

# Faithfully Flat Extensions of a Commutative Ring

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## Abstract

We show that a generalized monoid-ring over a commutative ring is faithfully flat over the base ring. We also find that under suitable conditions, certain ring properties such as coherence, Booleanness, finite conductor property and elementary divisor property are preserved on ascent.

## Introduction

Let  $A$  be a commutative ring with identity – simply a ring in the sequel – and  $S$  an additive monoid. The set of all finitely non-zero maps from  $S$  to  $A$  with point-wise addition and term by term (convolution) multiplication is the **monoid-ring (of  $S$  over  $A$ )**  $A[S]$ . The multiplication is well-defined because for each  $f \in A[S]$ , the **support** of  $f$ ,  $\text{Supp}(f) = \{s \in S : f(s) \neq 0\}$ , is a finite subset of  $S$  [7].

The classical work of Hahn [2] was the first to remove the finiteness of support, followed by that of Higman [3]. Recently Ribenboim systemized the study of these rings in a series of articles ([10] to [15] and the monograph [16]). He called them generalized power series rings. In this article, we shall call them generalized monoid-rings.

The main result of this article is that a generalized monoid-ring over a commutative ring is faithfully flat over the base ring (Theorem.2.3). We also study the stability of some ring-theoretic properties under the formation of generalized monoid-rings.

## 1. Preliminaries

A subset  $E$  of an **ordered monoid**  $(S, \leq)$  is **artinian (noetherian)**, if it satisfies the descending chain condition (DCC) (ascending chain condition (ACC)) with respect to the given order. It is **narrow** if it contains at most a finite number of mutually non-comparable elements. It is called **narrow-artinian (NA)** if it is narrow and artinian [Kru]. Since in general we shall consider only one order on a monoid; we shall drop the notation for order and write the ordered monoid simply as  $S$ .

Given  $s, t \in S$ ,  $t$  is a **difference** of  $s$ , if there is an element  $u$  of  $S$  such that  $t + u = s$ . Denote  $\mathbf{Diff}(s-t) = \{u \in S: u + t = s\}$  and  $\mathbf{Diff}(s) = \{t \in S: \text{There exists } u \in S: u + t = s\}$ .  $S$  **satisfies condition (D) if**

**(D):**  $\mathbf{Diff}(s-t) = \{u \in S: u + t = s\}$  is noetherian for  $s, t \in S$ .

A strictly ordered or cancellative monoid satisfies condition (D).

A subset  $E$  of  $S$  is **difference–narrow-artinian (DNA)**, if  $E \cap \mathbf{Diff}(s)$  is NA for every  $s \in S$ . An NA subset is also DNA.

*Hereafter all monoids in this article satisfy condition (D).*

Let  $A[[S]] = \{f: S \rightarrow A: \text{Supp}(f) \text{ is NA}\}$ . We define addition on  $A[[S]]$  coordinate wise and multiplication  $f * g$  by  $(f * g)(s) = \sum \{f(u)g(v): u + v = s\}$  for every  $s \in S$ . It is routine to check that with these operations  $A[[S]]$  is a commutative ring with identity. That the multiplication is well-defined follows from

**Proposition 1.1 (Ribenoim [16])** **Let  $S$  be an ordered monoid satisfying (D),  $s \in S$  and  $f_1, f_2, f_3, \dots, f_k \in A[[S]]$ . Let  $X_s = X_s(f_1, f_2, \dots, f_k) = \{(s_1, s_2, \dots, s_k) \in S^k : s_i \in \text{supp}(f_i) \text{ for } i = 1, \dots, k \text{ and } s_1 + s_2 + s_3 + \dots + s_k = s\}$ . Then the set  $X_s$  is finite.**

Similarly we define  $A\{[S]\} = \{f: S \rightarrow A: \text{Supp}(f) \text{ is DNA}\}$ . This is also a commutative ring with identity.

**Note:** The proof of the proposition indicates that the order of  $S$  – via condition (D) - is crucial to define multiplication.

In general, we shall consider only one monoid at a time, so we shall drop  $S$  from the notation for the monoid-ring under consideration and write  $[[A]]$ , etc;. Then we have  $A \subseteq [A] \subseteq [[A]] \subseteq \{[A]\}$ . We will hereafter state and prove results for the extension  $[[A]]$  only, it being understood that the results follow for all the three extensions, unless otherwise stated.

Given an  $A$ -module  $M$  we construct analogously an  $[[A]]$  module  $[[M]] = \{f: S \rightarrow M: \text{Supp}(f) \text{ is NA}\}$ .  $[[M]]$  is an  $[[A]]$  module with obvious addition and scalar multiplication by elements of  $[[A]]$ . A modified form of the proposition above assures that  $[[M]]$  is closed for this scalar multiplication.

For a given morphism  $\varphi: M \rightarrow N$ , we define  $[[\varphi]]: [[M]] \rightarrow [[N]]$  by  $[[\varphi]](f) = \varphi \cdot f$ . That this is indeed an  $[[A]]$ -morphism,  $[[\ ]]$  is a functor, and that this functor is additive, faithful, preserves exact sequences both ways, kernel, cokernel preserving both ways; hence preserves monic and epic both ways - All these - can be verified easily.

It also preserves arbitrary coproducts. Thus  $[[\bigoplus M_i]] \cong \bigoplus [[M_i]]$ . Hence

**Proposition 1.2**  $[[\ ]]$  preserves direct limits.

It also preserves finite generation (respectively presentation) of modules both ways. All the above facts will be used in the sequel without much ado.

## 2. Tensor Product

The bilinear correspondence  $\Phi: [[A]] \times M \rightarrow [[M]]$  given by  $\Phi(\mathbf{f}, \mathbf{m}) = \mathbf{f} \cdot \mathbf{m}$  induces a unique  $A$ -linear morphism  $\Theta_M: [[A]] \otimes M \rightarrow [[M]]$  such that  $\Theta_M(\mathbf{f} \otimes \mathbf{m}) = \mathbf{f} \cdot \mathbf{m}$ . Explicitly  $\Theta_M(\sum \mathbf{f}_i \otimes \mathbf{m}_i) = \sum \mathbf{f}_i \cdot \mathbf{m}_i$ .

**Proposition 2.1** The map  $\Theta_M: [[A]] \otimes M \rightarrow [[M]]$  is a natural transformation.

**Proof.** The following diagram, where  $\varphi: M \rightarrow N$  is a morphism, is commutative.

$$\begin{array}{ccc} [[A]] \otimes M & \rightarrow & [[M]] \\ \text{Id}_{[[A]]} \otimes \varphi \downarrow & & \downarrow [[\varphi]] \\ [[A]] \otimes N & \rightarrow & [[N]] \end{array}$$

**Theorem 2.2** Let  $A$  be ring,  $S$  monoid and  $M$  an  $A$ -module. Then  $\Theta_M: [[A]] \otimes M \cong [[M]]$ .

**Proof.** Clearly  $\Theta_M$  is surjective. Since  $[[\bigoplus A]] \cong \bigoplus [[A]]$ , the result is true when  $M$  is free. The general case follows by considering a free presentation  $F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$  of  $M$ .

$$\begin{array}{ccccc} [[A]] \otimes F_1 & \rightarrow & [[A]] \otimes F_0 & \rightarrow & [[A]] \otimes M \rightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ [[F_1]] & \rightarrow & [[F_0]] & \rightarrow & [[M]] \rightarrow 0. \end{array}$$

The 5-lemma shows that  $\Theta_M$  is an isomorphism.

**Proposition 2.2'** The functors  $[[A]] \otimes ( )$  and  $[[\ ]]$  are naturally isomorphic.

An  $A$ -module  $M$  is **flat** if the functor  $\otimes M$  carries an exact sequence into an exact sequence. It is **faithfully flat** if  $\otimes M$  preserves exact sequences both ways.

**Theorem 2.3** Let  $A$  be a commutative ring and  $S$  an ordered monoid, then the ring  $[[A]]$  is a faithfully flat extensions of  $A$ .

**Proof.** By 2.2 we have a natural isomorphism  $[[A]] \otimes (\ ) \cong [[ \ ]]$  of functors. And  $[[ \ ]]$  preserves exact sequence both ways.

**Remark:** A faithfully flat extension preserves descent [17]. Hence we study properties of ascent.

### 3. Boolean rings and coherent rings

A ring  $A$  is **boolean** if  $x^2 = x$  for  $x \in A$ . In a boolean ring  $2x = 0$  for every  $x \in A$ . A monoid  $S$  is **boolean** if  $s + s = s$  for all  $s$ .

**Theorem 3.1** Let  $A$  be a ring and  $S$  a monoid. Then  $[[A]]$  is boolean iff both the ring  $A$  and the monoid  $S$  are boolean.

**Proof.** Suppose  $A$  and  $S$  are boolean and  $f \in [[A]]$ . By definition,  $f^2(s) = \{\sum f(u)f(v) : u + v = s \text{ in } S\}$ . When  $u \neq v$  the term  $f(u)f(v)$  occurs twice in the sum and hence equal to zero. When  $u = v$  the surviving term is  $(f(u))^2 = f(u)$ , and  $s = u + u = u$ . Hence  $f(u) = f(s)$ , proving  $f^2(s) = f(s)$  and that  $[[A]]$  is boolean. Conversely, suppose that  $[[A]]$  is boolean. Then  $A$  is boolean. Let  $s \in S$ , consider the corresponding element  $s$  in  $[[A]]$ ;  $s^2 = s$  implies  $s + s = s$  in  $S$  and  $S$  is boolean.

**A ring  $A$  is coherent, if every finitely generated  $A$ -module is finitely presented or equivalently a product of copies of  $A$  is a-flat.**

Let us consider a projective system of monoids  $\{S_i ; f_{ij} : S_i \rightarrow S_j : j \leq i\}$  with the usual conditions on the family of maps  $\{f_{ij}\}$ . Then one can construct the projective limit of such a system in the usual way having the desired universal mapping properties.

A limit of a projective system of finite monoids is called **profinite**.

Let  $S$  and  $T$  be a pair of monoids with a monoid morphism  $\Phi : T \rightarrow S$  and let  $A[S]$  and  $A[T]$  denote the respective monoid-rings over a ring  $A$ . Then  $\Phi$  induces a ring homomorphism  $\underline{\Phi} : A[S] \rightarrow A[T]$  defined by  $(\underline{\Phi}(f))(t) = f(\Phi(t))$  for  $t \in T$  and  $f \in A[S]$ . Hence a projective system of monoids  $\{f_{ij} : S_i \rightarrow S_j : j \leq i : i, j \in I\}$  with limit  $S$  and  $A$  a ring induces an inductive system of rings  $([f_{ij}] : A[S_i] \rightarrow A[S_j] : i \leq j \in I)$ , whose limit is  $A[S]$ . Now we have:

**Theorem 3.2** Let  $A$  be coherent and  $S$  a profinite monoid. Then  $[A]$  is coherent.

Proof. Case i: let  $S$  be a finite monoid. Since  $A$  is coherent,  $\Pi A_i$  is  $A$ -flat. Then  $[A] \otimes (\Pi A_i) \cong [\Pi A_i]$  is  $[A]$ -flat. However  $[A]$  is  $A$ -finitely generated, hence finitely presented, so  $[A] \otimes$  commutes with direct products. It follows that  $[\Pi A_i] \cong [A] \otimes (\Pi A_i) \cong \Pi([A] \otimes A_i) \cong \Pi[A_i]$ . Thus  $\Pi[A_s]$  is  $[A]$  flat and that  $[A]$  is coherent.

Case ii:  $S$  is profinite, In the inductive system of rings induced by the projective system of finite monoids and monoid morphisms, each ring is coherent and each ring morphism is faithfully flat, hence the limit ring is coherent ([6] proposition 4. page 47, corollary on page 48 and exercise 12 (e), page 63).

**Remark** It is not known if the above theorem holds for other extensions  $[[A]]$  or  $\{\{A\}\}$

Let  $S$  be a monoid and  $x, y, s \in S$ . Define the congruence relation  $x C_s y$  on  $S$  if  $x+s = y+s$ . Let  $I_s = \{t \in S: C_s \Rightarrow C_t\}$ ; Then  $I_s$  is an ideal of the monoid  $S$ . If  $C_s$  implies  $C_t$ , then  $I_t \subseteq I_s$ . Hence for each  $s, t \in S$ ,  $C_s = C_t$  iff  $I_s = I_t$ , and in this case we write  $s \equiv t$ ; and  $\equiv$  is a congruence on  $S$ . An element  $s \in S$  is cancellative iff  $s \equiv 0$ . Let  $S/s$  be the quotient monoid defined by the congruence  $C_s$ . Then  $S/s \cong S/t$  iff  $s \equiv t$  and in particular  $S/s \cong S$  iff  $s \equiv 0$ .

Consider the set  $J = \{\underline{s} \in S/\equiv: \underline{s} \neq 0\}$  of equivalence classes, and order  $J$  by  $\underline{s} \leq \underline{t}$  iff  $I_t \subseteq I_s$ . Then  $\{J, \leq\}$  is a (left) directed set. Indeed given  $\underline{s}, \underline{t}$  in  $S/\equiv$ ,  $\underline{s+t} \leq \underline{s}, \underline{t}$  in  $S/\equiv$ . There is a monoid morphism  $f_{st}: S/t \rightarrow S/s$ , for  $\underline{s} \leq \underline{t}$  in  $J$ . And we have a projective system of monoids with the usual properties. Let  $\underline{S}$  denote the limit of the projective system  $\{f_{st}: S/t \rightarrow S/s: \underline{s} \leq \underline{t} \text{ in } J\}$ ; there is a monoid morphism  $\Phi: S \rightarrow \underline{S}$ . Now we have a partial converse of theorem 3.2.

**Theorem 3.3** Let  $A$  be a ring and  $S$  a monoid. Suppose the monoid-ring  $[A]$  of  $S$  over  $A$  has the following property:

(#) for any element  $f$  in  $[A]$  with support of cardinality  $\leq 1$ , the ideal-quotient  $(0: f)$  is finitely generated.

Then  $S$  is isomorphic to a submonoid of a profinite monoid.

**Proof.** Let  $\mathbf{s}$  be in  $[A]$  corresponding to  $s$  in  $S$ . Then  $f \in (0:\mathbf{s})$  iff the support of  $f$  has cardinality at least 2. In the case of cardinality 2, say  $f = \mathbf{x} + \mathbf{y}$ ,  $(\mathbf{x} + \mathbf{y})\mathbf{s} = 0$  in  $[A]$  iff  $x+s=y+s$  in  $S$ . So, the finite generation of ideal  $(0:\mathbf{s})$  implies that the congruence  $C_s$  as defined above is finitary and the quotient monoid  $S/s$  is finite  $\forall s \in S$ . Suppose  $x + s = y + s$  for some  $x, y$  and  $\forall s \in S$ . Then  $(\mathbf{x} + \mathbf{y})\mathbf{s} = 0$  in  $[A]$   $\forall s \in S$ , that is,  $\mathbf{s} \in (0:\mathbf{x}+\mathbf{y})$  in  $[A]$  for all  $s \in S$ . But then the ideal  $(0: \mathbf{x} + \mathbf{y})$  contains  $[A]$ , hence  $x=y$  in  $S$ . Thus given  $x \neq y$  in  $S$ , there is an  $s \in S$  such that

$x+s \neq y+s$ . The corresponding map from  $S$  to  $S/s$  carries  $x$  and  $y$  to distinct elements. Hence the map  $\Phi: S \rightarrow \underline{S}$  is a monoid monomorphism. Thus  $S$  is isomorphic to a submonoid of a profinite monoid  $\underline{S}$ . We deduce

**Theorem 3.4** Let  $A$  be a commutative ring and  $S$  a monoid such that  $[A]$  be coherent. Then  $S$  is isomorphic to a submonoid of a profinite monoid.

Remark:  $\underline{\mathbb{Z}}[X]$  shows that  $S$  in theorem 3.4 need not be isomorphic to a profinite monoid.

**Remark:** Similar results can be found in [8] in case of groups. (Chap. 8)

A ring  $A$  is called a **finite conductor ring**, if the ideal quotient  $(a:b)$  is finitely generated for  $\forall a, b \in A$  [9]. A coherent ring is a finite conductor ring.

Let  $S$  be a monoid and  $s, t \in S$  denote by  $(s:t) = \{u \mid u \in S: \text{there exists } v \in S \text{ such that } u+t = v+s\}$  Clearly it is an ideal in  $S$ , called the **conductor** (of  $t$  into  $s$ ). A monoid  $S$  is called a **finite conductor monoid**, if the conductor is finitely generated for  $s, t \in S$ .

**Theorem 3.5** The following conditions are equivalent:

- i.  $A$  is a finite conductor ring and  $S$  is a finite conductor monoid
- ii.  $[A]$  is a finite conductor ring.

**Proof.** Let  $F = \sum e(t_i, l_i)$  and  $G = \sum e(s_j, m_j) \in [A]$ , where  $e(t_i, l_i) \in [A]$  with support  $\{t_i\}$  and value  $l_i$ . Consider the finite family of related conductor ideals  $(l_i : m_j)$  and  $(t_i : s_j)$  in  $A$  and  $S$  respectively. By i., these ideals are finitely generated. Any conductor of  $F$  into  $G$  will have its "components" from these solutions. Hence  $(F:G)$  is also finitely generated.

Converse follows via the canonical embedding of  $S$  and  $A$  into  $[A]$

**A ring  $A$  satisfies (\*) if every finitely presented module is a direct sum of cyclic modules.**

**Theorem 3.6** Let  $A$  satisfies (\*) and  $S$  be a monoid and let  $X$  be one of the rings  $[A]$ ,  $[[A]]$  or  $\{\{A\}\}$ , then  $X$  satisfies (\*).

**Proof.** We consider a finite presentation of a  $X$ - module  $M$ . Let  $\{x_i\}$  be a finite family of generators of  $M$  and  $\{y_j\}$  be a finite family of generators of the kernel  $K$  of the given finite presentation. Let  $M^* = \sum Ax_i$  and  $K^* = \sum Ay_j$  be finite  $A$ - submodule of  $M$  and  $K$  respectively.  $K^*$  is evidently a kernel of a finite presentation of  $M^*$  induced by the finite presentation of  $M$ . This presentation splits and  $M^*$  is a direct sum of  $A$ -cyclic modules and that the functor  $[ \ ]$ ,  $([ [ \ ] ])$ ,

$\{\{ [ \ ] \}\}$  respectively) is faithfully flat implies that  $M$  is a direct sum of cyclic  $X$ -modules

A domain satisfying (\*) is called an elementary divisor ring [4].

If  $A$  is domain and the monoid  $S$  is cancellative and torsion free ( $nx=ny$  implies  $x=y$  for  $n$  a positive integer) then  $[A]$  and  $[[A]]$  are domains. ([11]; 2.4). Hence we have

**Theorem 3.7** Let  $A$  be an elementary divisor ring and  $S$  a torsion free, cancellative monoid then  $[A]$  and  $[[A]]$  are elementary divisor rings.

An ordered monoid is said to be **difference-closed**, if it is totally ordered and  $s < t$  in implies there exists  $u \in S$  with  $u+s=t$ . Let  $A$  be a domain  $S$  a difference-closed monoid. Then  $A$  is **generalized euclidean (modeled on  $S$ )** if  $\exists$  a map  $d: A^* \rightarrow S$  with the following properties:

$d_1 : d(a) \leq d(ab)$  for  $a, b \in A^*$

$d_2 : \text{given } a, b \text{ in } A \text{ with } d(a) \leq d(b)$ , there exists  $c, r$  in  $A$  such that  $b=ca+r$  with either  $r=0$  or else  $d(r) < d(a)$ .

**Theorem 3.8** let  $k$  be a field and  $S$  be a positively totally ordered, archimedean, torsion-free, cancellative monoid that is difference-closed, then  $([k], S)$  is a generalized euclidean domain (modeled on  $S$ ).

**Proof.**  $[k]$  is a domain ([11] 2.4). We can write the elements of  $[k]$  as sum of terms of increasing order of support. Accordingly consider  $\underline{a} = \sum e(t_i, l_i)$  and  $\underline{b} = \sum e(s_j, m_j)$ , where  $e(t_i, l_i) \in [A]$  and the  $\text{supp}(e(t_i, l_i))$  is  $t_i$  with value  $l_i$  etc;. Let  $\text{max.Supp } \underline{a}$  be  $t_n$  and  $\text{max.supp } \underline{b}$  be  $s_p$  respectively. Let us define  $d(\underline{a}) = t_n$  and  $d(\underline{b}) = s_p$ . then the map  $d$  satisfies  $d_1$ . For  $d_2$ , suppose  $d(\underline{a}) = t_n \leq d(\underline{b}) = s_p$ . there exists  $q \in S$  such that  $t_n + q = s_p$ . Let  $\underline{a}' = (\underline{b} - \underline{a} \cdot e(q, m_p/l_n))$ . Either  $\underline{a}' = 0$  or  $d(\underline{a}') < d(\underline{b}) = s_p$ . If  $d(\underline{a}') < d(\underline{a})$  we are done. Otherwise we repeat the process with  $d(\underline{a}')$  and  $d(\underline{a})$ .

**Theorem 3.9** The hypothesis same as in 3.8 suppose in addition  $S$  is artinian, then  $[k]$  is a GCD domain, hence a PID.

**Theorem 3.10** Let  $A$  be a UFD and  $S$  be an artinian archimedean difference-closed, torsion free, cancellative that is positively totally ordered then  $[A]$  is a UFD.

**Proof.** Gauss.

A commutative ring  $A$  is **absolutely flat** if for  $x \in A \exists y \in A$  such that  $xyx = x$ .

A monoid  $S$  is **absolutely flat** if for  $s \in S \exists t \in S$ , such that  $s+t+s = s$ .

A commutative ring is absolutely flat iff it is a subdirect product of fields.

The order of a monoid is **subtotal** if given  $s \in S$  there exists  $n \in \mathbb{N}$  such that  $ns \leq 0$  or  $0 \leq ns$ .

**Theorem 3.9** Let  $S$  be a torsion-free, cancellative monoid and  $A$  a commutative ring. Then the following are equivalent.

- a)  $[A]$  is absolutely flat
- b)  $A$  and  $S$  are absolutely flat.

**Proof:** a) implies b) is found in [15], where it is also proved that an absolutely flat torsion-free cancellative monoid is a group, whose order is subtotal. Hence for the converse we suppose  $A$  is absolutely flat and  $S$  is a torsion-free group, whose order is subtotal. Since  $A$  is absolutely flat, it is a subdirect product of fields. When  $k$  is a field and  $S$  is a torsion free group, whose order is subtotal,  $[k]$  is a field [1]. And  $[ ]$  preserves subdirect products.

A ring is called **semi-simple artinian (SSA)** if it is isomorphic to a coproduct of fields. A SSA ring is clearly absolutely flat.

**Theorem 3.10** Let  $A$  be a commutative ring  $S$  torsion-free, cancellative monoid then the following are equivalent.

- a)  $[A]$  is SSA
- b)  $A$  is SSA and  $S$  is absolutely flat.

**Remark:** as the results in [15] show most of the results above are also valid for non-commutative rings.

#### 4 Descent.

The ring extension  $A \subseteq B$  **descends property  $P$** , if for an  $A$ - module  $M$ , whenever the  $B$ - module  $M \otimes B$  has property  $P$ , so does  $M$ . (see [17]). Let  $A$  be a ring and  $M$  an  $A$ -module, if for a family of  $A$ - modules  $(Q_i)$ , the canonical morphism  $(\prod Q_i) \otimes M \rightarrow \prod (Q_i \otimes M)$  is injective, then  $M$  is called **an (Mittag-Leffler) ML- module**. ([17] page 71. proposition 2.1.5). An extension **descends ML- property**, if for a module  $M$  if its extension is ML, so is  $M$ . [17] defines and proves the descent of ML property under the additional assumption that  $M$  is  $A$ - flat. But we prove it without assuming flatness.

**Theorem 4.1** Let  $A$  be a commutative ring,  $M$ - an  $A$ -module and  $S$  a finite monoid, then the extension descends ML property.

**Proof.** Let us consider the following diagram, where  $M$  is an  $A$ - module and  $(Q_i)$  be a family of  $A$ -modules.

$$\begin{array}{ccc}
 ((\Pi Q_i) \otimes M) & \longrightarrow & \Pi(Q_i \otimes M) \\
 \downarrow & & \downarrow \\
 (\Pi[Q_i]) \otimes [M] & \longrightarrow & \Pi([Q_i] \otimes [M])
 \end{array}$$

We have to prove the canonical morphism at the top is a monomorphism. The bottom row is the canonical morphism for the module  $[M]$  with reference to the testing family of modules  $([Q_i])$ . Since  $[M]$  is a  $ML - [A]$ -module by hypothesis, this is a monomorphism.

The left vertical arrow is the composition of the following

$$(\Pi Q_i) \otimes M \rightarrow [(\Pi Q_i) \otimes M] \rightarrow ([\Pi Q_i]) \otimes [M] \rightarrow (\Pi([Q_i])) \otimes [M]$$

The first map is the canonical embedding of the module concerned: the second is due to the fact  $[M \otimes_A N] \cong [M] \otimes_{[A]} [N]$  for  $A$ -modules  $M$  and  $N$ . The third follows by hypothesis that  $\Pi$  commutes with the extension  $[ ]$ . Since each of these maps is a monomorphism, so is the left vertical arrow.

The right vertical arrow is composed of the following maps:  $\Pi(Q_i \otimes M) \rightarrow \Pi[Q_i \otimes M] \rightarrow \Pi([Q_i] \otimes [M])$  here the first map is the product of canonical embeddings of modules into their extensions, hence injective. The second map follows from  $[M \otimes_A N] \cong [M] \otimes_{[A]} [N]$  for  $A$ -modules  $M$  and  $N$ . Thus the resulting composition is injective. The above square is commutative. The top row is a monomorphism.

**Remark:** it is not known if this valid for a non-finite monoid. All our extensions descend projectivity. Indeed an  $A$ - module  $P$  is projective iff  $[P]$  ( $[[P]]$ ,  $\{[P]\}$  respectively) is so.

**Theorem 4.2** Let  $A$  be a commutative ring and we consider the extension  $[[A]]$   
**Then a module  $P$  is projective iff  $[[P]]$  is so.**

**Proof.** let  $P$  be projective then  $P \oplus Q \cong \oplus A$ . Hence  $[[P]] \oplus [[Q]] \cong \oplus [[A]]$ . Thus  $[[P]]$  is  $[[A]]$ -projective.

Conversely let us consider the following diagram.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & P \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & [[K]] & \longrightarrow & [[F]] & \longrightarrow & [[P]] \longrightarrow 0
 \end{array}$$

The top row is a free resolution of  $P$  and the bottom row is got from the top by applying  $[[\ ]]$ . Both sequences are exact. Since  $[[P]]$  is  $[[A]]$  projective, the bottom row splits by a map  $\beta$ . Consider the map  $\Pi\beta: [[P]] \rightarrow \bigoplus [[A]] \rightarrow [[P]]$ , where  $\Pi$  is the canonical epimorphism, the composition is the identity on  $[[P]]$ . Call a morphism  $f: [[M]] \rightarrow [[N]]$  **support-preserving** if for any  $x$ ,  $\text{Suppf}(x)$  is contained in  $\text{Supp}(x)$ . Clearly the map  $\Pi\beta$  being the identity on  $[[P]]$  is support preserving and  $\Pi$  is support preserving. So  $\beta$  is also support preserving. Finally  $\beta_{i_p}: P \rightarrow \bigoplus [[A]]$  is also support preserving. Then the  $\text{Im}(\beta_{i_p})$  is contained in  $\bigoplus A$ . Hence the top sequence is split by  $\beta_{i_p}$  and  $P$  is  $A$ -projective.

**Remark:** (see also [17]: page 82. Example 3.1.4). We shall give another proof of the above result using the notions of relative homological algebra.

An exact sequence is **pure-exact**, if it remains exact under tensoring by any module. Equivalently it is an inductive limit of system of split exact sequences. A module is **pure-projective**, iff it is projective with respect to the class of pure-exact sequences. A module is pure projective iff it is a summand of a coproduct of finitely presented modules. A module  $M$  always has a pure-projective resolution. ([17] page 55 Proposition 1.1.1).

**Theorem 4.3** Let  $A$  be a ring. Then a module  $M$  is pure projective iff  $[[M]]$  is so.

**Proof:** If  $M$  is pure-projective then  $M \oplus N \cong \bigoplus Q_i$ , where  $Q_i$  is a finitely presented  $A$ -module.. Then  $[[M]] \oplus [[N]] \cong \bigoplus [[Q_i]]$ .  $[[Q_i]]$  is  $[[A]]$  finitely presented. Hence  $[[M]]$  is  $[[A]]$  pure-projective.

For the converse: consider a pure-projective resolution of  $M$  and the corresponding diagram below.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \longrightarrow & Q & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & [[K]] & \longrightarrow & [[Q]] & \longrightarrow & [[M]] \longrightarrow 0
 \end{array}$$

The top sequence being pure-exact is an inductive limit of a system of split exact sequences. Hence the bottom is also the inductive limit of the corresponding system of split exact sequences, thus pure-exact. ( $[[\ ]]$  preserves inductive limit).

Then by hypothesis  $[[M]]$  is  $[[A]]$  pure projective, hence the bottom exact sequence splits. The splitting morphism can be lifted – as in the proof of the previous proposition - . So  $M$  is pure-projective.

**Remark:** the above result also gives another proof for the descent of projectivity. For, a module is projective iff it is flat and pure-projective. Hence descent of projectivity is equivalent to descent of both pure-projectivity and flatness. Most of the ring properties are hereditary because of the extension being faithfully flat.

**Theorem 4.4** Let  $A$  be a ring and  $S$  monoid. Then  $[[A]]$  is semi-hereditary (respectively hereditary, prufer, dedekind, noetherian) implies  $A$  is so (hereditary, prufer, dedekind, noetherian respectively).

**Proof.** A ring is (semi-hereditary) hereditary, if every (finitely generated) ideal is projective. A (semi-hereditary) hereditary domain is called (prufer) dedekind. Suppose  $[[A]]$  is a (semi-hereditary) hereditary and  $I$  be a (finitely generated) ideal of  $A$ , then  $[[I]]$  is projective. So is  $I$  and  $A$  is (semi-hereditary) hereditary. If  $I$  is an ideal of  $A$  and  $[[A]]$  is noetherian, the ideal  $[[I]]$  is finitely generated, hence  $I$  is also finitely generated. So  $A$  is noetherian.

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