Nonsingularity on Scaled Factor Circulant Matrices

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Abstract

In this paper, $M_n(F)$ will denote the set of all $n \times n$ matrices over any field. Matrices $A, R \in M_n(F)$ are called scaled factor circulant matrix and basic scaled factor circulant matrix respectively, where $AR = RA$. We give three discriminations by using only the elements in the first row of the scaled factor circulant matrix and the constants $d_1, d_2, \ldots, d_n$ in the diagonal matrix $D$ on non singularity.

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1 Introduction

Because of structure and many good properties of Circulant matrices, they have in recent years become one of the most important and active research field of applied mathematic and computation mathematic increasingly. They have been applied in many areas such as signal processing, digital image processing, coding Theory, and have a wide range of interesting applications\cite{4,5} as an important class of special matrices. Scaled factor circulant matrix, a kind of Circulant matrices have have a widely range of application in signal processing, code theory image processing, and so on.

J.L.Stuart and J.R.Weaver gave the definition of the Scaled factor circulant matrix and discussed the properties of them in \cite{8}. Jiangzhao Lin gave three

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discriminations by using only the elements in the first row of the scaled factor circulant matrix and the constants \( d_1, d_2, \cdots, d_n \) in the diagonal matrix \( D \) on non-singularity in \([1]\). Motivated by \([1]\), we give the discriminations by using only the element \( a_{\text{max}}, a_{\text{min}} \) and the constants \( d_1, d_2, \cdots, d_n \) in the diagonal matrix \( D \) on non-singularity continuously, where \( a_{\text{max}}, a_{\text{min}} \) are the maximum and minimum of the elements in the first row of the scaled factor circulant matrix respectively.

Throughout of the paper, let \( R \) be the basis scaled circulant factor matrix over \( F \):

\[
\begin{pmatrix}
0 & d_1 & 0 & \cdots & 0 \\
0 & 0 & d_2 & \cdots & 0 \\
0 & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & d_{n-1} \\
d_n & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\tag{1.1}
\]

where \( \prod_{j=1}^{n} d_j0 \). It is easily verified that the polynomial \( g(x) = x^n - d_1d_2\cdots d_n \) is both the minimal polynomial and the characteristic polynomial of the matrix \( R \). Moreover, \( R \) is nonderogatory.

Let also

\[
f(x) = \sum_{i=0}^{n-1} a_i k_i^{-1} x^i
\tag{1.2}
\]

where \( k_i = d_1d_2\cdots d_n, \ i = 1, 2, \cdots, n - 1, \ k_0 = 1, \omega_j (0 \leq j \leq n - 1) \) are the mutually different roots of equation \( x^n - d_1d_2\cdots d_n = 0 \). The polynomial (1.2) will be called the representor of the scaled factor circulant matrix \( A \).

**Definition 1.1.** An \( n \times n \) matrix \( A \) over \( F \) is called a scaled factor circulant matrix if \( A \) commutes with \( R \), that is,

\[
AR = RA,
\tag{1.2}
\]

where \( R \) is given in (1.1).

2 Preliminary Notes

In this section, we present some lemmas that are important to our main results.

**Lemma 2.1**\(^1\). \( A \in RCM_n \) if and only if \( A = f(R) \).

**Lemma 2.2**\(^2\). Let \( A = \text{scacirc}R(a_0, a_1, \cdots, a_{n-1}) \) be a scaled factor circulant matrix over the field \( F \). Then

\[
\sigma(A) = \{ \lambda_j | \lambda_j = f(d\omega^j) = a_0 + \sum_{i=1}^{n-1} a_i (d_i)^{-1} (d\omega^j)^i | 0 \leq j \leq n - 1 \} \tag{2.1}
\]
is the spectrum of $A$ and

$$A = f(R) = a_0 I + \sum_{i=1}^{n-1} a_i (\prod_{t=1}^{i} d_t)^{-1} R^i,$$  \hspace{1cm} (2.2)

Lemma 2.3$^{[1]}$. Suppose $A, B \in RCM_n$, then $AB = BA \in RCM_n$.

Lemma 2.4$^{[1]}$. Suppose $A, \in RCM_n$, if $A$ is nonsingular then $A^{-1} \in RCM_n$.

3 Main Results

Our main results are following theorems.

**Theorem 3.1.** Suppose $A = scacirc_R(a_0, a_1, \cdots, a_{n-1}) \in RCM_n$, if for $\forall M \in \{1, 2, \cdots, n\},$

$$\sum_{i=0, i \neq M}^{n-1} \left| \frac{d_1 d_2 \cdots d_M}{d_1 d_2 \cdots d_i} \right| \sqrt[\nu]{\left| d_1 d_2 \cdots d_n \right|} \left| i^{-M} \right| < 1,$$  \hspace{1cm} (3.1)

then $A$ is nonsingular.

**Proof.** If $A$ is singular, then by $[1, \text{Theorem 1}]$, we have

$$|a_M| \leq \sum_{i=0, i \neq M}^{n-1} \left| a_i \right| \left| \frac{d_1 d_2 \cdots d_M}{d_1 d_2 \cdots d_i} \right| \sqrt[\nu]{\left| d_1 d_2 \cdots d_i \right|} \left| i^{-M} \right|$$

for $M = 1, 2, \cdots, n$. So

$$|a_M| \leq \sum_{i=0, i \neq M}^{n-1} a_i \left| \frac{d_1 d_2 \cdots d_M}{d_1 d_2 \cdots d_i} \right| \sqrt[\nu]{\left| d_1 d_2 \cdots d_i \right|} \left| i^{-M} \right|$$

$$\leq \sum_{i=0, i \neq M}^{n-1} a_i \left| \frac{d_1 d_2 \cdots d_M}{d_1 d_2 \cdots d_i} \right| \sqrt[\nu]{\left| d_1 d_2 \cdots d_i \right|} \left| i^{-M} \right|$$

hence

$$1 \leq \sum_{i=0, i \neq M}^{n-1} \frac{a_i}{|a_M|} \left| \frac{d_1 d_2 \cdots d_M}{d_1 d_2 \cdots d_i} \right| \sqrt[\nu]{\left| d_1 d_2 \cdots d_i \right|} \left| i^{-M} \right|$$

which contradicts to inequality (3.1), therefore $A$ is nonsingular.

**Corollary 3.2.** Suppose $A = scacirc_R(a_0, a_1, \cdots, a_{n-1}) \in RCM_n$, if for $\forall M \in \{1, 2, \cdots, n\},$

$$\sum_{i=0}^{n-1} \left| \frac{d_1 d_2 \cdots d_M}{d_1 d_2 \cdots d_i} \right| \sqrt[\nu]{\left| d_1 d_2 \cdots d_n \right|} \left| i^{-M} \right| < 0,$$  \hspace{1cm} (3.2)
then $A$ is nonsingular.

**Corollary 3.3.** Suppose $A = scacirc_R(a_0, a_1, \cdots, a_{n-1}) \in RCM_n$, if

$$
\sqrt{n} \left[ 1 - \frac{n}{|d_1d_2 \cdots d_M|} \right]^M + \frac{n-1}{|d_1d_2 \cdots d_i|} \sum_{i=0, i \neq M}^{n-1} \frac{\sqrt{n}|d_1d_2 \cdots d_n|^i}{d_1d_2 \cdots d_i} < 1,
$$

(3.3)

then $A$ is nonsingular.

**Proof.** The proof is similar to Lemma 1. It is easy to see

$$
\sqrt{n} \left[ 1 - \frac{n}{|d_1d_2 \cdots d_M|} \right]^M + \frac{n-1}{|d_1d_2 \cdots d_i|} \sum_{i=0, i \neq M}^{n-1} \frac{\sqrt{n}|d_1d_2 \cdots d_n|^i}{d_1d_2 \cdots d_i}
$$

so

$$
\sqrt{n} \left[ 1 - \frac{n}{|d_1d_2 \cdots d_M|} \right]^M + \frac{n-1}{|d_1d_2 \cdots d_i|} \sum_{i=0, i \neq M}^{n-1} \frac{\sqrt{n}|d_1d_2 \cdots d_n|^i}{d_1d_2 \cdots d_i}
$$

by above inequality and (3.3), we have

$$
1 - \frac{\sqrt{n}|d_1d_2 \cdots d_n|^M}{|d_1d_2 \cdots d_M|} + \frac{n-1}{|d_1d_2 \cdots d_i|} \sum_{i=0, i \neq M}^{n-1} \frac{\sqrt{n}|d_1d_2 \cdots d_n|^i}{d_1d_2 \cdots d_i} < 1
$$

hence

$$
\frac{\sqrt{n}|d_1d_2 \cdots d_n|^M}{|d_1d_2 \cdots d_M|} > \frac{n-1}{|d_1d_2 \cdots d_i|} \sum_{i=0, i \neq M}^{n-1} \frac{\sqrt{n}|d_1d_2 \cdots d_n|^i}{d_1d_2 \cdots d_i}
$$

therefore $A$ is nonsingular.

**Theorem 3.4.** Suppose $A = scacirc_R(a_0, a_1, \cdots, a_{n-1}) \in RCM_n$, if for $\forall M \in \{1, 2, \cdots, n\}$,

$$
\frac{|d_1d_2 \cdots d_M|}{\sqrt{n}|d_1d_2 \cdots d_n|^M} < \frac{na}{n - 1},
$$

(3.2)
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and
\[
\frac{|d_1d_2\cdots d_i|}{\sqrt[n]{|d_1d_2\cdots d_n|^i}} < na, \quad i = 1, 2, \ldots, n - 1, \quad i \neq M \tag{3.3}
\]
then \(A\) is nonsingular.

**Proof.** If \(A\) is singular, then by [1, Theorem 2], we have
\[
|1 - \frac{\sqrt[n]{|d_1d_2\cdots d_n|^M}}{d_1d_2\cdots d_M}a_M| \leq \frac{1}{n}, \tag{3.4}
\]
or
\[
\frac{\sqrt[n]{|d_1d_2\cdots d_n|^i}}{|d_1d_2\cdots d_i|} a_i | \leq \frac{1}{n}. \tag{3.5}
\]
When inequality (3.4) is hold, then
\[
\frac{\sqrt[n]{|d_1d_2\cdots d_n|^M}}{d_1d_2\cdots d_M}a_M \leq 1 - \frac{1}{n}, \tag{3.6}
\]
or
\[
\frac{\sqrt[n]{|d_1d_2\cdots d_n|^M}}{d_1d_2\cdots d_M}a_M \geq 1 + \frac{1}{n} \geq -(1 - \frac{1}{n}), \tag{3.7}
\]
so
\[
1 - \frac{1}{n} \leq |\frac{\sqrt[n]{|d_1d_2\cdots d_n|^M}}{d_1d_2\cdots d_M}a_M| \leq \frac{\sqrt[n]{|d_1d_2\cdots d_n|^M}}{|d_1d_2\cdots d_M|} |a|,
\]
hence
\[
\frac{|d_1d_2\cdots d_M|}{\sqrt[n]{|d_1d_2\cdots d_n|^M}} \geq \frac{na}{n - 1}
\]
this contradicts to inequality (3.2).

On the other hand, when inequality (3.5) is hold, it is easy to see
\[
\frac{1}{n} \leq \frac{\sqrt[n]{|d_1d_2\cdots d_n|^i}}{|d_1d_2\cdots d_i|} |a_i| \leq \frac{\sqrt[n]{|d_1d_2\cdots d_n|^i}}{|d_1d_2\cdots d_i|} |a|
\]
so
\[
\frac{|d_1d_2\cdots d_i|}{\sqrt[n]{|d_1d_2\cdots d_n|^i}} \geq na, \quad i = 1, 2, \ldots, n - 1, \quad i \neq M
\]
this contradicts to inequality (3.3), therefore \(A\) is nonsingular.

**Theorem 3.5.** Suppose \(A = \text{scacirc}_R(a_0, a_1, \cdots, a_{n-1}) \in RCM_n\), if for \(\forall M \in \{1, 2, \cdots, n\}\),
\[
\frac{|d_1d_2\cdots d_M|}{\sqrt[n]{|d_1d_2\cdots d_n|^M}} < \min\left\{\frac{1}{(n-1)na}, \frac{na}{n - 1}\right\} \tag{3.8}
\]
then $A$ is nonsingular.

**Proof.** If $A$ is singular, then by Theorem 3.1, we have

$$1 \leq \sum_{i=0, i \neq M}^{n-1} \frac{|d_1d_2 \cdots d_M|}{|d_1d_2 \cdots d_i|^{\frac{i}{n}} |d_1d_2 \cdots d_i|^{i-M}}$$

(3.9)

by Theorem 3.4, we have

$$\frac{|d_1d_2 \cdots d_M|}{\sqrt[n]{|d_1d_2 \cdots d_n|}} \geq \frac{na}{n-1}$$

(3.10)

or

$$\frac{|d_1d_2 \cdots d_i|}{\sqrt[n]{|d_1d_2 \cdots d_n|}} \geq na, \ i = 1, 2, \cdots, n-1, \ i \neq M$$

(3.11)

When (3.9) and (3.11) is hold, then

$$1 \leq \sum_{i=0, i \neq M}^{n-1} \frac{|d_1d_2 \cdots d_M|}{|d_1d_2 \cdots d_i|^{\frac{i}{n}} |d_1d_2 \cdots d_i|^{i-M}}$$

$$\leq \frac{|d_1d_2 \cdots d_M|}{\sqrt[n]{|d_1d_2 \cdots d_n|}} n(n-1)a$$

this contradicts to inequality (3.8), therefore $A$ is nonsingular.

**References**


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