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Topological Properties of the Numerical Range of Matrix Polynomials

Takuma Kimura

Graduate School of Hirosaki University
Hirosaki 036-8561, Japan

Hiroshi Nakazato

Department of Mathematical Sciences
Faculty of Science and Technology
Hirosaki University, Hirosaki 036-8561, Japan
nakahr@cc.hirosaki-u.ac.jp

Abstract

Some fundamental examples are provided, which are useful to study the topological properties of the numerical range of matrix polynomials.

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1. Numerical range of a matrix polynomial and its holes

We consider a matrix polynomial

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \ldots + A_1\lambda + A_0$$

where $A_0, A_1, \ldots, A_{m-1}, A_m$ are $n \times n$ complex matrices with $A_m \neq 0$ and λ is a complex variable. Its spectrum $\sigma(P)$ and numerical range $W(P)$ is defined as

$$\sigma(P) = \{\lambda \in \mathbf{C} : \det(P(\lambda)) = 0\},$$

$$W(P) = \{\lambda \in \mathbf{C} : \xi^* P(\lambda)\xi = 0 \,\text{for some}\, \xi \in \mathbf{C}^n, \xi \neq 0\}$$

$$= \{\lambda \in \mathbf{C} : 0\lambda \in W(P(\lambda))\}$$

where $W(B)$ is defined as

$$W(B) = \{\xi^* B\xi : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}$$

for an $n\times n$ complex matrix B. It is known that $\sigma(P)$, $W(P)$ are closed subsets of the Gaussian plane $\mathbf{C}$ and satisfy $\sigma(P) \subset W(P)$ (cf. If the leading coefficient matrix A_m satisfies $0 \notin W(A_m)$, then the closed set $W(P)$ is bounded and it has at most m connected components (cf. [3]).

In [7], it is shown that $W(P)$ is simply connected if $m = 1$ and $0 \notin W(A_1)$. The numerical range of matrix polynomials attract many authors attension (cf. [1],[2],[3],[4],[5],
[6],[7] and the references therein). In [5,6], it is shown that the C-numerical range $W_C^J(B)$ on an operator on the 2-dimensional numerical range is connected and its fundamental group is trivial or it is isomorphic to the abelian group $\mathbf{Z}$. Motivated by this result, we study the number $h(P)$ of bounded connected components of the open set $\mathbf{C}\backslash W(P)$ under the condition $0 \notin W(A_m)$. Intuitively $h(P)$ is the number of the holes of $W(P)$. If the range $W(P)$ is connected, then the fundamental group $\pi(W(P))$ of the range $W(P)$ is isomorphic to the free group $F_{h(P)}$ with $h(P)$ generators (cf. [8]). In this note we give some fundamental examples of this problem for $n = 2$ or $m = 2$ under the condition $A_m = I_m$.

Example 1. We give an example for $n = 2$.

$$P_2(\lambda) = \begin{pmatrix} \lambda(\lambda-2) & 0 \\ 0 & (\lambda-4)(\lambda-6) \end{pmatrix}.$$

Then the range $W(P_2)$ is given by

$$W(P_2) = \{\lambda \in \mathbf{C} : |\lambda - 3| = \sqrt{3}\} \cup [0, 2] \cup [4, 6].$$

In fact $\lambda \in W(P_2)$ if the line segment $[\lambda(\lambda-2), (\lambda-4)(\lambda-6)]$ contains 0. We set $\lambda = x + iy$. If this condition is satisfied, then the matrix

$$\begin{pmatrix} \Re(\lambda(\lambda-2)) & \Re((\lambda-4)(\lambda-6)) \\ \Im(\lambda(\lambda-2)) & \Im((\lambda-4)(\lambda-6)) \end{pmatrix} = \begin{pmatrix} x^2-y^2-2x & x^2-y^2-10x+24 \\ 2xy-2y & 2xy-10y \end{pmatrix}$$

is singular, that is $\Im(\lambda) = y = 0$ or $|\lambda - 3|^2 = 3$. We assume that $\Im(\lambda) \neq 0$ and $|\lambda - 3|^2 = 3$. Then

$$\Re(\lambda) = 3 + \sqrt{3}\cos\theta, \Im(\lambda) = \sqrt{3}\sin\theta$$

for some $\theta \in [0, 2\pi]$ and

$$-\frac{\Re(\lambda(\lambda-2)}{\Re((\lambda-4)(\lambda-6))} = \frac{2\sqrt{3}+3\cos\theta}{2\sqrt{3}-3\cos\theta} > 0$$

for $\theta \in [0, 2\pi]$, and hence $0 \in [\lambda(\lambda-2), (\lambda-4)(\lambda-6)]$. Next we consider the case $\Im(\lambda) = 0$, that is, λ is a real number. Then $0 \in [\lambda(\lambda-2), (\lambda-4)(\lambda-6)]$ if and only if $\lambda(\lambda-2)$ and $(\lambda-4)(\lambda-6)$ have different signs or one of these two numbers vanishes. Hence $\lambda \in [0, 2] \cup [4, 6]$ is the necessary and sufficient condition. The set $W(P_2) = \partial W(P_2)$ is connected, and $\pi_1(W(P_2))$ is isomorphic to the additive group of integers $\mathbf{Z}$.

By using Example 1, we construct a slightly complicated example.

Example 2. For a natural number $k = 1, 2, 3, \ldots$, let

$$P_{2k}(\lambda) = \begin{pmatrix} \lambda^k(\lambda^k - 2) & 0 \\ 0 & (\lambda^k - 4)(\lambda^k - 6) \end{pmatrix}.$$

Then we have

$$W(P_{2k}) = \{\lambda \in \mathbf{C} : 0 \in W(P_{2k})\} = \{\lambda \in \mathbf{C} : \lambda^k \in W(P_2)\}.$$

The set $W(P_{2k}) = \partial W(P_{2k})$ is connected and the fundamental group $\pi(W(P_{2k}))$ is isomorphic to F_k, that is, the number of holes of $W(P_{2k})$ is k.

2. Quadratic diagonal matrix polynomials

In this section we consider the numerical range of quadratic diagonal matrix polynomials. First we consider a 3×3 quadratic polynomial

$$Q_3(\lambda) = \begin{pmatrix} (\lambda - 2a)(\lambda - (2a+2)) & 0 & 0 \\ 0 & (\lambda - 2b)(\lambda - (2b+2)) & 0 \\ 0 & 0 & (\lambda - 2c)(\lambda - (2c+2)) \end{pmatrix},$$

where a, b, c are integers satisfying $a > b+1 > b > c+1 > c$. Then the range $W(Q_3)$ is given by

$$W(Q_3) = \{\lambda \in \mathbf{C} : |\lambda - (a+c+1)| \leq \sqrt{(a-c)^2 - 1}, |\lambda - (a+b+1)| \geq \sqrt{(a-b)^2 - 1},$$

$$|\lambda - (b+c+1)| \geq \sqrt{(b-c)^2 - 1}\} \cup [2a, 2a+2] \cup [2b, 2b+2] \cup [2c.2c+2]. \quad (2.2)$$

In fact, we assume that $\Im(\lambda) \neq 0$. Then $\lambda \in W(Q_3)$ if and only if the equation

$$t_1(\lambda-2a)(\lambda-2a-2)+t_2(\lambda-2b)(\lambda-2b-2)+t_3(\lambda-2c)(\lambda-2c-2) = 0, \quad (2.1)$$

holds for some real numbers t_1, t_2, t_3 with $t_1 + t_2 + t_3 = 1$, $t_j \geq 0$ $(j = 1, 2, 3)$. The solution (t_1, t_2, t_3) of (2.1) with $t_1 + t_2 + t_3 = 1$ is given by

$$t_2 = \frac{[(a-c)^2 - 1] - (\Re(\lambda) - (a+c+1))^2 - \Im(\lambda)^2}{4(a-b)(b-c)},$$

$$t_1 = \frac{(\Re(\lambda) - (b+c+1))^2 + \Im(\lambda)^2 - [(b-c)^2 - 1]}{4(a-b)(a-c)},$$
$$t_3 = \frac{(\Re(\lambda) - (a+b+1))^2 + \Im(\lambda)^2 - [(a-b)^2 - 1]}{4(a-c)(b-c)}.$$

Since $a > b > c$, the folowing inequalities are the condition for w to belong to $W(Q_3)$:

$$|\lambda - (a+c+1)| \leq \sqrt{(a-c)^2 - 1}, |\lambda - (b+c+1)|$$
$$\geq \sqrt{(b-c)^2 - 1}, |\lambda - (a+b+1)| \geq \sqrt{(a-b)^2 - 1},$$

where the inequalities

$$a+b+1+\sqrt{(a-b)^2-1} < a+c+1+\sqrt{(a-c)^2-1},$$
$$a+c+1-\sqrt{(a-c)^2-1} < b+c+1-\sqrt{(b-c)^2-1},$$
$$b+c+1+\sqrt{(b-c)^2-1} < a+b+1-\sqrt{(a-b)^2-1}$$

hold. If $\Im(\lambda) = 0$, that is, $\lambda \in \mathbf{R}$, then the condition $\lambda \in W(Q_3)$ is equivalent to

$$0 \in [(\lambda-2a)(\lambda-2a-2), (\lambda-2b)(\lambda-2b-2)] \cup [(\lambda-2b)(\lambda-2b-2), (\lambda-2c)(\lambda-2c-2)]$$
$$\cup [(\lambda-2a)(\lambda-2a-2), (\lambda-2c)(\lambda-2c-2)].$$

This condition is rewritten as

$$\lambda \in [2a, 2a+2] \cup [2b, 2b+2] \cup [2c, 2c+2].$$

Thus we obtain the equation (2.2).

Example 3. We consider the following 4×4 quadratic matrix polynomial

$$Q_4(\lambda) = \begin{pmatrix} (\lambda-6)(\lambda-8) & 0 & 0 & \\ 0 & (\lambda-2)(\lambda-4) & 0 & 0 \\ 0 & 0 & (\lambda+4)(\lambda+2) & 0 \\ 0 & 0 & 0 & (\lambda+8)(\lambda+6) \end{pmatrix}.$$

Then the range $W(Q_4)$ is given by

$$W(Q_4) = \{\lambda \in \mathbf{C} : |\lambda| \leq 4\sqrt{3}, |\lambda| \geq 2\sqrt{2}, |\lambda - 5| \geq \sqrt{3}, |\lambda + 5| \geq \sqrt{3}\}$$
$$\cup [-8, -6] \cup [-4, -2] \cup [2, 4] \cup [6, 8].$$

This relation follows from the equation

$$W(Q_4) = \{\lambda \in \mathbf{C} : 0 \in \mathrm{Conv}((\lambda-6)(\lambda-8), (\lambda-2)(\lambda-4), (\lambda+4)(\lambda+2), (\lambda+8)(\lambda+6))\}$$

$$
\begin{aligned}
&= \{\lambda \in \mathbf{C} : 0 \in \mathrm{Conv}((\lambda-6)(\lambda-8), (\lambda-2)(\lambda-4), (\lambda+8)(\lambda+6))\} \\
&\cup \{\lambda \in \mathbf{C} : 0 \in \mathrm{Conv}((\lambda-6)(\lambda-8), (\lambda+4)(\lambda+2), (\lambda+8)(\lambda+6))\} \\
&\cup \{\lambda \in \mathbf{C} : 0 \in \mathrm{Conv}((\lambda-6)(\lambda-8), (\lambda-2)(\lambda-4), (\lambda+4)(\lambda+2))\} \\
&\cup \{\lambda \in \mathbf{C} : 0 \in \mathrm{Con}((\lambda-2)(\lambda-4), (\lambda+4)(\lambda+2), (\lambda+8)(\lambda+6))\}
\end{aligned}
$$

and (2.2). The set $W(Q_4)$ is connected and the fundamental group $\pi(W(Q_4))$ is isomorphic to F_3, that is, the number of holes of $W(Q_4)$ is 3.

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