_i, e_j, e_k) = 0 \text{ otherwise .}$$

Then $\frac{1}{3}\omega_L = e_1^*e_2^*e_3^* + de_1^*e_5^*e_6^* + de_2^*e_4^*e_6^* + de_3^*e_4^*e_5^* + e_7^*e_8^*e_9^*$ and it follows that

$$\omega = e_1e_2e_3 + de_1e_5e_6 + de_2e_4e_6 + de_3e_4e_5 + e_7e_8e_9 = \omega_{6,1,d} + e_7e_8e_9$$

because $\omega_{6,1,d} = e_1(e_3e_4 + e_5e_6) + e_2(e_3e_6 - de_4e_5)$ ([7]). We put $x = e_9 - e_1$, then $e_9 = x + e_1$ and $\overline{\omega}_{\langle x\rangle} = \overline{e}_1(\overline{e}_7\overline{e}_8 + \overline{e}_3\overline{e}_4 + \overline{e}_5\overline{e}_6) + \overline{e}_2(\overline{e}_3\overline{e}_6 - d\overline{e}_4\overline{e}_5)$ is equivalent to $\omega_{8,5,d}$.

If $L = K \times K \times K = K^3$ then $F = L^3 = K^9$, by the same method we obtain $\omega_{8,5}$.

In order to find the last trivector $\omega_{8,5,b,c}$, take $L = K(\alpha)$ a general separable field extension of degree 3 over K where $\alpha^3 + b\alpha + c = 0$ with $b, c \in K/bc \neq 0$, Since the trivectors $\omega_{8,5}, \omega_{8,5,d}$, $\omega_{8,5,a}$ and $\omega_{8,5,b,c}$ are 2-step splitting, using the invariant $\gamma_3(xu_1 + yu_2)$ which is a cubic forme; $u_1, u_2 \in \wedge^2 K^6$ we get :

$$\gamma_3(xu_1 + yu_2) = xy(x + y),\ x\left(x^2 - dy^2\right),\ a^2x^3 + y^3 \text{ and } x^3 + bxy^2 + cy^3$$

for $\omega_{8,5}, \omega_{8,5,d}$, $\omega_{8,5,a}$ and $\omega_{8,5,b,c}$ respectively. Clearly, the three last forms may be equivalent to the first form if the field is algebraically closed. □

Proof of Theorem 3.

Proof. As $H^1(G, X) = 0$ whenever X is one of the groups $\overline{K}_*^m$, $SP_{2m}(\overline{K})$, $GL_m(\overline{K})$, $SL_m(\overline{K})$ [14], by propositions 1 , 2, 3 and 6, and using the exact sequences of Galois cohomology sets, we get

$$H^1(G, A_i) = 0 \text{ if } i \in \{1, 2, 3, 6\}\,.$$

Then $|C_i| = 1$, consequently $\omega_{8,i}$ is the only K-form of $\omega_{8,i}$ for $i \in \{1, 2, 3, 6\}$. For $\omega_{8,4}$ by Lemma 1 we obtain two K-forms, $\omega_{8,4}$ and

$$\omega_{8,4,d} = e_5\left(e_1e_2 + e_3e_4\right) + e_6\left(e_1e_3 + de_2e_4\right) + e_7\left(e_1e_4\right) + e_8\left(e_2e_3\right)$$

where $d \notin K_*^2$. For the K-forms of $\omega_{8,5}$, Lemma 2 gives us four K-forms $\omega_{8,5}, \omega_{8,5,d}, \omega_{8,5,a}$ and $\omega_{8,5,b,c}$ where $bc \neq 0$. Notice that if $b = 0$, $\omega_{8,5,b,c}$ must be equivalent to $\omega_{8,5,a}$ and for $c = 0$, $\omega_{8,5,b,c}$ is equivalent to $\omega_{8,5,d}$. This establishes Theorem 3.

□

Corollary 1. *There are trivectors of rank 8 over K that are not 2-step splitting but they may become 2-step splitting by extension of scalars.*

Proof. If the field K is not quadratically closed, it is enough to consider trivectors $\omega_{8,4,d}$ which are not 2-step splitting and becomes equivalent to $\omega_{8,4}$ over the algebraic closure $\overline{K}$. Thus $(\omega_{8,4,d})_{\overline{K}}$ is 2-step splitting.

□

In order to deduce other results, the following definitions are necessary.

An ordered field K is said to be *real closed field* if any algebraic extension of K which is ordered must be equal to K. For example $\mathbb{R}$ the field of real numbers is a real closed field.

A field which is complete with respect to a discrete valuation is called a *local field* if its field of residue classes is finite. For example $\mathbb{Q}_p$ the field of $p-$adic Numbers for some prime p is a local field.
As an immediate consequence of Theorem 3, we deduce nine F_q-forms of 2 step-splitting of rank eight over a finite field F_q, indeed, we have $|H^1(G, S_3)| = 3$ because the last etale algebra L_4 does not exist [1] and $|H^1(G, \frac{\mathbb{Z}}{2\mathbb{Z}})| = 2$. But over the real closed field we get eight K-forms of rank 8 of 2-step splitting trivectors, indeed, $|H^1(G, S_3)| = 2$ because every polynomial of odd degree has a root in K so we obtain only two etale algebras L_1 and L_2. $|H^1(G, \frac{\mathbb{Z}}{2\mathbb{Z}})| = 2$, hence we deduce the known result [3] about real forms of rank 8 of 2-step splitting trivectors.
If K is local field such that its field of residue classes has a characteristic other than 2 and 3, then $|C_4| = |\frac{K_*}{K_*^2}| = |\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}| = 4$ [8]. Hence we deduce the following

Corollary 2. *a) There are nine F_q-forms of 2 step-splitting trivectors of rank 8 over a finite field F_q: $\omega_{8,1}$, $\omega_{8,2}$, $\omega_{8,3}$, $\omega_{8,4}$, $\omega_{8,4,d}$, $\omega_{8,5}$, $\omega_{8,5,d}$, $\omega_{8,5,a}$, $\omega_{8,6}$ where $d \notin (F_{q*})^2$, $a \notin (F_{q*})^3$.*
b) If K is real closed field (in particular if $K = \mathbb{R}$), then the K-forms of 2-step splitting trivectors of rank 8 are $\omega_{8,1}$, $\omega_{8,2}$, $\omega_{8,3}$, $\omega_{8,4}$, $\omega_{8,4,-1}$, $\omega_{8,5}$, $\omega_{8,5,-1}$, $\omega_{8,6}$.
c) If K is local field such that its field of residue classes has a characteristic different from 2 and 3. Then there are 12 K-forms of 2 step-splitting trivectors of rank 8. $\omega_{8,1}$, $\omega_{8,2}$, $\omega_{8,3}$, $\omega_{8,4}$, $\omega_{8,4,\alpha^i\beta^j}$, $\omega_{8,5}$, $\omega_{8,5,d}$, $\omega_{8,5,a}$, $\omega_{8,5,b,c}$, $\omega_{8,6}$ where $d \notin K_*^2$, $a \notin K_*^3$, $bc \neq 0$, and $\alpha\beta^{-1} \notin K_*^2$, $i, j \in \{0, 1\}$.

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