

A New Characterization of the Simple Group $G_2(5)$

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Abstract

The main aim of this article is to determine the structure of finite groups whose character degrees are those of $G_2(5)$. Consequently, the simple group $G_2(5)$ satisfies Huppert's conjecture.

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1 Introduction

For any finite group G , $Irr(G)$ denotes the set of all irreducible complex characters of G , and the set of all irreducible character degrees of G is defined by $cd(G) = \{\chi(1) | \chi \in Irr(G)\}$.

Classifying finite groups by the properties of their characters is an interesting problem in group theory. In the case where G is a solvable group, a bunch of results is known how properties of $cd(G)$ reflect information about the structure of G (see [4, 11]). If, on the other hand, G is a nonsolvable group, specially if G is simple, a new characterization concerned with the structure of simple groups and their character degrees were recently studied by B. Huppert.

In 2000, Huppert conjectured that each finite nonabelian simple group G is characterized by the set $cd(G)$ of degrees of its complex irreducible characters

(see below). In [6, 7, 8], he confirmed this conjecture for some nonabelian simple groups. For instance, we have the following important two infinite families $L_2(q)$ and $Sz(q)$, and a handful of other groups, for example, alternating groups A_n for $n \leq 10$, $L_3(q)$ for $q \leq 8$, $U_3(q)$ for $q \leq 9$, and a few others (See [6, 7, 8]). Moreover, he proved this conjecture for 19 out of 26 sporadic simple groups, and in [2, 3], six more of these groups were handled.

Conjecture. (Huppert) *Let H be any nonabelian simple group, and G a group such that $cd(G) = cd(H)$. Then $G \cong H \times A$, where A is an abelian group.*

The main aim of this paper is to prove Huppert's conjecture for the simple group $G_2(5)$:

Main Theorem. *Let G be a finite group with $cd(G) = cd(G_2(5))$, then $G \cong G_2(5) \times A$, where A is an abelian group.*

Outline of the proof

In order to prove the main result, we organize the proof as follows:

First of all, using Lemma 2.3 we prove that $G' = G''$. Second of all, by characterization of the simple groups whose orders are divisible by the primes 2, 3, 5, 7, 31, Lemma 2.9, we deduce that $G'/M \cong G_2(5)$, where G'/M is a chief factor of G . Next, we suppose that we show that all linear characters of M are invariant under G' , that is, $I_{G'}(\vartheta) = G'$, where $\vartheta \in \text{Lin}(M)$. In this step, the proof is based on Lemma 2.2, Lemma 2.5, Lemma 2.6 and the information about projective representations and Schur multipliers of related subgroups. Then, it is concluded that $M = M'$ and $M = 1$ using Lemma 2.4. Finally, we prove that $G \cong G' \times A$, where $A = C_G(G')$ is an abelian group, according to the group of outer automorphism of $G_2(5)$.

2 Preliminary Notes

In this section we state some elementary facts and prove some preliminary lemmas, which are useful to prove the main result.

Throughout this article, all groups are finite, and for a given group G , we denote $\text{Irr}(G)$ the set of its irreducible complex characters. If $\chi \in \text{Irr}(G)$, $\chi(1)$ is the degree of χ , and the set of all irreducible character degrees of G is denoted by $cd(G)$. An irreducible character of G whose its degree is equal to 1 is said to be linear, and the set of those characters is denoted by $\text{Lin}(G)$. The inner product of two characters χ and ψ of G is denoted by $(\chi, \psi)_G$. If N is

a subgroup of G , φ and χ are characters of N and G , respectively, then it is well-known that

$$(\chi, \varphi^G)_G = (\chi_N, \varphi)_N.$$

by Frobenius Reciprocity Law. Furthermore, if N is a normal subgroup of G and $\vartheta \in \text{Irr}(N)$, then the inertia group of ϑ in G is denoted by $I_G(\vartheta)$. The further unexplained notations are standard and can be found in [1, 5, 10].

Lemma 2.1. [5, Theorem 19.5, Theorem 21.3] *Suppose N is a normal subgroup of G and $\chi \in \text{Irr}(G)$.*

- (a) *If $\chi_N = \sum_{i=1}^k \vartheta_i$ with $\vartheta_i \in \text{Irr}(N)$, then k divides $|G/N|$. In particular, $\chi_N \in \text{Irr}(N)$, if $\chi(1)$ and $|G/N|$ are coprime;*
- (b) *If $\chi_N \in \text{Irr}(N)$, then $\chi\theta \in \text{Irr}(G)$, for every $\theta \in \text{Irr}(G/N)$.*

Lemma 2.2. [5, Theorem 19.6] *Suppose N is a normal subgroup of G , $\vartheta \in \text{Irr}(N)$ and $I := I_G(\vartheta)$, the inertia subgroup of ϑ in G .*

- (a) *If $\vartheta^I = \sum_{i=1}^k \varphi_i$, with $\varphi_i \in \text{Irr}(I)$, then $\varphi_i^G \in \text{Irr}(G)$ and $\varphi_i(1)|G : I| \in \text{cd}(G)$. Moreover, if $I = G$, then $(\varphi_i)_N = e\vartheta$, where $e \mid |G/N|$;*
- (b) *If ϑ is extended to $\vartheta_0 \in \text{Irr}(I)$, then $(\vartheta_0\theta)^G \in \text{Irr}(G)$ and $\vartheta(1)\theta(1)|G : I| \in \text{cd}(G)$, for all $\theta \in \text{Irr}(I/N)$.*

Lemma 2.3. *Let G/N be a solvable factor group of G , minimal with respect to being non-abelian. Then two following cases can occur.*

- (a) *G/N is a p -group, for some primes p . There exists $\psi \in \text{Irr}(G/N)$ such that $\psi(1) = p^b > 1$. If $\chi \in \text{Irr}(G)$ and $p \nmid \chi(1)$, then $\chi\theta \in \text{Irr}(G)$, for all $\theta \in \text{Irr}(G/N)$;*
- (b) *G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N . Then $|G/F| \in \text{cd}(G)$ and $|F/N| = p^a > 1$, for some primes p . Then F/N is an irreducible module for the cyclic group G/F , hence a is the smallest integer such that $p^a - 1 \equiv 0 \pmod{|G/F|}$. If $\psi \in \text{Irr}(F)$, then either $|G/F|\psi(1) \in \text{cd}(G)$ or $|F/N|$ divides $\psi(1)^2$. In the latter case p divides $\psi(1)$.
Moreover, if no proper multiple of $|G/F|$ is in $\text{cd}(G)$, then $\chi(1)$ divides $|G/F|$, for all $\chi \in \text{Irr}(G)$ such that $p \nmid \chi(1)$.*

Proof. See [6, Lemma 4] and [10, pp. 199-200]. \square

Lemma 2.4. [6, Lemma 6] *Suppose $M \trianglelefteq G'$, $G' = G''$, and for all linear character ϑ of M , $I_{G'}(\vartheta) = G'$. Then $M' = [M, G']$ and $|M/M'|$ divides the order of the Schur multiplier of G'/M .*

Lemma 2.5. *Let G be a finite group and $N \trianglelefteq G$, with $\vartheta \in Irr(N)$. If ϑ is extended to an irreducible character of S , a subgroup of $I := I_G(\vartheta)$ containing N , then there is an irreducible constituent φ of ϑ^I , with $\varphi(1) \geq \theta(1)\vartheta(1)$ for all $\theta \in Irr(S/N)$.*

Proof. Assume that $\vartheta^I = \sum_{i=1}^k \varphi_i$, where $I := I_G(\vartheta)$. By Mackey's theorem

$$(\vartheta^I)_S = \sum_{j=1}^m ((\vartheta^{r_j})_{Nr_j \cap S})^S,$$

where $I = \bigcup_{j=1}^m Nr_j S$. Since $N \trianglelefteq G$ and $r_j \in I$, we have $Nr_j = N$ and $\vartheta^{r_j} = \vartheta$. Thus $(\vartheta^I)_S = |I : S|\vartheta^S$. By the assumption, there exists $\vartheta_0 \in Irr(S)$ such that $\vartheta_{0N} = \vartheta$. By Gallagher theorem we have

$$(\vartheta^I)_S = |I : S| \sum_{\theta \in Irr(S/N)} \theta(1)\vartheta_0\theta.$$

Since $\sum_{i=1}^k ((\vartheta_0\theta)^I, \varphi_i)_I = ((\vartheta_0\theta)^I, \vartheta^I)_I = (\vartheta_0\theta, (\vartheta^I)_S) \neq 0$, for all $\theta \in Irr(S/N)$, there exists j such that $(\varphi_{jS}, \vartheta_0\theta) \neq 0$, for all $\theta \in Irr(S/N)$. Therefore $\varphi_j(1) \geq \theta(1)\vartheta(1)$, for all $\theta \in Irr(S/N)$. \square

Lemma 2.6. *Suppose $G = H : GL_2(q)$, where $q > 3$, K is a subgroup of G of index p^a , where $a = 1, 2$, and p is a prime number with $p \nmid |H|$, then $H \trianglelefteq K$. Moreover, if K is a normal subgroup of G , then $p^a \mid q - 1$ and $K = H : (SL_2(q) \cdot Z_{\frac{q-1}{p^a}})$.*

Proof. We prove the Lemma for $a = 2$, the proof for $a = 1$ is similar. Let K be a subgroup of G of index p^2 . If $G = KH$ or $|G : KH| = |KH : K| = p$, then p divides $|H : H \cap K|$, i.e., $p \mid |H|$ which is a contradiction. Therefore $HK = K$ and $H \trianglelefteq K$. To prove the last statement, suppose $K \trianglelefteq G$, put $\overline{K} := K/H$, $\overline{G} := G/H \cong GL_2(q)$ and $B := \overline{K} \cap SL_2(q)$. Thus by the assumption $|\overline{G} : \overline{K}| = p^2$ and we have $\overline{K} \trianglelefteq \overline{K}SL_2(q) \trianglelefteq GL_2(q)$. Assume that $GL_2(q) = \overline{K}SL_2(q)$. If $B \neq SL_2(q)$, then $B \leq Z(SL_2(q))$, so $|SL_2(q) : B| = p^2$ and $|L_2(q)| \cdot |Z(SL_2(q)) : B| = p^2$, which is impossible. Thus $SL_2(q) \leq \overline{K}$ and $\overline{G} = \overline{K}$, a contradiction. Let $|\overline{G} : \overline{K}SL_2(q)| = |\overline{K}SL_2(q) : \overline{K}| = p$, then $|SL_2(q) : B| = p$ and by the same method as in last case we must have $SL_2(q) \leq \overline{K}$, thus $\overline{K}SL_2(q) = \overline{K}$, so $|\overline{G} : \overline{K}| = p$, a contradiction. Therefore, $\overline{K} = \overline{K}SL_2(q)$, then $SL_2(q) \leq \overline{K}$, so $|\overline{G} : \overline{K}| \cdot |\overline{K} : SL_2(q)| = |\overline{G} : SL_2(q)| = q - 1$ because $GL_2(q)/SL_2(q) \cong Z_{q-1}$. Thus $p^2 \mid q - 1$ and $\overline{K}/SL_2(q)$ is isomorphic to a subgroup of Z_{q-1} of order $(q - 1)/p^2$. The proof is completed. \square

Lemma 2.7. [12] *If p is a prime and $n \geq 2$ be a natural number. Then one of the following holds.*

- (a) *There exists a prime q such that q divides $p^n - 1$ and q does not divide $p^i - 1$, for all $i < n$;*
- (b) *$p = 2$ and $n = 6$;*
- (c) *$p = 2^r - 1$ is a Mersenne prime, $n = 2$ and r is a prime number.*

Lemma 2.8. *Suppose that $p \geq 5$ is a prime, and that $n \geq 2$ is a positive integer. Then*

- (a) *if $p^{2n} - 1$ has at most three different prime divisors, then $p^n \in \{25, 49\}$;*
- (b) *$p^{6n} - 1$ has at least five different prime divisors.*

Proof. To prove (a), see [9, Lemma 1.4]. To prove (b), assume contrary, then $p^{6n} - 1$ has at most four prime divisors. By Lemma 2.7, there is a prime divisor q of $p^{6n} - 1$ such that $q \nmid p^{2n} - 1$. So $p^{2n} - 1$ has at most three different prime divisors. By (a), we must have $p^n = 25$ or 49 , so by a straightforward calculation of the prime factors of $p^{6n} - 1$ we get a contradiction. \square

In the following, for the positive integer n , $\pi(n)$ denotes the set of prime divisors of n , and for finite group G , $\pi(G)$ is defined by $\pi(|G|)$. Now assume that $p \geq 5$ is prime and $n \geq 2$. Then, by above lemma, $|\pi(p^{6n} - 1)| \geq 5$; moreover, if $|\pi(p^{2n} - 1)| \leq 3$, then $p^n \in \{25, 49\}$.

Lemma 2.9. *Let G be a finite simple group. Then $\pi(G) \subseteq \{2, 3, 5, 7, 31\}$ if and only if G is isomorphic to one of the following simple groups:*

- (a) *A_5, A_6 or $S_4(3)$, for $\pi(G) = \{2, 3, 5\}$;*
- (b) *$L_2(7), L_2(8)$ or $U_3(3)$, for $\pi(G) = \{2, 3, 7\}$;*
- (c) *$A_7, A_8, A_9, A_{10}, J_2, L_2(7^2), L_3(4), U_3(5), U_4(3), S_4(7), S_6(2)$ or $O_8^+(2)$, for $\pi(G) = \{2, 3, 5, 7\}$;*
- (d) *$L_2(31)$ or $L_3(5)$, for $\pi(G) = \{2, 3, 5, 31\}$;*
- (e) *$L_5(2), L_6(2), L_2(5^3)$ or $G_2(5)$, for $\pi(G) = \{2, 3, 5, 7, 31\}$.*

Proof. By [9], it is enough to classify finite simple groups G with $\pi(G) = \{2, 3, 5, 7, 31\}$. Since $31 \mid |G|$, G cannot be isomorphic to a sporadic simple group. if, moreover, G is isomorphic to an alternating group A_n , where $n \geq 5$, then $n \geq 31$ which is impossible because $11 \notin \pi(G)$.

Let G be isomorphic to a simple group of Lie type denoted by $Lie(q)$. We consider each type separately, and leave the proof of similar cases to the reader.

Suppose $G \cong L_n(q)$, then q is a power of 2, 3, 5, 7 or 31. Moreover, $n \leq 9$ because $11 \mid q^{10} - 1$, for all q . Let $q = 2^m$, since $127 \mid 2^7 - 1$ and $31 \in \pi(G)$,

$5|mi$, where $2 \leq i \leq 6$. If $5|m$, then $11 \in \pi(q^2 - 1)$, a contradiction. So $31 | q^5 - 1$, therefore $G \cong L_n(2^m)$, where $n = 5$ or 6 . Since $11 | 2^{10} - 1$, m is an odd number. By Lemma 2.7, $m = 1$ or $q^5 - 1$ has a prime divisor except 31. In the later case, since m is an odd number, we have $7 | q^5 - 1$, so $2^{15} - 1$ divides $|G|$, which is a contradiction. Hence $m = 1$, and $G \cong L_5(2)$ or $L_6(2)$. Let $q = 3^m$, since the order of 3 modulo 31 is 30, $30 | mi$, where $2 \leq i \leq 9$. So $3^{30} - 1$ divides $|G|$, a contradiction. Let $q = 5^m$, since $13 \in \pi(5^4 - 1)$, $n \leq 3$. Thus, $G \cong L_2(5^m)$ or $L_3(5^m)$. If $G \cong L_2(5^m)$, then $7 | 5^{2m} - 1$. Therefore, $3|m$ and $5^{6t} - 1 || G|$. If $t \geq 2$, by Lemma 2.8(b) we get a contradiction. Therefore, $m = 3$ and $G \cong L_2(5^3)$. If $G \cong L_3(5^m)$, then 7 divides $5^{2m} - 1$ or $5^{3m} - 1$, so 3 or 2 divides m , respectively. Then $5^9 - 1$ or $5^4 - 1$ divides $|G|$, respectively, which is impossible. Let $q = 7^m$, then $15 | mi$, where $2 \leq i \leq 9$. So $7^{15} - 1 || G|$, a contradiction. Let $q = 31^m$, since $331 | 31^3 - 1$, we have $G \cong L_2(31^m)$. As $7 \in \pi(G)$, then $3 | m$ and $331 \in \pi(G)$, a contradiction. Therefore, if $G \cong L_n(q)$, then $(n, q) = (2, 5)$, $(2, 6)$ or $(2, 5^3)$.

Let $G \cong U_n(q)$, $O_{2n+1}(q)$, $O_{2n}^+(q)$, $O_{2n}^-(q)$, $S_{2n}(q)$ or ${}^3D_4(q)$, by a similar method we get a contradiction. For example, let $G \cong U_n(q)$, then $3 \leq n \leq 9$ because $11 | q^{10} - 1$. If $q = 2^m$, then 31 divides $q^i - 1$ or $q^j + 1$, for some possibilities of i and j with $2 \leq i, j \leq 9$. Since $11 \in \pi(2^{10} - 1)$, then $5 \nmid m$. If $31 | q^i - 1$, where i is an even number, then $n \geq 10$, which is impossible. If $31 | q^j + 1$, where j is an odd number and $3 \leq j \leq 9$, then $j = 5$. As we have $31 | q^5 - 1$ and $\gcd(q^5 - 1, q^5 + 1) = 1$, a contradiction. If $q = 3^m$, then 31 divides $q^i - 1$ or $q^j + 1$, for some suitable i and j with $2 \leq i, j \leq 9$. If 31 divides $q^i - 1$ or $q^j + 1$, then 30 divides mi or $2mj$, respectively. Since $i, j \leq 9$, m is divisible by 3 or 5. Thus, 13 or 11 divides $q - 1$, respectively, which is a contradiction. If $q = 5^m$, since 13 divides $5^4 - 1$, $n = 3$ and m is an odd number. Thus, 31 divides $q^2 - 1$ or $q^3 + 1$. If $31 | q^2 - 1$, then $3|m$, so $5^9 + 1$ divides $|G|$, a contradiction. Since $31 | q^5 - 1$ and $\gcd(q^3 - 1, q^3 + 1) = 2$, $31 \nmid q^3 + 1$. If $q = 7^m$, then 31 divides $q^i - 1$ or $q^j + 1$, for some suitable i and j with $2 \leq i, j \leq 9$. Thus 15 divides mi or $2mj$, respectively. Then, 3 or 5 divides m , thus 19 or 2801 divides $q - 1$, respectively, a contradiction. If $q = 31^m$, since $13 \in \pi(q^4 - 1)$, $n = 3$. If m is an odd number, then $31^3 + 1 || G|$, a contradiction. Therefore, m is an even, so $31^4 - 1$ divides $|G|$, a contradiction.

Let $G \cong G_2(q)$, where $q = p^m$ and $p \in \{2, 3, 5, 7, 31\}$. If $p = 2$, then $2^{10} - 1$ divides $|G|$, a contradiction. If $p = 3$, then $3^6 - 1 | |G|$, a contradiction. Hence $p \geq 5$. By Lemma 2.8(b), we have $m = 1$. On the other hand, if $G \cong G_2(7)$ or $G_2(31)$, then 19 is a prime divisor of $7^6 - 1$ and $31^6 - 1$, which is a contradiction. Therefore, $G \cong G_2(5)$.

If G is isomorphic to ${}^2B_2(q)$, ${}^2F_4(q)$, where $q = 2^{2m+1}$, or ${}^2G_2(q)$, where $q = 3^{2m+1}$, with the same method we get a contradiction. For example, let $G \cong {}^2B_2(q)$, where $q = 2^{2m+1}$. Then $31 | q - 1$ or $q^2 + 1$; moreover, $5 | 2m + 1$. Thus, $2^{10} + 1$ divides $|G|$, a contradiction.

Since $13 \in \pi(q^{12} - 1)$, G cannot be isomorphic to the simple groups $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$ and ${}^2E_6(q)$.

Based on the above argument, $L_5(2)$, $L_6(2)$, $L_2(5^3)$ and $G_2(5)$, are the only simple groups G with $\pi(G) = \{2, 3, 5, 7 \text{ and } 31\}$, and the proof is completed. \square

3 Main Result

This section is devoted to prove the main theorem.

Proof. Let G be a finite group and $cd(G) = cd(G_2(5))$. By [1, p. 114], the character degrees of G are listed in the Table I.

Table I
The complex character degrees of $G_2(5)$

1	$1890 = 2 \cdot 3^3 \cdot 5 \cdot 7$	$13020 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 31$
$124 = 2^2 \cdot 31$	$2480 = 2^4 \cdot 5 \cdot 31$	$15500 = 2^2 \cdot 5^3 \cdot 31$
$280 = 2^3 \cdot 5 \cdot 7$	$2604 = 2^2 \cdot 3 \cdot 7 \cdot 31$	$15624 = 2^3 \cdot 3^2 \cdot 7 \cdot 31$
$651 = 3 \cdot 7 \cdot 31$	$3255 = 3 \cdot 5 \cdot 7 \cdot 31$	$15925 = 5^6$
$930 = 2 \cdot 3 \cdot 5 \cdot 31$	$3906 = 2 \cdot 3^2 \cdot 7 \cdot 31$	$16275 = 3 \cdot 5^2 \cdot 7 \cdot 31$
$960 = 2^6 \cdot 3 \cdot 5$	$10416 = 2^4 \cdot 3 \cdot 7 \cdot 31$	$17856 = 2^6 \cdot 3^2 \cdot 31$
$1085 = 5 \cdot 7 \cdot 31$	$12096 = 2^6 \cdot 3^3 \cdot 7$	$19530 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$
$1240 = 2^3 \cdot 5 \cdot 31$		

The proof is divided into the following steps.

Step 1. G' is a perfect group, i. e., $G' = G''$.

Assume contrary, then G has a normal subgroup N such that G/N is a solvable factor group of G , minimal with respect to being nonabelian.

Using Lemma 2.3; the structure of G/N is determined as follows.

(a) G/N is a p -group, for some primes p . Moreover, there exists $\psi \in Irr(G/N)$, with $\psi(1) = p^a > 1$ and $p^a \chi(1) \in cd(G)$, for $\chi \in Irr(G)$ such that $p \nmid \chi(1)$. If $p = 2$, we consider $\chi \in Irr(G)$, with $\chi(1) = 5 \cdot 7 \cdot 31$, so $2^a \cdot 5 \cdot 7 \cdot 31 \in cd(G)$ which is impossible. If $p = 3$ or 31 we get a contradiction by taking $\chi(1) = 2^2 \cdot 5^3 \cdot 31$ or $2^6 \cdot 3 \cdot 5$, respectively. Thus, $p \neq 2, 3$ and 31 , since $2^6 \cdot 3^2 \cdot 31 \in cd(G)$, we similarly reach to a forbidden degree again.

(b) G/N is a Frobenius group, with Frobenius kernel F/N of order $p^b > 1$, for a prime p , and $|G/F| \in cd(G)$.

Suppose that $|G/F| \in \{ 280, 960, 1890, 2480, 10416, 12096, 13020, 15500, 15624, 16275, 17856, 19530 \}$, then no proper multiple of $|G/F|$ is in $cd(G)$, so

$\chi(1) \mid |G/F|$, for all $\chi \in Irr(G)$ with $p \nmid \chi(1)$. If $p \neq 2, 3$ and 31 , since there is an irreducible character χ of G of degree $2^6 \cdot 3^2 \cdot 31$, we have $|G/F| = 2^6 \cdot 3^2 \cdot 31$. If $p = 5$, then $3 \cdot 7 \cdot 31$ divides $|G/F|$; otherwise, $2^6 \cdot 3 \cdot 5 \mid |G/F|$, a contradiction. Therefore, $p = 2, 3$ or 31 . Let $p = 2$, then $|F/N| = 2^a > 1$. Since $3 \cdot 5^2 \cdot 7 \cdot 31 \mid |G/F|$, $|G/F| = 3 \cdot 5^2 \cdot 7 \cdot 31$. So $a = 60$ because a is the order of 2 modulo $|G/F|$. If $\psi \in Irr(F) \setminus Lin(F)$, then $2^{30} \mid \psi(1)$, so 2^{30} divides some degrees of G , a contradiction, thus F is an abelian group. Put $\chi(1) = 2^2 \cdot 31$, then by Ito's theorem $|G/F|$ is an even number, a contradiction. Let $p = 3$, then there are characters χ and θ of G , with $\chi(1) = 2^2 \cdot 5^3 \cdot 31$ and $\theta(1) = 2^3 \cdot 5 \cdot 7$, so $\chi(1)$ and $\theta(1)$ divide $|G/F|$, which is impossible. If $p = 31$, we can choose $\chi(1) = 2^6 \cdot 3 \cdot 5$ and $\theta(1) = 2^6 \cdot 3^2 \cdot 7$ to get a contradiction.

If $|G/F| \in \{124, 651, 930, 1085, 1240, 2604, 3255, 3906\}$, then $|F/N| = p^a > 1$, where p is a prime number and $p \nmid |G/F|$. Let $\psi \in Irr(F)$, then $\psi(1) \mid |G/F| \in cd(G)$ or $p^a \mid \psi(1)^2$. If $|G/F| = 124 = 2^2 \cdot 31$, then $\psi(1) \leq 2^4 \cdot 3^2$ or $p^a \mid \psi(1)^2$. If $\chi \in Irr(G)$, with $\chi(1) = 2^6 \cdot 3^3 \cdot 7$, then by Lemma 2.1 we have $\chi(1) = k\psi(1)$, where $k \mid |G/F|$. So $3^3 \cdot 7$ divides $\psi(1)$, which is impossible. By the similar method we get a contradiction for other possible cases.

Step 2. The chief factor $\overline{G'} := G'/M$ of G , is isomorphic to $G_2(5)$.

Since $G' = G''$, and $\overline{G'}$ is characteristically simple group,

$$G'/M \cong S_1 \times \cdots \times S_k$$

where $S_i \cong S$ is a simple nonabelian group. The prime divisors of $|S|$ are exactly those primes in factorization of character degrees of S , and all degrees of S divide some degrees of G ; therefore, $\pi(S) \subseteq \{2, 3, 5, 7, 31\}$. By Lemma 2.9, the characterization of simple groups whose orders are divisible by the primes 2, 3, 5, 7 or 31, S is isomorphic to one of the following simple groups.

$$A_5, A_6, A_7, A_8, A_9, A_{10}, L_2(7), L_2(8), L_2(31), L_2(7^2), L_2(5^3), L_3(4), \\ L_3(5), L_5(2), L_6(2), U_3(3), U_3(5), U_4(3), S_4(3), S_4(7), S_6(2), O_8^+(2), \\ G_2(5) \text{ or } J_2.$$

In what follows, we frequently use *Atlas of Finite Groups* [1], for more information about character degrees, structure of maximal subgroups and outer automorphisms of the above finite simple groups. Now we discuss each possibility separately.

Suppose that S is isomorphic to one of the simple groups $U_4(3)$, $O_8^+(2)$, $L_5(2)$, $S_6(2)$ or $L_6(2)$. Then 2^7 divides some degrees of the irreducible characters of G , which is a contradiction. If $S \cong A_{10}$ or $S_4(3)$, then 3^4 divides some degrees of G . If $S \cong L_2(7^2)$ or $S_4(7)$, then G has a forbidden degree which is divided by 7^2 .

Let $S \cong A_6, A_7, A_8, A_9, L_2(7), L_2(8), L_2(31), L_2(5^3), L_3(4), L_3(5), U_3(3), U_3(5)$ or J_2 . Since G does not have a degree divided by $3^4, 7^2$ and $31^2, k = 1$ and $\overline{G'}$ is isomorphic to S . Let $\overline{H} := \overline{G'} \times C_{\overline{G'}}(\overline{G'})$, where $\overline{G} := G/M$, then $|\overline{G}/\overline{H}|$ divides $|Out(\overline{G'})|$. If $\psi \in Irr(\overline{H})$, and $\chi \in Irr(G)$ is a constituent of $\psi^{\overline{G}}$, then $\chi(1) = n\psi(1)$, for $n||Out(\overline{G'})|$. For each case, we can choose a suitable degree $\psi(1)$ such that $\chi(1) \notin cd(G)$. By this, we get a contradiction. For example, if $\overline{G'} \cong A_6$, we consider $\psi(1) = 9$, thus G has a degree of the form $3^2, 2 \cdot 3^2$ or $2^2 \cdot 3^2$, a contradiction.

Let $S \cong A_5$, then $k \leq 3$ because 3^a does not divide any degrees of G , where $a > 3$. If $k = 1$, then $2^a \cdot 5 \in cd(G)$, where $a = 0, 1$ or 2 , a contradiction. If $k = 2$, then $2^a \cdot 5 \in cd(G)$, where $a = 0, 1, 2$ or 3 , a contradiction. If $k = 3$, then $2^a \cdot 3^b \cdot 5 \in cd(G)$, where $a = 0, 1$ or 2 and $b = 0, 1$, a contradiction.

Therefore, $S \cong G_2(5)$. Since G does not have any degrees divisible by $7^2, k = 1$. Thus, $\overline{G'} \cong G_2(5)$.

Step 3. For all linear irreducible characters ϑ of M , we have $I_{G'}(\vartheta) = G'$ and $G' \cong G_2(5)$.

Suppose $I_{G'}(\vartheta) = I < G'$, for some $\vartheta \in Lin(M)$. If $\vartheta^I = \sum_{i=1}^k \varphi_i$ with $\varphi_i \in Irr(I)$, then by Lemma 2.2(a) we have $\varphi_i(1)|G' : I| \in cd(G')$. Let $\overline{U} := U/M$ be a maximal subgroup of $\overline{G'} \cong G_2(5)$ containing $\overline{I} := I/M$. By [1, p. 114], \overline{U} is isomorphic to one of the following groups:

- (1) $5_+^{1+4} : GL_2(5)$ or $5^{2+3} : GL_2(5)$ of index 3906;
- (2) $3 \cdot U_3(5) : 2$ of index 7750;
- (3) $L_3(5) : 2$ of index 7875;
- (4) $2 \cdot (A_5 \times A_5) \cdot 2$ of index 406875;
- (5) $U_3(3) \cdot 2$ of index 484375;
- (6) $2^3 \cdot L_3(2)$ of index 4359375.

Since $|G' : U|$ divides some degrees of G , the last four cases cannot hold. Therefore, we have only the following two possibilities:

- (1) $\overline{U} \cong A : GL_2(5)$ of index 3906, where $A = 5_+^{1+4}$ or 5^{2+3} ; then, $|U : I|\varphi_i(1) = 1, 2^2$ or 5 .

Suppose $\varphi_i(1) = 1$, for some i , then $|U : I| = 1, 2, 4$ or 5 . Thus, φ_i is an extension of ϑ to I , and by Lemma 2.2(b) we have $|G' : I|\varphi_i(1)\theta(1) \in cd(G')$, for all $\theta \in Irr(I/M)$. If $|U : I| = 1$, then $I = U$, so G has a degree divided by $2^2 \cdot 3^3 \cdot 7 \cdot 31$, a contradiction. If $|U : I| = 2$, then by Lemma 2.6, \overline{I} has a subgroup which is isomorphic to $SL_2(5)$. So G has a degree which is divisible

by $2^2 \cdot 3^3 \cdot 7 \cdot 31$, a contradiction. If $|U : I| = 4$, then by Lemma 2.6, we have $A \leq \bar{I}$, then $\bar{I}/A \leq \bar{U}/A \cong GL_2(5)$. Thus, \bar{I} is contained in a maximal subgroup T of $GL_2(5)$ of order $2^4 \cdot 3 \cdot 5$, which is nonabelian. Therefore, there exists a character degree of G divided by a proper multiple of $2^3 \cdot 3^2 \cdot 7 \cdot 31$, a contradiction. Let $|U : I| = 5$ and $B := A \cap \bar{I}$. So by the same method as before \bar{I}/B must be a maximal subgroup of $GL_2(5)$ of order $2^5 \cdot 3$, which is nonabelian, so is \bar{I} . Moreover, there is an irreducible character of G which is divided by $2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 31 \cdot \theta(1)$, where $\theta \notin Lin(\bar{I})$, a forbidden degree.

Therefore, $\varphi_i(1) \neq 1$, for all i . Then, $U = I$ or $|U : I| = 2$, and we have $(\varphi_i)_M = e\vartheta$, where $e = 2, 4, 5$. Moreover, by Lemma 2.6, I has a subgroup S , where $\bar{S} := S/M$ is isomorphic to $SL_2(5)$. Since the Schur multiplier of \bar{S} is 1, ϑ is extended to an irreducible character ϑ_0 of S . So by Lemma 2.5, there exists i , such that $\varphi_i(1) \geq \theta(1)$, for all $\theta \in Irr(\bar{S})$. Since \bar{S} has an irreducible character of degree 6, we get a contradiction.

(2) $\bar{U} \cong 3 \cdot U_3(5) : 2$ of index 7750; then, $|U : I|\varphi_i(1) = 2$.

If there exists $\varphi_i \in Irr(I)$, with $\varphi_i(1) = 1$, by the same method as above we view a contradiction. Therefore, for all i , $\varphi_i(1) = 2$ and $I = U$, so \bar{U} has a normal subgroup of structure $T/M := 3 \cdot U_3(5)$. Hence there exists $B \leq T$ such that $B/M \cong 3 \cdot P$, where P is a nonabelian Sylow 5-subgroup of $U_3(5)$. By Ito's theorem, the character degrees of B are power of 5. We claim that $(\varphi_i)_B \notin Irr(B)$. Otherwise, Lemma 2.1(a) implies that $(\varphi_i)_M \in Irr(M)$, so by Lemma 2.1(b), $\varphi_i(1)\eta(1) \in cd(B)$ for all $\eta \in Irr(B/M)$, which is a contradiction. So $(\varphi_i)_B = \lambda_1 + \lambda_2$, where $\lambda_i(1) = 1$, then ϑ is extended to λ_i of $Irr(B)$. By Lemma 2.5 we deduce that $\varphi_i(1) \geq \theta(1)$, where $\theta \in Irr(B/M)$; therefore, $\varphi_i(1) \geq 5$, a contradiction.

According to the above argument, we have $I = G'$. By Lemma 2.4, $|M/M'|$ divides the order of Schur multiplier of $\bar{G}' \cong G_2(5)$, which is equal to 1, so $M = M'$. If $M \neq 1$, then there is $\vartheta \in Irr(M)$ which is nonlinear and it is extended to an irreducible character ϑ_0 of G' because the Schur multiplier of \bar{G}' is trivial. Thus, by Lemma 2.1(b) we have $\vartheta_0\theta \in Irr(G')$, for all $\theta \in Irr(\bar{G}')$. Therefore, $2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 31 \cdot \vartheta(1)$ divides some degrees of G , a contradiction. Hence, $M = 1$ and $G' \cong G_2(5)$.

Step 4. $G \cong G' \times A$, where A is an abelian group.

Let $A = C_G(G')$. Since $G' \cap A = 1$ and $G'A = G' \times A$,

$$G' \cong \frac{G'}{G' \cap A} \cong \frac{G'A}{A} \trianglelefteq \frac{G}{A} \leq Aut(G').$$

Hence, $|\frac{G}{G' \times A}|$ divides $|Out(G_2(5))|$, which is equal to 1. Therefore $G \cong G_2(5) \times A$, where $A = C_G(G_2(5))$ is an abelian group. \square

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