Numerical Ranges of $4 \times 4$ Reducible Companion Matrices

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Abstract

In this paper, $A$ will denote an $n \times n$ complex matrix with complex number entries, the numerical range of $A$ is defined to be $W(A) = \{ x^*Ax : x \in \mathbb{C}^n, |x| = 1 \}$. We study some properties of numerical ranges of $4 \times 4$ reducible companion matrices.

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1 Introduction

Throughout this paper, $A$ will denote an $n \times n$ complex matrix with complex number entries, $\mathbb{C}$ and $\mathbb{C}^n$ denote respectively the field of complex numbers, the space of all complex vectors of length $n$. The numerical range of $A$ is defined to be

$$ W(A) = \{ x^*Ax : x \in \mathbb{C}^n, |x| = 1 \} $$

where $x^*$ denotes the complex conjugate transpose of $x$. The numerical range of $A$ is the range of the quadratic form $x^*Ax$, where $x$ varies overall vectors on the unit sphere $\{ x \in \mathbb{C}^n, |x| = 1 \}$. Following diagram illustrates the numerical range as an image of the unit sphere.

In 1918 O. Toeplitz gave a new concept on complex matrix: numerical range and in 1919 F. Hausdorff proved that the numerical range of a complex matrix is convex. The extensive study of the numerical range of complex matrix represents one of the most active and fertile research topics in the matrix theory in the last few decades. The subject is related and has applications to many different branches of pure and applied science such as operator theory, $C^*$-algebras, Banach algebras, matrix norms, inequalities, numerical analysis,
perturbation theory, matrix polynomials, systems theory, quantum physics, etc. It may be used to draw surprisingly strong conclusions about the spectrum and the combinatorial structure of a matrix, in particular a nonnegative matrix. In this paper, we study some properties of numerical ranges of $4 \times 4$ reducible companion matrices.

Let 
\[ f(z) = z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n \]  \hspace{1cm} (1)

be a complex polynomial and
\[
\begin{pmatrix}
0 & 1 \\
\vdots & \ddots \\
0 & 1 \\
-a_n & -a_{n-1} & \cdots & -a_1
\end{pmatrix}
\]  \hspace{1cm} (2)

be a $n \times n$ companion matrix associated with $f(z)$.

Lemma 1.1 Let $A = \{a_{ij}\} \in M_n$ be given, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of $A$, and let $\mu_1, \mu_2, \ldots, \mu_n$ be given positive real numbers such that $\mu_1 + \mu_2 + \cdots + \mu_n = 1$, the following are equivalent:

(a) $A$ is not a scalar multiple of the identity;
(b) $\mu_1a_{11} + \mu_2a_{22} + \cdots + \mu_na_{nn}$ is in the relative interior of $W(A)$;
(c) $\mu_1\lambda_1 + \mu_2\lambda_2 + \cdots + \mu_n\lambda_n$ is in the relative interior of $W(A)$.

Lemma 1.2 The numerical range of a matrix $A$ is invariant under unitary transformations.

Lemma 1.3[1, Theorem 3.3] Let $A$ be a $4 \times 4$ companion matrix with $\sigma(A) = \{a, ai, -1/a, -i/a\}$, where $|a| > 1$. Then $W(A)$ is an elliptic disc if and only if $a > \sqrt{1 + \sqrt{2}}$.

2 Main Results

In this section, we present our main results.

Theorem 2.1 Let $A$ be a $4 \times 4$ companion matrix. Then the necessary Condition of $W(A)$ being an elliptic disc is $|A| = -1$.

Proof. By [1], when $A$ is a $4 \times 4$ reducible companion matrix with elliptic numerical range, then we have either $\sigma(A) = \{a, -a, i/a, -i/a\}$ or $\sigma(A) = \{a, ai, -1/a, -i/a\}$ where $|a| > 1$.

When $\sigma(A) = \{a, -a, i/a, -i/a\}$ then $|A| = a \times (a) \times i/2 \times (-i/a) = -1$

the other hand $\sigma(A) = \{a, ai, -1/a, -i/a\}$ then $|A| = a \times ai \times (-1/a) \times (-i/a) = 1$

The proof of Theorem 2.1 is complete.
Theorem 2.2  Let $A$ be a $4 \times 4$ companion matrix. Then the necessary Condition of $W(A)$ being an elliptic disc is One of the following conditions holds, at least:

1. either $a_1 = 0$ or $\arg(a_1) = -\pi/4$;
2. $a_2$ is real number or pure imaginary number;
3. either $a_3 = 0$ or $\arg(a_3) = -\pi/4$;
4. $a_4 = -1$.

Proof. By [1], when $A$ is a $4 \times 4$ reducible companion matrix with elliptic numerical range, then we have either $\sigma(A) = \{a, -a, i/a, -i/a\}$ or $\sigma(A) = \{a, ai, -1/a, -i/a\}$ where $|a| > 1$.

when $\sigma(A) = \{a, -a, i/a, -i/a\}$ then

$$a_1 = -(a - a + i/a - i/a) = 0$$

$$a_2 = a \times (-a) + a \times i/a + a \times (-i/a) + (-a) \times i/a + (-a) \times (-i/a) + i/a \times (-i/a)$$
$$= -a^2 + i - i + i + 1/a^2$$
$$= 1/a^2 - a^2$$

$$a_3 = -(a \times (-a) \times i/a + a \times (-a) \times (-i/a) + a \times i/a \times (-i/a) + (-a) \times i/a \times (-i/a))$$
$$= -(ai + a - 1/a)$$
$$= 0$$

the other hand $\sigma(A) = \{a, ai, -1/a, -i/a\}$ then

$$a_1 = -(a + ai - 1/a - i/a) = -(a - 1/a)(1 + i)$$

so, we have

$$\arg(a_1) = 4/\pi$$
\[ a_2 = a \times (ai) + a \times (-1/a) + a \times (-i/a) + (ai) \times (-1/a) + (ai) \times (-i/a) \\
+(-1/a) \times (-i/a) \\
= a^2i - 1 - i - i + 1 + i/a^2 \\
= (a - 1/a)^2i \]

\[ a_3 = -(a \times (ai)) \times (-1/a) + a \times (ai) \times (-i/a) + a \times (-1/a) \times (-i/a) \\
+(ai) \times (-1/a) \times (-i/a)) \\
= -(-ai + a + i/a - 1/a) \\
= -(a - 1/a)(1 - i) \]

so, we have
\[ \arg(a_3) = -\pi/4 \]

(4) when \( \sigma(A) = \{a, -a, i/a, -i/a\} \) then
\[ a_4 = -(-1)^{4+1}a \times (-a) \times i/a \times (-i/a) = -1 \]

The other hand \( \sigma(A) = \{a, ai, -1/a, -i/a\} \) then
\[ a_4 = -(-1)^{4+1}a \times (ai) \times (-1/a) \times (-i/a) = -1 \]

The proof of Theorem 2.2 is complete.

**Corollary 2.3** Let \( A \) be a \( 4 \times 4 \) companion matrix. Then the necessary Condition of \( W(A) \) being an elliptic disc is either
\[ f(z) = z^4 + pz^2 - 1 \]
where \( p < 0 \), or
\[ f(z) = z^4 - \alpha z^3 - \beta z^2 - \gamma z - 1 \]
where \( \alpha = q(1 + i), \beta = q^2i, \gamma = q(1 - i), q > 0. \)

The proof of Corollary 2.3 is easy to see by Theorem 2.3.

By Lemma 3 and Corollary 2.3, we have the following result.

**Theorem 2.5** Let
\[ f(z) = a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 \]
and $A$ be a $4 \times 4$ companion matrix associated with $f(z)$. Where $a_0, a_1, a_2, a_3, a_4 \neq 0$. $W(A)$ is an elliptic disc if and only if the following conditions hold together:

1. $a_0 = -a_4$;
2. $|a_2| = a_1 \cdot a_3$
3. $\sigma(A) = \{a, ai, -1/a, -i/a\}$ where $a > \sqrt{1 + \sqrt{2}}$

References


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