

Influence of the Semi Cover-Avoiding Property on the Structure of Finite Groups¹

Yaqiong Cui and Xiuyun Guo²

Department of Mathematics, Shanghai University
99 Shangda Road, Shanghai, 200444, P. R. China

Abstract

In this paper, some sufficient conditions for a finite group to be supersolvable or nilpotent are given by means of some kinds of generalized normality of cyclic subgroups of order prime and 2^2 , and several known results are generalized .

Mathematics Subject Classification: Primary 20D10, 20D20

Keywords: semi cover-avoiding property, semi-normal subgroups, supersolvable groups, nilpotent groups

1 Introduction

Throughout this paper, all groups considered are finite.

In 1988, X. Y. Su introduced the definition of the semi-normal subgroup and discussed the structure of the group whenever its some subgroups are semi-normal. Recently, Y. Fan and X. Y. Guo gave the definition of the semi cover-avoiding property in [2], which covered both the C-normality property and the cover-avoiding property and obtained some meaning results(see [2, 3, 4]). Based on the former results, in this paper, we will study the influence of some subgroups either having the semi cover-avoiding property or being semi-normal on the structure of groups.

¹The research was supported by the National Natural Science Foundation of China(10771132), SGRC(GZ310) and the research grant for graduate students of Shanghai University.

²Corresponding author. e-mail: xyguo@staff.shu.edu.cn

2 Basic Definitions and Preliminaries

Definition 1 [1] Let A be a subgroup of a group G . If there is a subgroup B of G such that $G = AB$ and $AB_1 < G$ for every proper subgroup B_1 of B , then A is called a semi-normal subgroup of G , and the set of all such B is denoted by $S_G(A)$.

Definition 2 [2] Let H be a subgroup of a group G and M/N be a normal factor of G . Then H is said to cover M/N if $HM = HN$. On the other hand, if $H \cap M = H \cap N$, then H is said to avoid M/N .

Definition 3 [2] Let H be a subgroup of a group G . H is said to have the semi cover-avoiding property in G if there is a chief series of G

$$1 = G_0 < G_1 < \cdots < G_l = G$$

such that H either covers or avoids G_i/G_{i-1} for every $i = 1, \dots, l$.

Definition 4 [5] Let G be a group, $S(G)$ be the set of supersolvable subgroups of G and

$$\mathfrak{A}(G) = \{N \mid N \trianglelefteq G \text{ and } NX \in S(G) \text{ whenever } X \in S(G)\}.$$

Set $U(G) = \langle N \mid N \in \mathfrak{A}(G) \rangle$. It is obvious that $\mathfrak{A}(G) \neq \phi$ and $U(G) \text{ char } G$.

Lemma 1 [1] Let H be a semi-normal subgroup of a group G , then:
 (1) H is a semi-normal subgroup of K if $H \leq K \leq G$,
 (2) HN/N is a semi-normal subgroup of G/N if N is a normal subgroup of G .

Lemma 2 [3] Let H be a subgroup of a group G having the semi cover-avoiding property in G , then:

- (1) H has the semi cover-avoiding property in K if $H \leq K \leq G$,
- (2) H/N has the semi cover-avoiding property in G/N if N is a normal subgroup of G and $N \leq H$,
- (3) HN/N has the semi cover-avoiding property in G/N if $(|H|, |N|) = 1$, where $(\ , \)$ denotes the greatest common divisor.

Recall that a group G is called an inner supersolvable group if G is not supersolvable but every proper subgroup of G is supersolvable.

If G is an inner supersolvable group, then:

- (1) G has the unique normal Sylow p -subgroup P , and there exists $M \leq G$ such that $G = PM, P \cap M = 1$, and M is supersolvable,
- (2) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$,
- (3) if $p > 2$, then $\exp P = p$, and if $p = 2$, then $\exp P \leq 4$,
- (4) there exists an element c in $P \setminus \Phi(P)$ such that $\langle c \rangle$ is not normal in G ,
- (5) $P/\Phi(P)$ is not cyclic.

Lemma 3 [6] Let G be an inner supersolvable group and P be a normal Sylow p -subgroup of G . If there is a normal subgroup N of G such that G/N is supersolvable, then $P \in \text{Sly}_p(N)$.

Lemma 4 Let G be an inner supersolvable group and P be a normal Sylow p -subgroup of G , then there exists an element c in $P \setminus \Phi(P)$ such that $\langle c \rangle$ does not have the semi cover-avoiding property in G .

Proof Assume that the result is false. Then, for every element x in $P \setminus \Phi(P)$, $\langle x \rangle$ has the semi cover-avoiding property in G , that is, there is a chief series of G

$$1 = G_0 < G_1 < \dots < G_l = G$$

such that either $\langle x \rangle G_i = \langle x \rangle G_{i-1}$ or $\langle x \rangle \cap G_i = \langle x \rangle \cap G_{i-1}$ for every $i = 1, 2, \dots, l$. Since x is in G , there must exist an integrate number k such that x is in G_{k+1} but not in G_k . It follows that $\langle x \rangle G_k = \langle x \rangle G_{k+1}$, and therefore G_{k+1}/G_k is a cyclic group of order p or 4 .

On the other hand, since $(P \cap G_k)\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, we will see that $(P \cap G_k)\Phi(P) = P$ or $\Phi(P)$. If $(P \cap G_k)\Phi(P) = P$, then $P \cap G_k = P$, in contradiction to that x is not in $P \cap G_k$, so $P \cap G_k = \Phi(P)$. By using the same arguments, we can have that $P \cap G_{k+1} = P$. Thus $P/\Phi(P)$ is a cyclic group of order p or 4 , the final contradiction.

Lemma 5 [1] If a p -subgroup P of a group G is semi-normal in G , then $PQ = QP$, for every q -subgroup Q of G with $q \neq p$.

Lemma 6 [5] Let H be a subgroup of a group G , then $U(G) \cap H \leq U(H)$.

Lemma 7 [5] Let G be a group, then G is supersolvable if and only if $G/U(G)$ is supersolvable.

Lemma 8 [1] If A is a semi-normal subgroup of a group G , then A^x is

also a semi-normal subgroup of G for all x in G and $S_G(A^x) = S_G(A)$. And if $B \in S_G(A)$, then $B^y \in S_G(A)$ for all y in G .

Lemma 9 Let H be a subgroup of a group G having the semi cover-avoiding property in G , then H^x has also the semi cover-avoiding property in G for all x in G .

Proof Since H has the semi cover-avoiding property in G , then there is a chief series of G

$$1 = G_0 < G_1 < \cdots < G_l = G$$

such that either $HG_i = HG_{i-1}$ or $H \cap G_i = H \cap G_{i-1}$ for every $i = 1, \dots, l$. We can see that $H^x G_i = (HG_i)^x = (HG_{i-1})^x = H^x G_{i-1}$ and $H^x \cap G_i = (H \cap G_i)^x = (H \cap G_{i-1})^x = H^x \cap G_{i-1}$ for all x in G . Hence the result holds.

Lemma 10 [7] Suppose that every maximal subgroup of a Sylow subgroup of a group G either is semi-normal or has the semi cover-avoiding property in G , then G is supersolvable.

3 Main Results

Theorem 1 Let N be a normal subgroup of a group G such that G/N is supersolvable. If every cyclic subgroup of N of order prime and 2^2 either is semi-normal or has the semi cover-avoiding property in G , then G is supersolvable.

Proof Assume that the theorem is false and let G be a counterexample of the smallest order. Then

(1) G is inner supersolvable:

Let H be a proper subgroup of G . Since $HN/N \cong H/H \cap N$, we see that $H/H \cap N$ is supersolvable. By Lemma 1 and Lemma 2, we can see that every cyclic subgroup of $H \cap N$ of order prime and 2^2 either is semi-normal or has the semi cover-avoiding property in H . Thus H satisfies the hypotheses of the theorem for its normal subgroup $H \cap N$, and the minimality of G implies that H is supersolvable. So G is inner supersolvable.

(2) Let P be the unique normal Sylow p -subgroup of G . By Lemma 3 and the property of the inner supersolvable group, we see that $P \leq N$ and therefore every subgroup of P of order prime and 2^2 (when $p = 2$) either is semi-normal or has the semi cover-avoiding property in G .

(3) The final contradiction:

By (2) and Lemma 4, there exists an element x in $P \setminus \Phi(P)$ such that $\langle x \rangle$ is semi-normal in G , and therefore there exists $B \in S_G(\langle x \rangle)$ such that $G = \langle x \rangle B$

and $\langle x \rangle B_1 < G$ for every proper subgroup B_1 of B . Set $B = \langle B_p, B_{p_1}, \dots, B_{p_r} \rangle$, wherever $B_p \in \text{Syl}_p(B)$, $B_{p_i} \in \text{Syl}_{p_i}(B)$, for every $i = 1, 2, \dots, r$. By applying Lemma 5, we see $\langle x \rangle B_{p_i} \leq G$, and therefore $\langle x \rangle = \langle x \rangle B_{p_i} \cap P$ is normal in $\langle x \rangle B_{p_i}$, thus $B_{p_i} \leq N_G(\langle x \rangle)$. Noticing that P is the unique normal Sylow p -subgroup of G and $P/\Phi(P)$ is abelian, we know $\langle x \rangle \Phi(P)/\Phi(P)$ is normal in $\langle x \rangle B_p \Phi(P)/\Phi(P)$. It follows that $\langle x \rangle \Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. Therefore $\langle x \rangle \Phi(P) = P$, in contradiction to that $P/\Phi(P)$ is not cyclic.

The proof of the theorem is completed.

Theorem 2 Let N be a normal subgroup of a group G such that G/N is supersolvable. If every subgroup of N of order prime is contained in $U(G)$, and every cyclic subgroup of N of order 2^2 either is semi-normal or has the semi cover-avoiding property in G , then G is supersolvable.

Proof Suppose that the theorem is false and let G be a counterexample of the smallest order. Then

(1) G is inner supersolvable:

Let K be a proper subgroup of G , it follows from $KN/N \cong K/K \cap N$ that $K/K \cap N$ is supersolvable. By the hypotheses and Lemma 6, we see that every subgroup of $K \cap N$ of order prime is contained in $U(K)$. By using Lemma 1 and Lemma 2 once again, we get that every cyclic subgroup of $K \cap N$ of order 2^2 either is semi-normal or has the semi cover-avoiding property in K . So K satisfies the hypotheses of the theorem for its normal subgroup $K \cap N$. It follows that K is supersolvable by the minimality of G , and therefore G is inner supersolvable. So we can let $G = PM$ and $P \cap M = 1$, where P is the unique normal Sylow p -subgroup of G and M is supersolvable.

(2) By $P \leq N$ and Lemma 3, we see that every subgroup of P of order prime is contained in $U(G)$, and every cyclic subgroup of P of order 2^2 (when $p = 2$) either is semi-normal or has the semi cover-avoiding property in G .

(3) $p = 2$ and P is not cyclic:

Otherwise, assume $p > 2$. By the property of the inner supersolvable group G , we get $\exp P = p$. Since G/P is supersolvable, we can conclude that $P \leq U(G)$ by (2). So it follows from $G/U(G) \cong (G/P)/(U(G)/P)$ that $G/U(G)$ is supersolvable. By Lemma 7, we have that G is supersolvable, a contradiction. So $p = 2$. Then by the supersolvability of G/P , we know P is not cyclic.

(4) For all x in $P \setminus \Phi(P)$, $\circ(x) = 4$:

Otherwise, then there exists an element y in $P \setminus \Phi(P)$ such that $\circ(y) = 2$. Let $L = \langle y \rangle^G$, then L is normal in G and $L \leq P$, and therefore $L\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. The minimality of $P/\Phi(P)$ implies that $L\Phi(P)/\Phi(P) = P/\Phi(P)$ and so $P = L\Phi(P) = L$. It follows that $P = L \leq U(G)$ and $G/U(G)$ is supersolvable. By using Lemma 7 once again, we can conclude that G is supersolvable, a contradiction.

(5) The final contradiction:

By the hypotheses of the theorem and Lemma 4, $\langle x \rangle$ is semi-normal in G for all x in $P \setminus \Phi(P)$, so there is $B \in S_G(\langle x \rangle)$ such that $G = \langle x \rangle B$. Since G is solvable, there exists an element a in G such that $M^a \leq B$. By the Lemma 8, we can assume $M \leq B$. Noticing that $\langle x \rangle < P$, we see that $\langle x \rangle M < G$ and $\langle x \rangle = \langle x \rangle (M \cap P)$ is normal in $\langle x \rangle M$. Since $P/\Phi(P)$ is abelian, we have that $\langle x \rangle \Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. The minimality of $P/\Phi(G)$ implies $\langle x \rangle \Phi(P)/\Phi(P) = P/\Phi(P)$, namely $P = \langle x \rangle$, in contradiction to that $P/\Phi(P)$ is not cyclic.

The proof of the theorem is completed.

Theorem 3 If there exists a Sylow p -subgroup P of a group G for any prime p dividing the order of G such that

- (i) $N_G(P)/C_G(P)$ is a p -group,
 - (ii) every maximal subgroup of P either is semi-normal or has the semi cover-avoiding property in G ,
- then G is nilpotent.

Proof Assume the theorem is false and let G be a counterexample of the smallest order. Then

(1) Let P^* be a Sylow p -subgroup of G , and we have that P^* satisfies (i) and (ii):

By the Sylow Theorem, there exists an element x in G such that $P^* = P^x$. Since $N_G(P^*) = N_G(P^x) = (N_G(P))^x$ and $C_G(P^*) = C_G(P^x) = (C_G(P))^x$, it follows from $N_G(P^*)/C_G(P^*) \cong N_G(P)/C_G(P)$ that $N_G(P^*)/C_G(P^*)$ is a p -group. And by Lemma 8 and Lemma 9, every maximal subgroup of P^* either is semi-normal or has the semi cover-avoiding property in G . So P^* satisfies (i) and (ii).

(2) By Lemma 10, we can get that G is supersolvable. Let q be the maximal prime dividing the order of G and Q be a Sylow q -subgroup of G , then Q is normal in G .

(3) Let N be a minimal normal subgroup of G , and we see that G/N is nilpotent:

Considering the quotient group $\bar{G} = G/N$, let $\bar{P} = PN/N$ be a Sylow p -subgroup of \bar{G} .

Since $N_{\bar{G}}(\bar{P}) = N_G(P)N/N$ and $C_{\bar{G}}(\bar{P}) \geq C_G(P)N/N$, we see that $N_{\bar{G}}(\bar{P})/C_{\bar{G}}(\bar{P})$ is a p -group.

Assume that N is an r -subgroup of G and P_0/N is a maximal subgroup of PN/N . If $r = p$, then $N \leq P$ and P_0 is a maximal subgroup of P either being the semi-normal or having the semi cover-avoiding property in G . Therefore, by Lemma 1 and Lemma 2, we have that P_0/N either is semi-normal or has the semi cover-avoiding property in G/N . If $r \neq p$, then $P_0 = P_0 \cap PN = (P_0 \cap P)N$. We can conclude that $P_0 \cap P$ is a maximal subgroup of P by

comparing the orders, so $P_0 \cap P$ either is semi-normal or has the semi cover-avoiding property in G . By using Lemma 1 and Lemma 2 once again, we can see that $P_0/N = (P_0 \cap P)N/N$ either is semi-normal or has the semi cover-avoiding property in G/N .

From the above statements, we know G/N satisfies the hypotheses of the theorem, and therefore G/N is nilpotent by the minimality of G .

(4) N is the unique minimal normal subgroup of G and $N = Q$:

Let H be another minimal normal subgroup of G , and we see that G/H is nilpotent by (3). It follows from $G/H \cap N \lesssim G/H \times G/N$ that G is nilpotent, a contradiction. Thus N is the unique minimal normal subgroup of G .

Moreover, if $\Phi(G) \neq 1$, then by the minimality of N , we get $N \leq \Phi(G) \neq 1$. It concludes that $G/\Phi(G) \neq 1$ is nilpotent by (3), and therefore G is nilpotent, a contradiction. Thus, by Lemma 14, we see $\Phi(G) = 1$ and $F(G) = N$. Since $Q \leq F(G) = N$ and N is the unique minimal normal subgroup of G , we get $N = Q$.

(5) The final contradiction:

We know that $Q = N$ is a cyclic subgroup of order prime by (2) and (4), so $Q \leq C_G(Q)$. By the hypotheses, $N_G(Q)/C_G(Q)$ is a q -group, then $C_G(Q) = N_G(Q) = G$, that is $Q \leq Z(G)$. It follows from $G/Z(G) \cong (G/Q)/(Z(G)/Q)$ that $G/Z(G)$ is nilpotent, and therefore G is nilpotent, the final contradiction.

Thus the counterexample of the smallest order does not exist, and therefore G is nilpotent.

Reference

- [1] X. Y. Su. The Semi-Normal Subgroups of Finite Groups[J]. J. Math(Wuhan), 1988, 8(1): 5-9.
- [2] Y. Fan, X. Y. Guo, K. P. Shum. Remarks on Two Generalizations of Normality of Subgroups[J]. Chinese Ann. Math, 2006, 27A(2): 169-176.
- [3] X. Y. Guo, P. F. Guo, K. P. Shum. On Semi Cover-Avoiding Subgroups of Finite Groups[J]. Pure Appl. Algebra, 209(2007): 151-158.
- [4] X. Y. Guo, J. X. Wang, K. P. Shum. On Semi Cover-Avoiding Maximal Subgroups and Solvability of Finite Groups[J]. Communications in Algebra, 34(2006): 3235-3244.
- [5] F. H. Zeng, S. R. Li. Influence of Semi-Normal Subgroups and C-Normal Subgroups on the Supersolvability of Groups[J]. Guangxi Science 2003, 10(3): 161-164, 168.

- [6] P. C. Wang, F. T. Wen, W. X. Li. C-Normal Subgroups of a Finite Group[J]. Journal of Qufu Normal University(Natural Science), 1997, 23(4): 5-7.
- [7] X. Y. Guo, Y. Q. Cui. On the Semi Cover-avoiding Property and the Structure of Finite Groups[J]. Journal of Shanghai University(Natural Science)(to appear).

Received: November 20, 2007