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On the Brauer Monoids

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Abstract. The Brauer monoid is defined to be the set of all equivalence classes under a certain relation on the set of all weak 2-cocycles. It is not an easy task to compute such a monoid in general. In [HLS], it was shown that this monoid can be given as a disjoint union of groups. Again, it is quite difficult to compute such groups. Stimets [S] gave a condition under which these groups can be given in terms of the well-known Ext-groups which are relatively easy to deal with. In this work, I give situations in which Stimets' condition is always satisfied. I also introduce a dimension notion for the graded module that is used in Stimets' condition. At the end of this paper, I use the dimension notion to deduce the dimension for the graded module in some classes explicitly, and demonstrate some idempotents for which the chain complex is not exact.

0. Introduction

In classical group cohomology theory (see [HS]), one considers a group G, a G-module A, and a sequence of boundary operators ∂_n mapping functions f on n-tuples of elements of G taking values in A to such functions on $(n+1)$-tuples. One considers the kernel of the map ∂_n, up to the image of the map ∂_{n-1}, and examines the resulting algebraic structure. The object in question, denoted $H^n(G, A)$, is an Abelian group which is an invariant of the G-module A. The elements of $H^n(G, A)$ are classes of n-cocycles.

If G is the Galois group of a field K over some other field F, then K^* is a G-module, and an element f of $H^2(G, K^*)$ can be thought of as determining an algebra A_f, via the so-called "crossed-product" construction. The algebra A_f is simple with center F, and represents an element of the relative Brauer group $Br(K/F)$. There is also a reverse correspondence: each element of the Brauer group is suitably represented by a "crossed-product" algebra (see [FD]).

In 1979, [HLS] examined the question of what happens in the case $n = 2$ and when the functions f take on values not in the G-module K^* but in the full field K. That is, f could occasionally (but not always) take on zero values. Given some small restrictions to discard any uninteresting manifestations, the resulting algebraic structure is again considered. In this case, however, the object in question is no longer a group, but is merely a commutative monoid, denoted $M^2(G,K)$. The elements here are called "weak 2-cocycles." The Brauer monoid $M^2(G,K)$ is thus a generalization of the classical cohomology $H^2(G,K^*)$. Again one has a crossed-product construction and the resulting algebras are called "strongly primary" (not necessarily central-simple) F-algebras. Haile [H2] showed any strongly primary algebra is equivalent to one arising from a weak cocycle. For this reason, we will refer to the "Brauer monoid" as the set of all weak cocycles modulo the classical co-boundary relation given by ∂, i.e. elements of the image of ∂_1 on cochains which take on *all non-zero values.*

The monoid $M^2(G,K)$ is "regular" in the sense that it splits into the disjoint union of subsets, each of which has a unique idempotent which acts as an identity, and these sets can be regarded as groups with respect to this idempotent. In addition to the theory on idempotents in an Abelian, regular monoid, the idempotents themselves impose partial orderings on the group G (or more accurately sets of cosets of a certain subgroup H, called the inertial subgroup, of G) which satisfy certain properties known collectively as "lower subtractivity." Let e be such an idempotent cocycle. Then $eM^2(G,K)$ is a group in its own right, with e as the identity element — we denote the group more simply as $M^2_e(G,K)$. The idempotent, and hence the group, is associated to a particular lower subtractive partial order $\mathcal{P}$. It is the partial order and its effects on the structure of $M^2_e(G,K)$ that we will be most concerned with.

1. Preliminaries

Let K/F be a finite Galois extension of fields and let G be its Galois group. A weak 2–cocycle is a function $f : G\times G \to K$ satisfying

(i) $f^\sigma(\tau,\nu)f(\sigma,\tau\nu) = f(\sigma\tau,\nu)f(\sigma,\tau)$

(ii) $f(\sigma,1) = f(1,\sigma) = 1$

for all $\sigma,\tau,\nu\in G$.

We define a relation "$\leq$" on G by $\sigma\leq\tau$ if and only if $f(\sigma,\sigma^{-1}\tau)\neq 0$. This relation is lower subtractive, that is if $\sigma\leq\tau$ then $\sigma\leq\alpha\leq\tau$ if and only if $\sigma^{-1}\alpha\leq\sigma^{-1}\tau$. The set $H = \{\sigma\in G : \sigma\leq 1\}$ is a subgroup of G called the inertial subgroup of f. The relation induces a partial order on G/H given by $\sigma H\leq\tau H$ if $\sigma\leq\tau$, and H has the property $f(H\times H)\subseteq K^*$. It turns out that the relation on G/H is still lower subtractive with a unique minimal element H [A].

Given a weak 2-cocycle f, define a function $e : G\times G\to\{0,1\}$ by $e(\sigma,\tau) = 0$ if and only if $f(\sigma,\tau) = 0$. Then e is a weak 2-cocycle called the idempotent weak 2-cocycle associated to f. Two weak 2-cocycles f, g are called cohomologous (or equivalent) if there is a function $\alpha : G\to K^*$ such that

$$f(\sigma,\tau) = \frac{\alpha(\sigma)\alpha^\sigma(\tau)}{\alpha(\sigma\tau)}\,g(\sigma,\tau) \quad \text{for all } \ \sigma,\tau\in G.$$

Any two cohomologous weak 2-cocycles have the same associated idempotent cocycle. Under the equivalence relation introduced above the set of classes of weak 2-cocycles from $G \times G$ to K forms a monoid denoted by $M^2(G, K)$. The subgroup of invertible elements of this monoid is the usual cohomology group $H^2(G, K^*)$.

Let e be an idempotent weak 2-cocycle. If f is a weak 2-cocycle associated to e then we can define a function $g : G \times G \to K$ by

$$g(\sigma, \tau) = \begin{cases} (f(\sigma, \tau))^{-1} & \text{if } f(\sigma, \tau) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then g is a weak 2-cocycle associated to e. If $[\,\cdot\,]$ denotes the equivalence class in the relation above, let $M_e^2(G, K) = \{[f] \in M^2(G, K) \mid [f][e] = [f]$ and there is a weak 2-cocycle g such that $[f][g] = [e]\}$. Then $M_e^2(G, K)$ is a group with identity $[e]$ and $M^2(G, M) = \cup_e M_e^2(G, K)$ (disjoint) where the union is over all idempotent weak 2-cocycles, [H1]. It is worthwhile to compute such groups to obtain the monoid and hence the corresponding Brauer monoid.

The set of all idempotent cocycles on $G \times G$ with inertial subgroup H is in $1 - 1$ correspondence with all lower subtractive graphs on G/H with a unique root H. Whenever we refer to the graph for e, we mean the graph associated to the weak 2-cocycle e. If $[f] \in M_e^2(G, K)$ then f and e have the same associated graph. The following results and definitions are from [S]. Each lower subtractive graph determines a ring $R_e = \mathbb{Z}\{x_\sigma : \sigma \in G\}/I_e$ where I_e is the ideal generated by $\{x_\sigma x_\tau - x_{\sigma\tau} \mid \sigma \leq \sigma\tau\}$. The ring R_e is called the derived ring of e.

Define a graded R_e–module $\mathcal{M}$ by $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} M_n$ where

$$M_n = \begin{cases} \displaystyle\bigoplus_{g_1 \leq \cdots \leq g_n} R_e[g_1, \ldots, g_n] & n \geq 1 \\ R_e & n = 0 \\ \mathbb{Z} & n = -1 \\ 0 & n \leq -2 \end{cases}$$

with differentials

$$\partial_n[g_1, \ldots, g_n] = x_{g_1}[g_1^{-1}g_2, \ldots, g_1^{-1}g_n] + \sum (-1)^i [g_1, \ldots, \hat{g}_i, \ldots, g_n].$$

The pair $(\mathcal{M}, \partial)$ is a chain complex called the chain complex of the idempotent e. Suppose f is a function from G^n to a field K which satisfies

(i) if $f(g_1, g_2, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_n) \neq 0$ then $f(g_1, g_2, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_n) = 1$ for all $g_1, g_2, \ldots, g_n$ in G and all $1 \leq i \leq n$,

(ii) $f(1, 1, \ldots, 1, g, 1, \ldots, 1) = 1$ for all $g \in G$, and

(iii) for each $g_1, g_2, \ldots, g_{n+1}$ in G,

$$\begin{aligned} & f^{g_1}(g_2,\ldots,g_{n+1})\prod_{i \text{ even}} f(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}) \\ = \quad & f(g_1,\ldots,g_n)\prod_{i \text{ odd}} f(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}) \end{aligned}$$

if n is even, and

$$\begin{aligned} & f^{g_1}(g_2,\ldots,g_{n+1})f(g_1,\ldots,g_n)\prod_{i \text{ even}} f(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}) \\ = \quad & \prod_{i \text{ odd}} f(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}) \end{aligned}$$

if n is odd. We then call f a *weak n-cocycle*. The first condition is analogous to the standard degeneracy conditions used in homological algebra, but still allows for the possibility that certain cochains may take on non-invertible values.

The second condition ensures that the cochains (cocycles) have a sufficient amount of invertibility.

If f is a weak n-cocycle and there is another weak n-cocycle g and an invertible cochain $\beta : G^{n-1} \to K^*$ such that $f = \partial\beta \cdot g$, where

$$\begin{aligned} & \partial\beta(g_1,\ldots,g_n) \\ = \quad & \beta^{g_1}(g_2,\ldots,g_n)\beta^{(-1)^n}(g_1,\ldots,g_{n-1})\prod \beta^{(-1)^i}(g_1,\ldots,g_ig_{i+1},\ldots,g_n), \end{aligned}$$

then we say f is cohomologous to g and write $f \sim g$. Let $M^n(G,K)$ be the monoid of weak n-cocycles modulo the equivalence relation $\sim$. Then $M^n(G,K)$ is called the weak Galois cohomology monoid. The cocycles equivalent (cohomologous) to the identity are known as *weak coboundaries*.

The pair $(\mathcal{M},\partial)$ is always exact at M_0.

If $(\mathcal{M},\partial)$ is exact then $M^i_e(G,K) \simeq Ext^i_{R_e}(\mathbb{Z},K^*)$. It is quite difficult to check exactness in general but using the "contraction process" makes the situation somewhat easier. If $g_1 < \cdots < g_n$, $g_i \neq 1$ call $[g_1,\cdots,g_n]$ an (n)–cell in M_n.

Proposition 1.1 (Contraction process)**.** (See [S]). *Let $C_n \in M_n$, $C_{n-1} \in M_{n-1}$ be two cells such that C_n does not appear in the boundary of M_{n+1} and C_{n-1} appears only once in the boundary of C_n, then the chain complex $(\mathcal{M}',\partial)$ obtained by removing C_n and C_{n-1} is exact at M'_n if and only if $(\mathcal{M},\partial)$ is exact at M_n. Moreover, $(\mathcal{M},\partial)$ and $(\mathcal{M}'.\partial)$ have the same homology groups.*

This procedure allows us to cancel cells gradually until we reach a point after which we can not proceed any further. Then we can investigate exactness at fewer modules.

2. Trees

Since the lower subtractive relation on G/H is a partial order, there is a unique graph rooted at H and associated to f. It is the same graph as the graph of e where $[f] \in M_e^2(G,K)$. In this section, suppose this graph is a tree.

Lemma 2.1. *A lower subtractive chain: $1 \leq 2 \leq \cdots \leq k$ defined on the cyclic group $\mathbb{Z}_{k+1} = \{0, 1, \ldots, k\}$ can be contracted to a point.*

Proof. We only have one k–cell, so

$$\partial_k[1,2,\ldots,k] = x[1,2,\ldots,k-1] + \sum_{i=1}^{k}(-1)^i[1,2,\ldots,\hat{i},\ldots,k].$$

Excise the cells $[2,3,\ldots,k]$ and $[1,2,\ldots,k]$. Now, we have $k-1$ $(k-1)$–cells, and

$$\begin{aligned}
&\partial_{k-1}[1,3,\ldots,k] = x[2,3,\ldots,k-1] - \underline{[3{,}4,\ \ldots\ {,}k]} + \text{cells begin with 1}\\
&\partial_{k-1}[1,2,4,\ldots,k] = x[1,3,\ldots,k-1] - \underline{[2{,}4,\ \ldots\ {,}k]} + \text{cells begin with 1}\\
&\vdots\\
&\partial_{k-1}[1,2,\ldots,k-1] = x[1,2,\ldots,k-2] - \underline{[2{,}3,\ \ldots\ {,}k\text{-}1]} + \text{cells begin with 1.}
\end{aligned}$$

All underlined cells are excised with the left hand side cells. Continue in this process and note that in each step of the excision operation we can excise all cells with coefficient ± 1 which begin with an element different from 1. This can be done because such cells can not be repeated in the right hand side except for one cell which can be excised at the end. □

Theorem 2.2. *Let Γ_e be a lower subtractive tree with two generators over a finite Abelian group G where e is an idempotent weak 2-cocycle. Then $M_e^i(G,K) = 0$ for $i > 1$. Moreover, $M_e^1(G,K) \approx \mathrm{Ext}^1_{\mathbb{Z}\{x,y\}}(\mathbb{Z}, K^*)$ and $M_e^0(G,K) \approx Ext^0_{\mathbb{Z}\{x,y\}}(\mathbb{Z}, K^*)$.*

Proof. The graph has two generators with no relations between them since it is a tree, so the derived ring obtained from Γ_e is $\mathbb{Z}\{x,y\}$. Now, we construct such a graph over an arbitrary finite Abelian group G. We only have two elements above the identity a, b (say). The height–2 elements above a could be ab and a^2, and the height two elements above b could be ba and b^2. But $ab = ba$, so this element can not be above both. Without loss of generality, let ab, a^2 be above a and b^2 above b. Using this restriction to build such a graph will finally give a graph of this form

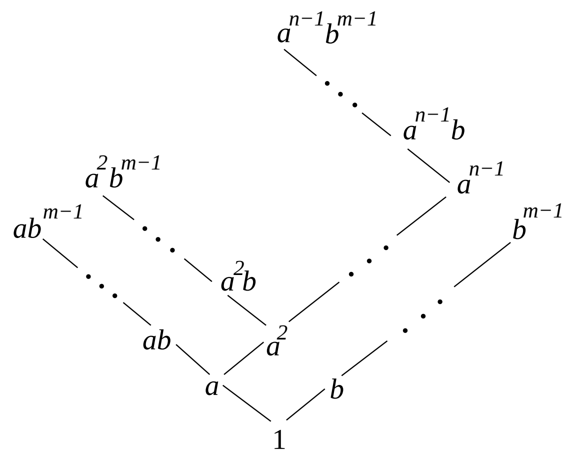

FIGURE 1

where order $a = n$, order $b = m$, or any lower subtractive subgraph of it. This graph is over the group $\mathbb{Z}_n \times \mathbb{Z}_m$. Using the same method, it turns out that we can not construct a lower subtractive tree with two generators over $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_j}$ where $n_i > 1$, $j > 2$. Now, each n–cell starting with an element different from a or b can be excised as in Lemma 2.1. So, we get the 2–cells $[b, b^i]$, $2 \leq i < m$ and $[a, g]$, $g \notin \{1, a, b, b^2, \ldots, b^{m-1}\}$, which obviously can be excised to get only the two generators a and b.

We need to check exactness of the chain complex $(\mathcal{M}', \partial) : 0 \to M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ only at $M_1 = \mathbb{Z}\{x, y\}[a] \oplus \mathbb{Z}\{x, y\}[b]$. In contrast to the case in which the derived ring is commutative, this chain complex has non-zero Euler characteristic, nevertheless it is exact. We show injectivity of ∂_1. Let $\partial_1(\alpha[a] - \beta[b]) = 0$, this implies $\alpha(x_a - 1) = \beta(x_b - 1)$. Let $\sum_{i=1}^{s} a_i f_i$, $\sum_{i=1}^{t} b_i g_i$ be the sums of the "words" of the highest degrees (in the usual definition) in α, β respectively. The equation $\alpha x_a - \beta x_b = \alpha - \beta$ implies $\sum_{i=1}^{s} a_i f_i x_a = \sum_{i=1}^{t} b_i g_i x_b$. But this can not happen unless $\sum_{i=1}^{s} a_i f_i = \sum_{i=1.}^{t} b_i g_i = 0$. So, $\alpha = \beta = 0$, thus d_1 is injective. So, $\mathcal{M}'$ is exact and hence the original chain complex $(\mathcal{M}, \partial)$ is exact and they have the same cohomology groups. And we get $M_e^i(G, K) \approx \mathrm{Ext}^i_{\mathbb{Z}\{x,y\}}(\mathbb{Z}, K^*)$ for all i. □

Lemma 2.3. *If Γ_e is a lower subtractive tree over a finite group where e is an idempotent weak 2-cocycle, then it can be contracted to the height–1 elements.*

Proof. In Lemma 2.1, we showed that this is true if the graph is a chain. Let C_f, C_g, $C_h, \ldots$ be m–cells of the maximal length.

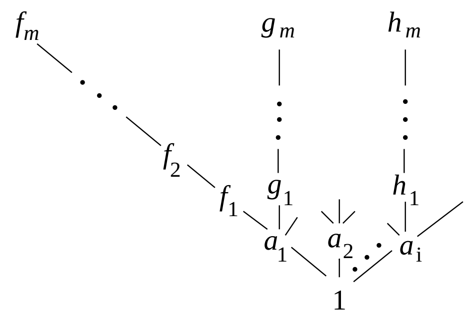

FIGURE 2

$$\partial_{m+1}(C_f) = x_{a_1}[a_1^{-1}f_1, \dots, a_1^{-1}f_m] - [f_1, \dots, f_m] + [a_1, f_2, \dots, f_m] - \dots$$
$$\partial_{m+1}(C_g) = x_{a_1}[a_1^{-1}g_1, \dots, a_1^{-1}g_m] - [g_1, \dots, g_m] + [a_1, g_2, \dots, g_m] - \dots$$
$$\vdots$$
$$\partial_{m+1}(C_h) = x_{a_i}[a_i^{-1}h_1, \dots, a_i^{-1}h_m] - [h_1, \dots, h_m] + [a_i, h_2, \dots, h_m] - \dots$$

All m–cells which do not begin with generators (elements of height 1) can be excised with all $(m+1)$–cells.

In the next step, we only express the images of the m–cells which begin with generators, because any other m–cell has been already excised.

We now excise all $(m-1)$–cells which begin with non-generator elements and end with elements of the maximal height. This can be done because such cells can not be among the "inverse" terms and can not be repeated either since the graph is a tree. After excising these cells, we will be able to excise the rest of $(m-1)$–cells which begin with non-generators. By the same method, we excise all i–cells that do not begin with generators. At the same time, we have excised i–cells $(i > 1)$ which begin with generators by excising the left hand side of the equations. We now end up with the 2–cells: $[a_1, t_{1i}]$, t_{1i} is right above a_1. $[a_2, t_{2i}]$, t_{2i} is right above $a_2 \dots$. But in this stage we can excise t_{ji} gradually. ☐

Example 2.1. *Let e be given over $\mathbb{Z}_{10}$ by:*

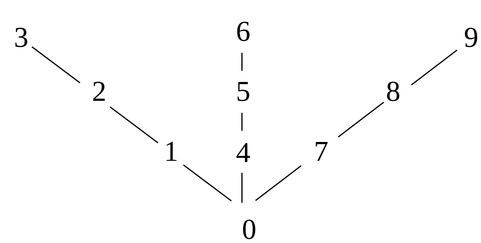

FIGURE 3

$$\begin{aligned}\partial_3[1,2,3] &= x[1,2] - \underline{[2,3]} + [1,3] - [1,2]\\ \partial_3[4,5,6] &= y[1,2] - \underline{[5,6]} + [4,6] - [4,5]\\ \partial_3[7,8,9] &= z[1,2] - \underline{[8,9]} + [7,9] - [7,8]\end{aligned}$$

Next step:

$$\begin{aligned}\partial[1,2] &= x[1] - \underline{[2]} + [1]\\ \partial[1,3] &= x[2] - \underline{[3]} + [1]\\ \partial[4,5] &= y[1] - \underline{[5]} + [4]\\ \partial[4,6] &= y[2] - \underline{[6]} + [4]\\ \partial[7,8] &= z[1] - \underline{[8]} + [7]\\ \partial[7,9] &= z[2] - \underline{[9]} + [7]\end{aligned}$$

Notice that we must excise [9], [6] and [3] first in order to be able to excise [2].

Theorem 2.4. *Let Γ_e be a lower subtractive tree over a finite Galois group G with m generators $a_1, a_2, \ldots, a_m$. Then $M_e^i(G,K) = 0$ for $i > 1$, and $M_e^i(G,K) \approx \operatorname{Ext}^i_{\mathbb{Z}\{x_1,\ldots,x_m\}}(\mathbb{Z}, K^*)$ for $i = 0, 1$.*

Proof. Since the given graph is a tree, there is no relation between the generators and hence the derived ring is $R_e = \mathbb{Z}\{x_1, \ldots, x_m\}$. So, the original chain complex is exact if and only if the following chain complex is exact:

$$0 \longrightarrow M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0, \quad \text{where } M_0 = R_e,\ M_1 = \bigoplus_{i=1}^{m} R_e[a_i].$$

We only need to check that ∂_1 is injective. Let $\delta_1(\sum_{i=1}^m \alpha_i a_i) = 0$. This implies that $\sum_{i=1}^m \alpha_i(x_i - 1) = 0$ or $\sum_{i=1}^m \alpha_i x_i = \sum_{i=1}^m \alpha_i$. Let $\sum_j \lambda_{ij} f_{ij}$ be the sum of words of the highest degree in α_i. This implies the terms $\sum_j \lambda_{1j} f_{1j} x_1$, $\sum_j \lambda_{2j} f_{2j} x_2, \ldots, \sum_j \lambda_{mj} f_{mj} x_m$ must cancel one another, but this can not happen unless $\sum_j \lambda_{ij} f_{ij} = 0$ for all i. So ∂_1 is injective and the chain complex is exact. The proof is complete since the original module and the reduced module have the same cohomology groups. □

3. A Dimension for the Graded Module $\mathcal{M}$

Let e be an idempotent weak 2-cocycle defined on a finite Galois group G.

Definition 3.1. *If R is a ring, M is an R–module, the projective dimension of M, denoted by $P\text{-}\dim_R M$, is defined by*

$$\begin{aligned}&P\text{-}\dim_R M\\ = \ &\inf\{n \mid 0 \to P_n \to \cdots \to P_0 \to M \to 0 \ \ \textit{is a projective resolution of } M\}.\end{aligned}$$

It can be shown that the following statements are equivalent:

(i) $P\text{-}\dim_R M \leq n$.
(ii) $\operatorname{Ext}^i_R(M, -) = 0$ *for* $i > n$.
(iii) $\operatorname{Ext}^{n+1}_R(M, -) = 0$

Definition 3.2. *We define the dimension of the graded module $\mathcal{M}_e$, denoted by* $\dim \mathcal{M}_e$*, by*

$$\dim \mathcal{M}_e = P\text{-}\dim_{R_e}\mathbb{Z},$$

provided the chain complex $(\mathcal{M}_e, d)$ is exact.

Remark 3.1. *Although P-$\dim_{R_e}\mathbb{Z}$ is always meaningful, we are only interested in the case when $\mathcal{M}_e$ is exact which gives a connection between the groups $M_e^i(G,K)$ and Ext-groups.*

From this definition, it is clear that

$$\begin{aligned} \dim \mathcal{M}_e &= \inf\left\{n \mid \operatorname{Ext}^i_{R_e}(\mathbb{Z}, -) = 0 \quad \text{for } i > n\right\} \\ &= \sup\left\{n \mid \operatorname{Ext}^n_{R_e}(\mathbb{Z}, M) \neq 0 \quad \text{for some } R_e\text{–module } M\right\}. \end{aligned}$$

Now, if the inertial subgroup $H = \{1\}$, then $\dim \mathcal{M}_e \leq n$ for some integer n because e admits an exact free R_e–module for $\mathbb{Z}$, namely $0 \to M_n \to M_{n-1} \to \cdots \to M_0 = R_e \to \mathbb{Z} \to 0$.

Example 3.1. *If G is a nontrivial finite group, then P-$\dim_{\mathbb{Z}G}\mathbb{Z} = \infty$. To see this, note that G contains a cyclic subgroup $H = \langle t \rangle$, $|H| = m > 1$. Consider the projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$:*

$$\cdots \xrightarrow{N} \mathbb{Z}H \xrightarrow{t-1} \mathbb{Z}H \xrightarrow{N} \mathbb{Z}H \xrightarrow{t-1} \mathbb{Z}H \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0 \quad \textit{where } N = \sum_0^{m-1} t^i.$$

Take $Hom(\cdot, \mathbb{Z})$ where H acts trivially on $\mathbb{Z}$. Replace the first term by 0 and use the identification $Hom_{\mathbb{Z}H}(\mathbb{Z}H, \mathbb{Z})] \approx \mathbb{Z}$ to get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{t-1} \mathbb{Z} \xrightarrow{N} \cdots \xrightarrow{t-1} \mathbb{Z} \xrightarrow{N} \mathbb{Z} \xrightarrow{t-1} \cdots.$$

So $H^n(H, \mathbb{Z}) = \mathbb{Z}^H/N\mathbb{Z} = \mathbb{Z}/m\mathbb{Z} \neq 0$ for any positive even integer $n \Longrightarrow P$-$\dim_{\mathbb{Z}H}\mathbb{Z} = \infty$. Since $H < G$, any projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ can be regarded as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. This yields P-$\dim_{\mathbb{Z}H}\mathbb{Z} \leq P$-$\dim_{\mathbb{Z}G}\mathbb{Z} \Longrightarrow P$-$\dim_{\mathbb{Z}G}\mathbb{Z} = \infty$.

Proposition 3.2. *If e is an idempotent with inertial subgroup H, then $\mathbb{Z}H$ is a subring of R_e.*

Proof. The set of generators of $\mathbb{Z}H$ is contained in the set of generators of R_e and each relation in $\mathbb{Z}H$ holds in R_e, so there is a well-defined map $\mathbb{Z}H \xrightarrow{i} R_e$ taking x to x. Now the set of generators of R_e is the same as the set of generators of $\mathbb{Z}G$ and each relation in R_e holds in $\mathbb{Z}G$ ($\mathbb{Z}G$ has more relations). So there is a well-defined map $R_e \xrightarrow{i'} \mathbb{Z}G$ taking x to x. The composition $i' \circ i$ takes $x \in \mathbb{Z}H$ to $x \in \mathbb{Z}G$ which shows the injectivity of i. □

Theorem 3.3. *Let e be a lower subtractive graph with inertial subgroup H. Then R_e is a free $\mathbb{Z}H$–module.*

Proof. Let $g_1H, g_2H, \dots, g_kH$ be of height-1 elements. Let $S = \{x_{\sigma_1}x_{\sigma_2}\dots x_{\sigma_r} \mid r \geq 1,\, \sigma_i \in \{g_1, g_2, \dots, g_k\}\} \cup \{1\}$.

Define an equivalence relation on S by $u \sim v$ if there exists $h \in H$ such that $ux_h = v$. Let $\widetilde{S} = \{[s] \mid s \in S\}$ be the set of all equivalence classes and T be a set of class representatives. If $m \in R_e$ is a monomial in $\{x_g \mid g \in G\}$, then there is a unique element u of T such that $m = ux_h$ for some $h \in H$. Denote u by $\widetilde{m}$ and x_h by $\widehat{m}$ so that $m = \widetilde{m}\widehat{m}$. We claim the following statements:

(1) If m_1, m_2 are monomials in $\{x_g \mid g \in G\}$, then $\widetilde{m_1\widetilde{m}_2} = \widetilde{m_1m_2}$.
(2) $\bigcup_{h\in H} Tx_h$ is a disjoint union.
(3) $\bigcup_{h\in H} Tx_h$ is a $\mathbb{Z}$–basis for R_e.

For (1), let $m_2 = \widetilde{m}_2x_h$ for some $h \in H$, so $\widetilde{m_1\widetilde{m}_2} = m_1m_2x_{h^{-1}} = \widetilde{m_1m_2}$. For (2), notice that if $t_1x_{h_1} = t_2x_{h_2}$ then $t_1x_{h_1}x_{h_2^{-1}} = t_2$ which means that $t_1 \sim t_2$ or $t_1 = t_2$ in T. To show (3), form the free $\mathbb{Z}$-module M on the set $\{Y_{t,x_h} \mid t \in T, h \in H\}$ where Y_{t,x_h} is corresponding to tx_h in R_e. Define an action of R_e on M via

$$x_\sigma \cdot Y_{t,x_h} = Y_{\widetilde{x_\sigma t},\widehat{x_\sigma t}\cdot x_h}, \quad \sigma \in G.$$

We need to show that this is a well-defined module action. Notice that for $\sigma, \tau \in G$ we have

$$\begin{aligned} x_\tau(x_\sigma \cdot Y_{t,x_h}) &= x_\tau \cdot (Y_{\widetilde{x_\sigma t},\, \widehat{x_\sigma t}\cdot x_h}) = Y_{\widetilde{x_\tau\widetilde{x_\sigma t}},\, \widehat{x_\tau\widetilde{x_\sigma t}}\,\widehat{x_\sigma t}\,x_h}. \\ (x_\tau x_\sigma)\cdot Y_{t,x_h} &= Y_{\widetilde{x_\tau x_\sigma t},\, \widehat{x_\tau x_\sigma t}\cdot x_h}. \end{aligned}$$

Statement (1) shows that $\widetilde{x_\tau\widetilde{x_\sigma t}} = \widetilde{x_\tau x_\sigma t}$. Also, $\widehat{x_\tau\widetilde{x_\sigma t}} = \widehat{x}_\tau\,\widehat{\widetilde{x_\sigma t}} = \widehat{x}_\tau$ (since $\widehat{\widetilde{x_\sigma t}} = 1$) so $\widehat{x_\tau\widetilde{x_\sigma t}}\cdot\widehat{x_\sigma t} = \widehat{x}_\tau\widehat{x_\sigma t} = \widehat{x_\tau x_\sigma t}$. Now define a homomorphism $R_e \xrightarrow{\varphi} \mathrm{End}_{\mathbb{Z}}(M)$ by

$$a \mapsto (\mu \mapsto a\cdot\mu).$$

If $q = \sum_{\substack{t\in T\\ h\in H}} b_{t,x_h}t\cdot x_h = 0$ (finite sum), then

$$\begin{aligned} 0 &= \varphi(q)(1) = \sum_{\substack{t\in T\\ h\in H}} b_{t,x_h}tx_h\cdot 1 = \sum_{\substack{t\in T\\ h\in H}} b_{t,x_h}Y_{\tilde{t},\hat{x}_h} \\ &= \sum_{\substack{t\in T\\ h\in H}} b_{t,x_h}Y_{t,x_h} \Longrightarrow b_{t,x_h} = 0 \quad \text{for all } t, h. \end{aligned}$$

This shows claim (3) and hence the freeness of R_e as a $\mathbb{Z}$–module. Statement (2) above tells that R_e is a free $\mathbb{Z}H$–module with basis T. □

Theorem 3.4. *(See Theorem 5.1 in [O]). Let U be an exact functor: ${}_{R^*}\mathcal{M} \to {}_R\mathcal{M}$ which is covariant, then for all $M \in {}_{R^*}\mathcal{M}$:*

$$P\text{-}\dim_R UM \leq P\text{-}\dim_{R^*} M + P\text{-}\dim_R UR^*.$$

Corollary 3.5. *For any lower subtractive graph e over a finite group G with inertial subgroup H, we have*

$$P\text{-}\dim_{\mathbb{Z}H}\mathbb{Z} \leq dim\,\mathcal{M}_e.$$

Proof. In the theorem above, let $R^* = R_e$, $R = \mathbb{Z}H$ to obtain

$$P\text{-}\dim_{\mathbb{Z}H}\mathbb{Z} \leq P\text{-}\dim_{R_e}\mathbb{Z} + P\text{-}\dim_{\mathbb{Z}H}R_e.$$

But since R_e is a free $\mathbb{Z}H$–module (by Theorem 3.3), so it is projective and we find $P\text{-}\dim_{\mathbb{Z}H}R_e = 0$ by considering $0 \to R_e \xrightarrow{i} R_e \longrightarrow 0$ as a projective resolution of R_e over $\mathbb{Z}H$. Now, $P\text{-}\dim_{R_e}\mathbb{Z} = \dim\mathcal{M}_e$ by definition, so the inequality follows. □

Theorem 3.6. $\dim\mathcal{M}_e = \infty$ *if and only if* $H \neq \{1\}$.

Proof. (⇒) If $H = \{1\}$ then the discussion after Definition 3.2 shows that $\dim\mathcal{M}_e$ is finite.

(⇐) If $H \neq \{1\}$, then Example 3.1 gives that $\dim_{\mathbb{Z}H}\mathbb{Z} = \infty$. Now Corollary 3.5 completes the proof. □

Corollary 3.7. *Let e be an idempotent weak 2-cocycle whose chain complex is exact with trivial inertial subgroup, then the groups $M_e^i(G,K) = 0$ for all $i >$ the maximal chain in the graph of e.*

Theorem 3.8. *Let e, e' be lower subtractive graphs with trivial inertial subgroups. If $R_{e'} = R_e/(w)$ where w is central and is neither a unit nor a zero divisor, then*

$$\dim\mathcal{M}_e = 1 + \dim\mathcal{M}_{e'}.$$

Proof. This was shown in a more general version in [O] (Proposition 5.8). □

Example 3.2. *Let e, e' be weak 2-cocycles defined on $\mathbb{Z}_4$ and given by their graphs as follows*

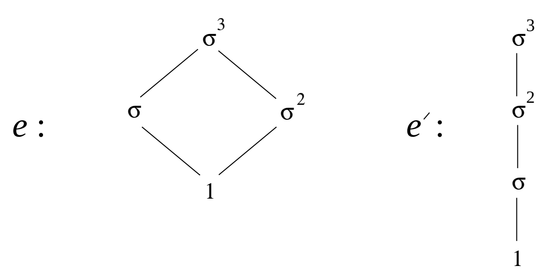

FIGURE 4

So, $R_e = \mathbb{Z}[x,y]$ and $R_{e'} = \mathbb{Z}[x] \simeq \mathbb{Z}[x,y]/(y)$ where y is a non-unit, non-zero divisor in R_e. Thus, by the theorem above $\dim\mathcal{M}_e = 1 + \dim\mathcal{M}_{e'}$. *Clearly* $\dim\mathcal{M}_{e'} = 1$ *so* $\dim\mathcal{M}_e = 2$.

We state a standard result, (see, for instance, Theorem 12.1 in [HS]):

Proposition 3.9. *If $U : {}_{R^*}\mathcal{M} \to_R \mathcal{M}$ is an exact functor and has a left adjoint $F : {}_R\mathcal{M} \to_{R^*} \mathcal{M}$, then F sends projectives to projectives.*

Let $\varphi : R \to R^*$ be a ring homomorphism. Then any R^*–module M can be regarded as an R–module by defining $r \cdot m = \varphi(r) \cdot m$. Likewise, for any R^*–homomorphism $\gamma : M \to N$, it can be regarded as an R-homomorphism by

$$\begin{aligned} U^\varphi\gamma &: U^\varphi M \to U^\varphi N \\ U^\varphi\gamma(rm) &= \varphi(r)\gamma(m). \end{aligned}$$

We define the change-of-ring functor to be the functor induced by φ. It was shown in [HS] that U^φ is exact and has a left adjoint $FM = R^* \otimes_R M$.

If $P : \cdots P_n \to P_{n-1} \to \cdots \to P_0$ is a projective resolution for $\mathbb{Z}$ as an R–module, assume that R^* is a free R–module, then

$$FP : \cdots FP_n \longrightarrow FP_{n-1} \longrightarrow \cdots \longrightarrow FP_0$$

is a projective resolution for $\mathbb{Z}$ as an R^*–module, so we have the following:

Proposition 3.10. *(Proposition 1.2 [HS]) If R^* is free as a right R–module via φ, then*

$$\theta : \mathrm{Ext}^i_{R^*}(R^* \otimes_R \mathbb{Z}, K^*) \longrightarrow \mathrm{Ext}^i_R(\mathbb{Z}, U^\varphi K^*)$$

is an isomorphism.

Take $R = \mathbb{Z}H$ where H is the inertial subgroup of e, $R^* = R_e$, we get

$$\mathrm{Ext}^i_{R_e}(R_e \otimes_{\mathbb{Z}H} \mathbb{Z}, K^*) \approx H^i(H, K^*) \tag{3.3.1}$$

We showed that if e has non trivial inertial subgroup H, then $\mathbb{Z}H$ is a subring of R_e. Define the restriction map:

$$Res : M^i_e(G, K) \longrightarrow H^i(H, K^*)$$

by

$$f \mapsto f|_H .$$

Now, we have the short exact sequence $0 \to \ker\varepsilon \to R_e \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ where ε is the augmentation map. Tensor it with $\mathbb{Z}$ over $\mathbb{Z}H$ to get:

$$0 \longrightarrow \ker\varepsilon' \longrightarrow R_e \otimes_{\mathbb{Z}H} \mathbb{Z} \xrightarrow{\varepsilon'} \mathbb{Z} \longrightarrow 0.$$

Take $\hom(\cdot, K^*)$ over R_e and pass to long exact sequence to obtain:

Proposition 3.11. *If e has a non-trivial inertial subgroup H, ε' is the augmentation map $\varepsilon' : R_e \otimes_{\mathbb{Z}H} \mathbb{Z} \to \mathbb{Z}$, then the following sequence is exact:*
$\cdots \longrightarrow \mathrm{Ext}^{n-1}_{R_e}(\ker\varepsilon', K^*) \longrightarrow \mathrm{Ext}^n_{R_e}(\mathbb{Z}, K^*) \longrightarrow \mathrm{Ext}^n_{R_e}(R_e \otimes_{\mathbb{Z}H} \mathbb{Z}, K^*) \longrightarrow \mathrm{Ext}^n_{R_e}(\ker\varepsilon', K^*) \longrightarrow \cdots$ *or equivalently by (3.3.1),*

$\cdots \longrightarrow \mathrm{Ext}^{n-1}_{R_e}(\ker\varepsilon', K^*) \longrightarrow M^n_e(G, K) \longrightarrow H^n(H, K^*) \longrightarrow \mathrm{Ext}^n_{R_e}(\ker\varepsilon', K^*) \longrightarrow M^{n+1}_e(G, K) \longrightarrow \cdots$ *is exact.*

Corollary 3.12. *If the inertial subgroup H is trivial, then*

$$M_e^n(G,K) \simeq \mathrm{Ext}_{R_e}^{n-1}(\ker \varepsilon', K^*) \quad \textit{for all } n.$$

4. Idempotents with Non-Exact Chain Complexes

In this section we provide some classes of idempotents whose chain complexes are not exact and in this situation, it is hard to compute the Brauer monoid components $M_e^2(G,K)$.

Definition 4.1. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (r-times), and e be defined on G by*

$$(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r) < (\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_r')$$

if and only if $\varepsilon_i \leq \varepsilon_i'$ for all i, $\varepsilon_i, \varepsilon_i' \in \{0,1\}$, then e is a lower subtractive graph which is called an r-cube graph.

Example 3.2 can be generalized by using induction on r to show that $\dim \mathcal{M}_{e_r} = r$ where e_r is the r-cube.

In [S1], it was shown that the chain complex for any r-cube is exact. We show that exactness can be easily missed, by giving classes of examples.

Definition 4.2. *Using an r-cube graph, we define a lower subtractive graph e' on $\mathbb{Z}_{2^r-1}$ by omitting the top element $(1,1,\ldots,1)$ and replacing each element $(\varepsilon_1, \ldots, \varepsilon_r)$ by $\varepsilon_1 + \varepsilon_2 \cdot 2 + \varepsilon_3 \cdot 2^2 + \cdots + \varepsilon_r \cdot 2^{r-1}$. Call the resulting graph "The top broken r-cube".*

Proposition 4.1. *The chain complex for any idempotent e whose graph is the "top broken" r-cube, $r \geq 3$, is not exact.*

Proof. By taking the boundaries of all $(r-1)$–cells in the original r-cube, we get $(r+1)!$ $(r-2)$–cells. Among them, there are $(r+1)! - 2r!$ which end with the top element $(1,1,\ldots,1)$ and they are ordered in a symmetric way so that they can be cancelled by adding them with alternating signs. The remaining $(r-2)$–cells do not cancel out and the image of this combination is zero. Since $M_n' = 0$ in the top broken graph, the graded module is not exact. □

Example 4.1. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, and define e on G by*

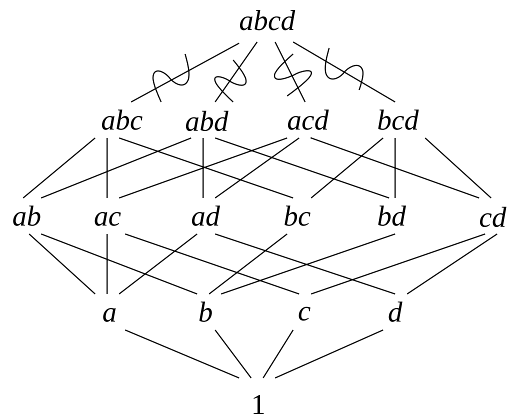

FIGURE 5

There are 4! *3–cells in the complete 4–cube. Define* e' *on* $G' = \mathbb{Z}_{15}$ *by*

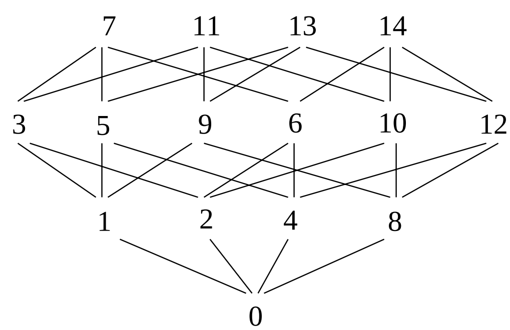

FIGURE 6. e' (obtained by dropping the top element in the r–cube).

The chain complex $(\mathcal{M}', \partial)$ *for* e' *is* $0 \to M_3' \to M_2' \to M_1' \to M_0' \to \mathbb{Z} \to 0$. *Define*

$$
\begin{array}{lll}
w_1 = \partial_4[a, ab, abc, abcd] & w_2 = \partial_4[a, ab, abd, abcd] & w_3 = \partial_4[a, ac, acd, abcd] \\
w_4 = \partial_4[a, ac, abc, abcd] & w_5 = \partial_4[a, ad, abd, abcd] & w_6 = \partial_4[a, ad, acd, abcd] \\
w_7 = \partial_4[b, ab, abd, abcd] & w_8 = \partial_4[b, ab, abc, abcd] & w_9 = \partial_4[b, bc, abc, abcd] \\
w_{10} = \partial_4[b, bc, bcd, abcd] & w_{11} = \partial_4[b, bd, bcd, abcd] & w_{12} = \partial_4[b, bd, abd, abcd] \\
w_{13} = \partial_4[c, ac, abc, abcd] & w_{14} = \partial_4[c, ac, acd, abcd] & w_{15} = \partial_4[c, bc, bcd, abcd] \\
w_{16} = \partial_4[c, bc, abc, abcd] & w_{17} = \partial_4[c, cd, acd, abcd] & w_{18} = \partial_4[c, cd, bcd, abcd] \\
w_{19} = \partial_4[d, ad, acd, abcd] & w_{20} = \partial_4[d, ad, adb, abcd] & w_{21} = \partial_4[d, bd, abd, abcd] \\
w_{22} = \partial_4[d, bd, bcd, abcd] & w_{23} = \partial_4[d, cd, bcd, abcd] & w_{24} = \partial_4[d, cd, acd, abcd]
\end{array}
$$

Then $w = \sum_{i=1}^{24} (-1)^{i-1} w_i \in M_3' - \{0\}$ *which in turn implies that the chain complex of* e' *is not exact. It is not necessary to have a combination of all* *these 4–cells, for instance, you may notice that:*

$$u = w_1 + w_9 + w_{13} - w_4 - w_8 - w_{16} \in M_3' - \{0\},$$

and $\partial(u) = 0$. This also shows non-exactness.

Example 4.2. *Let e be an idempotent weak 2-cocycle given by*

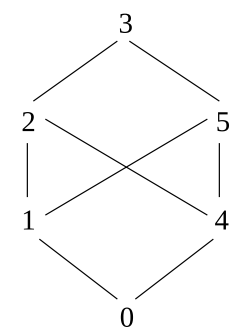

FIGURE 7

over $\mathbb{Z}_6$. Consider

$$w = (1-y)[1,2] - (1-y)[4,2] - (x+1)[1,5] + (x+1)[4,5]$$

then $\partial(w) = 0$ and if

$$\partial_3(\alpha_1[1,2,3] + \alpha_2[1,5,3] + \alpha_3[4,2,3] + \alpha_4[4,5,3]) = w$$

then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha$ (say) and so

$$\alpha(x-1) = (1-y)$$

which has no non-trivial solution for α in $R_e = \mathbb{Z}[x,y]/(x^2 = y^2)$. So $(\mathcal{M}_e, \partial)$ is not an exact chain complex.

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