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A Characterizaton of ${}^2D_n(p^k)$ by Order of Normalizer of Sylow Subgroups

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Abstract

Let G be a finite group. If $|N_G(R)| = |N_{{}^2D_n(p^k)}(\bar{R})|$ for every prime r, where $R \in Syl_r(G)$, $\bar{R} \in Syl_r({}^2D_n(p^k))$ and $n \geq 2$, then $G \cong {}^2D_n(p^k)$.

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1 Introduction

Characterization of finite groups is one of the central themes of research in group theory. There are various characterizations of finite groups by given properties, such as order of their elements or order of their solvable subgroup, etc (see [6, 9, 13, 14]). Order's special subgroups could be used to characterize a finite group. One class of such subgroups is the normalizer of Sylow subgroups. In 1992, Bi [1] showed that $L_2(p^k)$ can be characterized only by the order of normalizer of its Sylow subgroups. This type of characterization is done for the following groups:

$L_2(q)$ [1], $L_n(q)$ [2], $S_4(q)$ [3], alternating groups [4], $U_n(q)$ [5], Janko groups [11], Mathieu simple groups [12]. In this paper, we have characterized ${}^2D_n(p^k)$ by the order of normalizer of Sylow subgroups, for $n \geq 2$. In fact we have proved the following theorem:

Main Theorem . Let G be a finite group. If $|N_G(R)| = |N_{{}^2D_n(p^k)}(\bar{R})|$ for every prime r, with $n \geq 2$, then $G \cong {}^2D_n(p^k)$, where $R \in Syl_r(G)$ and $\bar{R} \in Syl_r({}^2D_n(p^k))$.

If G is a finite group, then $\pi(G)$ is the set of all prime divisors of $|G|$ and $\pi_{m_1,\dots,m_n}(G)$ is the set of all prime divisors of $|G|/(\prod_{i=1}^n(p^{m_i}+1))$, where $\prod_{i=1}^n(p^{m_i}+1)||G|$. Also $GF(p^k)$ is the finite field with p^k elements. If m is a

natural number and r is a prime, then $|m|_r$ denotes the r-part of m. In other words $|m|_r = r^k$ if $r^k \| m$. All further unexplained notations are standard and could be found in [7] and [10].

2 Preliminary results

First, let us bring some useful lemmas.

Lemma 2.1 *[8]. (Frattini Argument) Let G be a finite group and H a normal subgroup of G. If P is a p-Sylow subgroup of H, then $G = HN_G(P)$.*

Lemma 2.2 *[2]. Let G be a finite simple group. If $p^e \| |G|$ and $|G| < p^{5e/2}$, then G is isomorphic to a simple group of Lie type in characteristic p (including ${}^2F_4(2)'$ for $p = 2$).*

Lemma 2.3 *[2]. Let G be a finite group and $N \lhd G$. If $r \mid |G/N|$, $r \nmid |N|$ (r is prime and $r \neq p$), and if in addition $p^e \| |N|$ and $p^t \| |C_N(R)|$, where $R \in Syl_r(G)$, then $r \mid p^{e-t} - 1$.*

Lemma 2.4 *[2]. Let r and s be prime numbers ($r \neq s$) and $exp_r(p) = mk$ $(n/2 < m \leq n)$. If $x, y \in L_n(p^k)$ with $|x| = r$, $|y| = s^t$ and if in addition $s \nmid p^{km} - 1$ and $xy = yx$, then $s^t \mid |L_{n-m}(p^k)|$ and $m \leq n - 2$.*

Lemma 2.5 *[2]. Let r and s be prime numbers ($r \neq s$), $exp_r(p) = mk$ $(n/2 < m \leq n)$ and $K \leq L_n(q)$. If $exp_r(p) = mk$ and $R \in Syl_r(K)$, and if in addition, there is no element of order rs in K and $s^t \mid |N_K(R)|$, then $s^t | m$.*

Lemma 2.6 *[2]. Let $n > 2$ be a natural number. If $p^n \neq 2^6$, then there is a prime factor r of $p^n - 1$ such that $exp_r(p) = n$. And if p is not a Mersenne prime, then there is a prime factor r of $p^2 - 1$ such that $exp_r(p) = 2$.*

3 On the order of Sylow normalizer of ${}^2D_n(p^k)$

In this section, we assume that $n \geq 2$ and $(SO^-_{2n}(p^k))' = \Omega^-_{2n}(p^k)$.

Lemma 3.1 . *Suppose that $\bar{R} = RZ/Z \in Syl_r({}^2D_n(p^k))$, for every prime r, where $Z := Z(\Omega^-_{2n}(p^k))$ and $R \in Syl_r(\Omega^-_{2n}(p^k))$, then $|N_{{}^2D_n(p^k)}(\bar{R})| = |N_{SO^-_{2n}(p^k)}(R)|(2, p^{nk} + 1)/2(4, p^{nk} + 1)$.*

Proof. Straightforward. □

Lemma 3.2 . *Let $exp_r(p) = 2nk$ and $\bar{R} \in Syl_r({}^2D_n(p^k))$. If $p^t \mid |N_{{}^2D_n(p^k)}(\bar{R})|$, then $p^t | 2n$.*

Proof. It follows from Lemma (2.5) and Lemma (3.1). □

Lemma 3.3 . *If* $P \in Syl_p(SO^-_{2n}(p^k))$ *and* $n \geq 2$, *then* $|N_{SO^-_{2n}(p^k)}(P)| = p^{n(n-1)k}(p^k+1)(p^k-1)^{(n-1)}$, *where* $p \neq 2$.

Proof. Let $G_n = SO^-_{2n}(p^k)$ and ß $= \{v_1, ..., v_{2n}\}$ be an orthogonal basis of $V_{2n}(p^k)$ such that

$$J_n = [f]_{ß} = \begin{bmatrix} 0 & O & 1 \\ O & J_{n-1} & O \\ 1 & O & 0 \end{bmatrix},$$

where f is an orthogonal form over $V_{2n}(p^k)$. We can see

$$(G_n)_{[v_1]} = \{ \begin{bmatrix} a & -av^tJ_{n-1}B & -av^tJ_{n-1}v/2 \\ O & B & v \\ 0 & O & a^{-1} \end{bmatrix} | a \in GF^*(p^k), v \in V_{2(n-1)}(p^k)\},$$

where $B \in G_{n-1}$. Thus $|(G_n)_{[v_1]}| = p^{n(n-1)k}(p^{nk} - p^{(n-1)k} + p^k - 1)(p^{2(n-2)k} - 1)...(p^{2k} - 1)$. Let

$$P = \{ \begin{bmatrix} 1 & -v^tJ_{n-1}B & -v^tJ_{n-1}v/2 \\ O & B & v \\ 0 & O & 1 \end{bmatrix} | v \in V_{2(n-1)}(p^k), B \in P'\} \leq (G_n)_{v_1},$$

where $P' \in Syl_p(G_{n-1})$. Since $|P| = p^{n(n-1)k}$, we have $P \in Syl_p(G_n)$. Suppose that $g \in N_{G_n}(P)$. If $v_1.g = v = \sum_{i=1}^{2n} a_iv_i$, then $P \leq (G_n)_v$. We can see $v = a_1v_1$, where $a_1 \in GF^*(p^k)$. Hence $g \in (G_n)_{[v_1]}$. If

$$h = \begin{bmatrix} a & -av^tJ_{n-1}E & -av^tJ_{n-1}v/2 \\ O & E & v \\ 0 & O & a^{-1} \end{bmatrix} \in N_{G_n}(P),$$

then we have $E \in N_{G_{n-1}}(P')$. Thus

$$N_{G_n}(P) = \{ \begin{bmatrix} a & -av^tJ_{n-1}E & -av^tJ_{n-1}v/2 \\ O & E & v \\ 0 & O & a^{-1} \end{bmatrix} | a \in GF^*(p^k), v \in V_{2(n-1)}(p^k)\},$$

where $E \in N_{G_{n-1}}(P')$. So $|N_{G_n}(P)| = p^{2(n-1)k}(p^k-1)|N_{G_{n-1}}(P')| = p^{n(n-1)k}(p^k-1)^{(n-1)}(p^k+1)$. □

Lemma 3.4 . *If* $P \in Syl_2(\Omega^-_{2n}(2^k))$ *and* $n \geq 2$, *then* $|N_{\Omega^-_{2n}(2^k)}(P)| = 2^{n(n-1)k}(2^k+1)(2^k-1)^{(n-1)}$.

Proof. Let $G_n = SO^-_{2n}(2^k)$ and ß $= \{v_1, ..., v_{2n}\}$ be an orthogonal basis of $V_{2n}(2^k)$ such that $Q^-(\sum_{i=1}^{2n} x_i v_i) = \sum_{i=1}^{n-1} x_i x_{2n-i+1} + kx_n^2 + x_{n+1}^2 + x_n x_{n+1} = x_1 x_{2n} + Q^-(\sum_{i=2}^{2n-1} x_i v_i)$, Where Q^- is an orthogonal form over $V_{2n}(2^k)$ and $x^2 + x + k \in GF(2^k)[x]$ is irreducible over $GF(2^k)$. We can see:

$$(G_n)_{[v_1]} = \{ \begin{bmatrix} a & u & b \\ O & B & w \\ 0 & O & a^{-1} \end{bmatrix} | a \in GF^*(2^k), u, w \in V_{2(n-1)}(2^k), b = aQ^-(wv'^t)\},$$

such that $B \in G_{n-1}$ and $u_i = b + aQ^-(v'A^i + wv'^t)$, where $1 \leq i \leq 2n-2$ and $i \notin \{n-1, n\}$. Moreover $v' = (v_2, ..., v_{2n-1})$, $u_{n-1} = ak+b+aQ^-(v'A^{n-1}+wv'^t)$ and $u_n = a + b + aQ^-(v'A^n + wv'^t)$, where A^j is j-th column of A . Now We have:

$$(\Omega^-_{2n}(2^k))_{v_1} = \{ \begin{bmatrix} 1 & u & b \\ O & B & w \\ 0 & O & 1 \end{bmatrix} \in (G_n)_{[v_1]} | B \in \Omega^-_{2(n-1)}(2^k)\}.$$

Let

$$P = \{ \begin{bmatrix} 1 & u & b \\ O & B & w \\ 0 & O & 1 \end{bmatrix} \in (\Omega^-_{2n}(2^k))_{v_1} | B \in P'\},$$

where $P' \in Syl_2(\Omega^-_{2(n-1)}(2^k))$. Since $|P| = 2^{n(n-1)k}$, we have $P \in Syl_2(\Omega^-_{2n}(2^k))$. Suppose that $g \in N_{G_n}(P)$. If $v_1.g = v = \sum_{i=1}^{2n} a_i v_i$, then $P \leq (G_n)_v$. We can see $v = a_1 v_1$, where $a_1 \in GF^*(2^k)$. Hence $g \in (G_n)_{[v_1]}$. If

$$h = \begin{bmatrix} a & u & b \\ O & E & w \\ 0 & O & a^{-1} \end{bmatrix} \in N_{G_n}(P) \leq (G_n)_{[v_1]},$$

then $E \in N_{G_{n-1}}(P')$. Thus:

$$N_{G_n}(P) = \{ \begin{bmatrix} a & u & b \\ O & E & w \\ 0 & O & a^{-1} \end{bmatrix} \in (G_n)_{[v_1]} | E \in N_{G_{n-1}}(P')\}.$$

So $|N_{\Omega^-_{2n}(2^k)}(P)| = |N_{G_n}(P)|/2 = 2^{2(n-1)k}(2^k-1)|N_{G_{n-1}}(P')|/2 = 2^{n(n-1)k}(2^k-1)^{(n-1)}(2^k+1)$. □

Lemma 3.5 . *Let $G_n = SO^-_{2n}(p^k)$, $v \in V^\sharp_{2n}(p^k)$ and $p \neq 2$.*
I) If $f(v,v) = 0$, then $|(G_n)_v| = p^{n(n-1)k}(p^{(n-1)k}+1)(p^{2(n-2)k}-1)...(p^{2k}-1)$.
II) If $f(v,v) = a \neq 0$, then $(G_n)_v \cong SO_{2n-1}(p^k)$.

Proof. I) Let $f(v,v)=0$. Since G_n acts transitively over $\{w \in V_{2n}^{\sharp}(p^k) \mid f(w,w)=0\}$, there is $g \in G_n$ such that $v_1.g = v$. Thus $(G_n)_v = ((G_n)_{v_1})^g$. Hence $|(G_n)_v| = |(G_n)_{v_1}| = p^{n(n-1)k}(p^{(n-1)k}+1)(p^{2(n-2)k}-1)...(p^{2k}-1)$.
II) If $f(v,v)=a \neq 0$, then $V =< v >\perp< v >^{\perp}$ and $(< v >^{\perp}, f\mid_{<v>^{\perp}})$ is an orthogonal space. If $ß' = \{w_1, ..., w_{2n-1}\}$ is an orthogonal basis of $< v >^{\perp}$, then $ß = \{v, w_1, ..., w_{2n-1}\}$ is a basis of $V_{2n}(p^k)$. Let

$$J_n = [f]_{ß} = \begin{bmatrix} a & O \\ O & J'_{n-1} \end{bmatrix},$$

where

$$J'_{n-1} = [f|_{<v>^{\perp}}]_{ß'}.$$

If

$$g = \begin{bmatrix} 1 & u \\ O & E \end{bmatrix} \in (G_n)_v,$$

then $g^t J_n g = J_n$. So $E \in SO_{2n-1}(p^k)$ and $u = 0$. Hence

$$(G_n)_v = \{\begin{bmatrix} 1 & O \\ O & E \end{bmatrix} | E \in SO_{2n-1}(p^k)\}.$$

Therefore $(G_n)_v \cong SO_{2n-1}(p^k)$ and the proof is completed. □

Lemma 3.6 . *Let $G_n = SO_{2n}^{-}(2^k)$ and $v \in V_{2n}^{\sharp}(2^k)$. If $exp_r(2) = 2nk$, then $r \nmid |(G_n)_v|$.*

Proof. We Know $SL_{2n}(2^k)$ act transitively over $V_{2n}^{\sharp}(2^k)$. Since $G_n \leq SL_{2n}(2^k)$, we have $|(G_n)_v| \mid |(SL_{2n}(2^k))_v| = 2^{n(n-1)k/2}\prod_{i=2}^{2n-1}(2^{ki}-1)$. Thus $r \nmid |(G_n)_v|$. □

Corollary 3.7 . *If $exp_r(p) = 2nk$, then $det(A-I) \neq 0$, for every $r-$element A of $SO_{2n}^{-}(p^k)$.*

Proof. By Lemma (3.5) and Lemma (3.6), we have $r \nmid |(G_n)_v|$ for every $v \in V_{2n}^{\sharp}(p^k)$. Thus $v.A \neq v$. □

Lemma 3.8 . *If $exp_{r_1}(p) = 2(n-1)k$, then*

$$|N_{\Omega_{2n}^{-}(p^k)}(R_1)| = (p^k-1)|N_{SO_{2(n-1)}^{-}(p^k)}(R'_1)|,$$

where $R_1 \in Syl_{r_1}(SO_{2n}^{-}(p^k))$ and $R'_1 \in Syl_{r_1}(SO_{2(n-1)}^{-}(p^k))$.

Proof. We assume that $p \neq 2$ and $ß = \{v_1, ..., v_{2n}\}$ is an orthogonal basis of $V_{2n}(p^k)$ such that

$$J_n = [f]_ß = \begin{bmatrix} J_1' & O \\ O & J_{n-1} \end{bmatrix},$$

where

$$J_1' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and f is an orthogonal form over $V_{2n}(p^k)$. If

$$R_1 = \{\begin{bmatrix} I_2 & O \\ O & B \end{bmatrix} | B \in R_1'\},$$

where $R_1' \in Syl_{r_1}(SO^-_{2(n-1)}(p^k))$, then $R_1 \in Syl_{r_1}(SO^-_{2n}(p^k))$. Let

$$g = \begin{bmatrix} A & u \\ w & E \end{bmatrix} \in N_{GO^-_{2n}(p^k)}(R_1).$$

Thus for every

$$x = \begin{bmatrix} I_2 & O \\ O & B \end{bmatrix} \in R_1,$$

there is

$$x' = \begin{bmatrix} I_2 & O \\ O & B' \end{bmatrix} \in R_1,$$

such that $x.g = g.x'$. Hence $Bw = w$, $BE = EB'$ and $uB' = u$. By Corollary (3.7), u and w = 0. We have, $A \in GO_2^+(p^k)$ and $E \in N_{GO^-_{2(n-1)}(p^k)}(R_1')$. Thus $|N_{SO^-_{2n}(p^k)}(R_1)| = |GO_2^+(p^k)||N_{GO^-_{2(n-1)}(p^k)}(R_1')|/2 = 2(p^k-1)|N_{SO^-_{2(n-1)}(p^k)}(R_1')|$. If $p = 2$, the lemma is proved similar to the above procedure. □

Remark 3.9 . *In Lemma (3.8), if $(n, k) \in \{(2, 3), (4, 1)\}$, then there isn't any prime number as s such that $exp_s(2) = 2(n-1)k$. Therefore, in Lemma (3.8), we have $(n, k) \notin \{(2, 3), (4, 1)\}$, where $p = 2$.*

Lemma 3.10 . *If $exp_{r_2}(p) = 2(n-2)k$, for $n \geq 4$ and $R_2 \in Syl_{r_2}(SO^-_{2n}(p^k))$, then*

$$|N_{\Omega 2n^-(p^k)}(R_2)| = p^{2k}(p^{2k} - 1)^2|N_{SO^-_{2(n-2)}(p^k)}(R_2')|,$$

where $R_2' \in Syl_{r_2}(SO^-_{2(n-2)}(p^k))$.

Proof. It is proved similar to Lemma (3.8). □

Remark 3.11 . *In Lemma (3.10), if* $(n,k) = (5,1)$*, then there isn't any prime number as* s *such that* $exp_s(2) = 2(n-2)k$*. Therefore, in Lemma (3.10), we have* $(n,k) \neq (5,1)$*, where* $p = 2$*.*

Lemma 3.12 . *Let* G *be a finite group and let* H *be a subgroup of* G*. If* H *is a simple group of Lie type in characteristic* p *and* $r \in \pi(H)$ *such that* $exp_r(p) = ak$ *and* $|G|_r = |p^{ak} - 1|_r$*, then* $r \notin \pi(G/H)$*.*

Proof. The lemma can be proved by checking the orders of the simple groups of Lie type in characteristic p. □

Lemma 3.13 . *Let* G *be a simple group of Lie type in characteristic* p *and* $n \geq 2$*.*
Case 1. $p \neq 2$*. If* $|G|_p = p^{n(n-1)k}$*,* $Max\{exp_r(p)|\ r \in \pi(G)\} = 2nk$*,* $Max\{exp_r(p)|\ r \in \pi_{2nk}(G)\} = 2(n-1)k$ *and* $|G| \mid |{}^2D_n(p^k)|$*, then* $G \cong {}^2D_n(p^k)$*.*
Case 2. $p = 2$*. If* $2^{n(n-1)k-1} \leq |G|_2 \leq 2^{n(n-1)k}$*,* $Max\{exp_r(2)|\ r \in \pi(G)\} = 2nk$*,* $Max\{exp_r(2)|\ r \in \pi_{nk}(G)\} = 2(n-1)k$ *and* $|G| \mid |{}^2D_n(2^k)|$*, for* $(n,k) \notin \{(4,1),(3,1),(2,3)\}$*, or* $Max\{exp_r(2)|\ r \in \pi(G)\} = 4$ *and* $|G| \mid |{}^2D_n(p^k)|$*, for* $(n,k) = (3,1)$*, or* $Max\{exp_r(2)|\ r \in \pi(G)\} = 8$ *and* $|G| \mid |{}^2D_n(p^k)|$*, for* $(n,k) = (4,1)$ *or* $Max\{exp_r(2)|\ r \in \pi(G)\} = 12$ *and* $|G| \mid |{}^2D_n(p^k)|$*, for* $(n,k) = (2,3)$*, then* $G \cong {}^2D_n(2^k)$*.*

Proof. The Lemma can be proved by checking the orders of the simple groups of Lie type in characteristic p. □

4 The Main Theorem

In this section first we bring some useful lemmas. In what follows, we assume that $|N_G(S)| = |N_{{}^2D_n(p^k)}(\bar{S})|$, for every prime s, where $S \in Syl_s(G)$, $\bar{S} \in Syl_s({}^2D_n(p^k))$ and $exp_{r_i}(p) = 2(n-i)k$. Moreover, we suppose that $r := r_0$. In the following lemma n and k are natural numbers such that r and r_1 exist. Furthermore r_2 exists, where $n \geq 3$. Therefore $(n,k) \notin \{(5,1),(4,1),(3,3),(3,1),(2,3)\}$, where $p = 2$.

Lemma 4.1 . *If* $N \lhd G$ *and* $p \notin \pi(G/N)$*, then* $r_i \notin \pi(G/N)$ *for all* $0 \leq i < n/2$*.*

Proof. Let $P \in Syl_p(N)$. Since $p \notin \pi(G/N)$, we have $P \in Syl_p(G)$ and by Frattini argument, $|G/N| \mid |N_G(P)|$. Thus by Lemma (3.3) and Lemma(3.4), we have $r_i \notin \pi(G/N)$ for all $0 \leq i < n/2$. $\square$

Corollary 4.2 . *Let $1 \leq M \leq N \leq G$ be a normal series of G. If $r \in \pi(N/M)$ and $p \notin \pi(G/N)$, then $p \in \pi(N/M)$.*

Proof. If not, $p \notin \pi(N/M)$, then by Lemma (4.1), we have $r \notin \pi(G/M)$, it is a contradiction. $\square$

Lemma 4.3 . *If $1 \leq H \leq K \leq G$ be a normal series of G, $p \notin \pi(G/K)$ and $n \geq 3$, then:*
i. $r \in \pi(K/H)$ iff $r_1 \in \pi(K/H)$.
ii. If $r_1 \in \pi(K/H)$, then $r_2 \in \pi(K/H)$.
iii. If $r_2 \in \pi(K/H)$, then $r_1 \in \pi(K/H)$, where $n \geq 4$.

Proof. i. Let $r \in \pi(K/H)$. We must prove that $r_1 \in \pi(K/H)$. For this purpose, we argue by contradiction. Suppose $r_1 \notin \pi(K/H)$. By Frattini argument $|G/H| \mid |N_G(R_1)|$, where $R_1 \in Syl_{r_1}(H)$. We have $R_1 \in Syl_{r_1}(G)$. Thus, by Lemma (3.8), $r \mid |N_{{}^2D_{n-1}(p^k)}(R_1)|$, which is a contradiction with $r \nmid |{}^2D_{n-1}(p^k)|$. Therefore $r_1 \in \pi(K/H)$. Conversely, let $r_1 \in \pi(K/H)$ and $R \in Syl_r(H)$. If $r \notin \pi(K/H)$, then $R \in Syl_r(G)$. Hence, by Lemma (3.2), $r_1 \mid n$. On the other hand $exp_{r_1}(p) = 2(n-1)k$. So $2(n-1)k \mid r_1 - 1$. Therefore $2(n-1)k < n$, which is a contradiction.
ii. Let $r_1 \in \pi(K/H)$, thus $r \in \pi(K/H)$. If $r_2 \notin \pi(K/H)$, then similar to the above argument, we obtain $r \mid |N_{{}^2D_{n-2}(p^k)}(R_2)|$, which is a contradiction with $r \nmid |{}^2D_{n-2}(p^k)|$.
iii. Let $r_2 \in \pi(K/H)$. If $r_1 \notin \pi(K/H)$, then similar to the above argument $r_2 \mid |N_G(R_1)|$, where $R_1 \in Syl_{r_1}(G)$. Thus $r_2 \mid n-1$. Therefore $2(n-2)k < n-1$, which is a contradiction with $n \geq 4$. $\square$

For $p = 2$, we have the following lemma:

Lemma 4.4 . *Let $N \trianglelefteq G$. If $r \in \pi(N)$ and $r \notin \pi(G/N)$, then $|G/N|_2 \leq 2$.*

Proof. Let $R \in Syl_r(N)$. By Frattini argument $|G/N| \mid |N_G(R)|$. But $R \in Syl_r(G)$. So, by Lemma (2.5), $2^t \mid 2n$, where $|G/N|_2 = 2^t$. Thus $2^{t-1} \mid n$. On the other hand, by Lemma(4.3), $r_1 \notin \pi(G/N)$. Similar to the above procedure, $2^{t-1} \mid n-1$. Therefore $t \leq 1$. $\square$

Lemma 4.5 . *Let $1 \leq H \leq K \leq G$ be a normal series of G and $p \neq 2$. If $p \in \pi(K/H)$ and $p \notin \pi(G/K)$, then $r \in \pi(K/H)$. Furthermore, $r_1 \in \pi(K/H)$, for $n \geq 3$ and $r_2 \in \pi(K/H)$, for $n \geq 4$.*

Proof. First, we assume that $(p,n)=1$. It is sufficient to prove $r \in \pi(K/H)$. For this, we suppose $r \notin \pi(K/H)$ and obtain a contradiction. Let $r \notin \pi(K/H)$ and $R \in Syl_r(H)$. By Frattini argument, $|G/H| \mid |N_G(R)|$. Hence $p \mid |N_G(R)|$, because $p \in \pi(G/H)$. On the other hand, by Lemma (4.1), we have $R \in Syl_r(G)$. Thus by Lemma (3.2), $p \mid n$, which is a contradiction. Now assume that $p \mid n$. So $p \nmid n-1$. Since $p \neq 2$ and $p \mid n$, we have $n \geq 3$. By Lemma (4.3), it is sufficient to prove $r_1 \in \pi(K/H)$. For this, we assume that $r_1 \notin \pi(K/H)$ and obtain a contradiction. Let $r_1 \notin \pi(K/H)$ and $R_1 \in Syl_{r_1}(H)$. By Frattini argument, $|G/H| \mid |N_G(R_1)|$. Thus $p \mid |N_G(R_1)|$. On the other hand, by Lemma (4.1), we have $R_1 \in Syl_{r_1}(G)$. Hence, by Lemma (3.2) and Lemma(3.8), $p \mid n-1$, which is a contradiction. Therefore $r_1 \in \pi(K/H)$ and the proof is completed. $\square$

Lemma 4.6 . *Let $1 \leq H \leq K \leq G$ be a normal series of G and $p \neq 2$. If $p \nmid n$, $p \in \pi(G/K)$ and $p \in \pi(K/H)$, then at least one of r_i $(0 \leq i \leq 2)$ is in $\pi(K/H)$.*

Proof. Let $r_i \notin \pi(K/H)$ for all $0 \leq i \leq 2$. Suppose $\bar{K} = K/H$ and $\bar{G} = G/H$, then $\bar{K} \lhd \bar{G}$. Since $p \nmid n$, we have $|N_G(R)|_p = 1$, where $R \in Syl_r(G)$. We can see $r \in \pi(G/K)$. Thus by Lemma (2.3) , $r \mid p^t - 1$, where $|\bar{K}|_p = p^t$ and $1 \leq t < n(n-1)k$. Hence $2nk \mid t$. So $n \geq 3$. Now, we have two cases:
I. $p \nmid n-1$. Similar to the above argument $2(n-1)k \mid t$. Therefore $2n(n-1)k \mid t$ contrary to $1 \leq t < n(n-1)k$.
II. $p \mid n-1$. Hence $p \nmid n-2$ and therefore we have two subcases:

i. $n \geq 6$. Since $p \nmid n-2$, by Lemma (3.2) and Lemma (3.10), we have $|N_G(R_2)|_p = p^{2k}$, where $R_2 \in Syl_{r_2}(G)$. Thus, by Lemma (2.3), $r_2 \mid p^{t-e} - 1$, where $0 \leq e \leq 2k$. So $2(n-2)k \mid t-e$. On the other hand, there is a natural number m such that $2 \leq 2m < (n-1)$ and $t = 2nkm$, because $2nk \mid t$. Therefore $2(n-2)k \mid 4km - e$. It follows that $2(n-2)k \leq 4km - e$ or $4km = e$. Since $e \leq 2k$, $k \geq 1$ and $m \geq 1$, we have $4km \neq e$. So $2(n-2)k \leq 4km - e$ and $2 \leq 2m < n-1$. Hence $4km - e \leq 2(n-2)k$. Thus $e = 0$ and $t = 2nkm = n(n-2)k$. Moreover $r_1 \notin \pi(K/H)$. Therefore, similar to the above argument $2(n-1)k \mid t - e'$, where $p^{e'} \mid n-1$. So $2(n-1)k \mid n(n-2)k - e'$ and $e' < (n-1)/2$. Hence $e' = (n-2)k$, which is a contradiction with $e' < (n-1)/2$.

ii. $(n,p) = (4,3)$. We have $2nk = 8k \mid t$ and $1 \leq t < 12k$. Thus $t = 8k$. On the other hand, by Lemma (3.8), $6k \mid 8k - e$, where $e \leq 1$. Hence $e = 8k$, contrary to $e \leq 1$. $\square$

Lemma 4.7 . *Let $1 \leq H \leq K \leq G$ be a normal series of G, $p = 2$, $n \geq 5$ and n is odd. If $r_i \in \pi(G/K)$ for all $0 \leq i \leq 2$ and $4 \in \pi(K/H)$, then at least one of r_i $(i \in \{0,2\})$ is in $\pi(K/H)$.*

Proof. If not, assume that $r_i \notin \pi(K/H)$ for all $0 \leq i \leq 2$. Since $r \notin \pi(K/H)$, by Lemma (2.3), $2nk \mid t-e$, where $2 \leq t < n(n-1)k$ and $0 \leq e \leq 1$. Thus, there is a natural number as m such that $t-e = 2nkm$ and $2 \leq 2m < n-1$. On the other hand, $r_2 \notin \pi(K/H)$. So $2(n-2)k \mid t-e_2$, where $0 \leq e_2 \leq 2k+1$. Therefore $2(n-2)k \mid 4km+e-e_2$. Since $0 \leq e_2 \leq 2k+1$, $0 \leq e \leq 1$, $k \geq 1$, and $m \geq 1$, we have $2(n-2)k \leq 4km+e-e_2 \leq 2(n-2)k+1$. So $e-e_2 = 0$ and $2m = n-2$, it is a contradiction . Therefore at least one of r_i $(i \in \{0,2\})$ is in $\pi(K/H)$. $\square$

Remark 4.8 . *In Lemma (4.7), if $(n,k) = (5,1)$, then $r = 11$ and $r_1 = 17$. We can see $|N_{2D_n(2^k)}(R)| = 165$ and $|N_{2D_n(2^k)}(R_1)| = 136$, where $R \in Syl_r({}^2D_n(2^k))$ and $R_1 \in Syl_{r_1}({}^2D_n(2^k))$. So, similar to the above argument, one of r_i $(0 \leq i \leq 1)$ is in $\pi(K/H)$.*

Lemma 4.9 . *Let $1 \leq H \leq K \leq G$ be a normal series of G and $p \neq 2$. If $p \mid n$, $p \in \pi(G/K)$ and $p \in \pi(K/H)$, then at least one of r_i $(0 \leq i \leq 2)$ is in $\pi(K/H)$.*

Proof. We have two cases:
i. $n \geq 5$. Since $p \neq 2$, we have $p \nmid n-1$ and $p \nmid n-2$. Let r, r_1 and $r_2 \notin \pi(K/H)$. Similar to the proof of Lemma (4.6), we have $2(n-1)k \mid t$ and $2(n-2)k \mid t-e$, where $1 \leq t < n(n-1)k$ and $0 \leq e \leq 2k$. Thus there is a natural number as m such that $t = 2(n-1)km$ and $2 \leq 2m < n$. So $2(n-2)k \mid 2km-e$. Since $2km-e < nk < 2(n-2)k$, we have $2km = e$. Moreover $e \leq 2k$, therefore $e = 2k$, $m = 1$ and $t = 2(n-1)k$. On the other hand, $2nk \mid 2(n-1)k-e'$, where $e' < n/2$. So $2(n-1)k = e'$, which is a contradiction with $e' < n/2$. Thus at least one of r_i $(i = 0,1,2)$ is in $\pi(K/H)$.
ii. $n \leq 4$. Since $p \mid n$ and $p \neq 2$, we have $(p,n) = (3,3)$. If $r \notin \pi(K/H)$ and $r_1 \notin \pi(K/H)$, then similar to the above argument and the proof of Lemma (4.6), we have $6k \mid t-e$ and $4k \mid t$, where $1 \leq t < 6k$ and $0 \leq e \leq 1$, which is a contradiction. Therefore at least one of r_i $(0 \leq i \leq 1)$ is in $\pi(K/H)$. $\square$

Lemma 4.10 . *Let $1 \leq H \leq K \leq G$ be a normal series of G, $p = 2$, $n \geq 4$ and n is even. If $r_i \in \pi(G/K)$ for all $(0 \leq i \leq 1)$ and $4 \in \pi(K/H)$, then at least one of r_i $(0 \leq i \leq 1)$ is in $\pi(K/H)$.*

Proof. We assume that $r \notin \pi(K/H)$ and $r_1 \notin \pi(K/H)$, then we obtain a contradiction. By Lemma (2.5), $2nk \mid t-e$ and $2(n-1)k \mid t-e_1$, where $1 \leq t < n(n-1)k$, $0 \leq e < 1+n/2$ and $0 \leq e_1 \leq 1$. It follows, there is a natural number as m such that $t-e = 2nkm$ and $2 \leq 2m < n-1$. Hence $2(n-1)k \mid 2km+e-e_1$. Since $0 \leq e \leq 1+n/2$, $k \geq 1$ and $m \geq 1$, we have $2km+e-e_1 \geq 2(n-1)k$, it is a contradiction. Therefore at least one of r_i $(0 \leq i \leq 1)$ is in $\pi(K/H)$. $\square$

Remark 4.11 . *In Lemma (4.10), if* $(n,k)=(4,1)$*, then* $r=17$ *and* $r_2=5$*. We can see* $|N_{{}^2D_n(2^k)}(R)|=68$ *and* $|N_{{}^2D_n(2^k)}(R_2)|=720$*, where* $R\in Syl_r({}^2D_n(2^k))$ *and* $R_2\in Syl_{r_2}({}^2D_n(2^k))$*. So, similar to the above argument, one of* r_i $(i\in\{0,2\})$ *is in* $\pi(K/H)$*.*

Proof of the main Theorem . We know that ${}^2D_2(p^k)\cong L_2(p^{2k})$ and ${}^2D_3(p^k)\cong U_4(p^k)$. Moreover $L_2(p^{2k})$ and $U_4(p^k)$ are characterized in [1] and [5], respectively. Thus we assume that $n\geq 4$. We prove this theorem in two cases:

Case1. $p\neq 2$. Let $1=G_0\leq G_1\leq ...\leq G_f=G$ be a chief series of G and $j_0=Max\{1\leq i\leq f|\ p\in\pi(G_i/G_{i-1})\}$. Assume that $H{:=}\ G_{j_0-1}$ and $K{:=}\ G_{j_0}$. So $p\notin\pi(G/K)$ and $p\in\pi(K/H)$. First, we prove that $|K/H|_p=p^t=p^{n(n-1)k}$. If not, then by Lemma (4.1), we have $r_i\notin\pi(G/K)$, where $0\leq i<n/2$. By Lemma (4.5), we have r, r_1 and r_2 are in $\pi(K/H)$. On the other hand, by Lemma (4.6) and Lemma (4.9), if $p\in\pi(G_j/G_{j-1})$, where $j<j_0$, then at least one of r_i $(0\leq i\leq 2)$ is in $\pi(G_j/G_{j-1})$. Assume that $|G_j/G_{j-1}|_p=p^{t_j}$, then $|G_j/G_{j-1}|\geq p^{5t_j/2}$. If not, since there is one of r_i $(0\leq i\leq 2)$ in $\pi(G_j/G_{j-1})$, we have G_j/G_{j-1} is a simple group of Lie type in characteristic p. Thus by Lemma (3.12), $r_i\notin\pi(K/H)$, which is a contradiction. Hence $|G_j/G_{j-1}|\geq p^{5t_j/2}$ for all $j<j_0$, where $|G_j/G_{j-1}|_p=p^{t_j}$. So $|K/H|<p^{5t/2}$ because $|G|<p^{5n(n-1)k/2}$ and $p\notin\pi(G/K)$. Therefore K/H is a simple group of Lie type in characteristic p, because $r\in\pi(K/H)$. Thus $r_i\notin\pi(G_j/G_{j-1})$ because $r_i\in\pi(K/H)$, where $0\leq i\leq 2$ and $j<j_0$. Hence $p\notin\pi(G_j/G_{j-1})$ for all $j<j_0$. It follows that $|K/H|_p=p^{n(n-1)k}$. Therefore by Lemma (3.13), we have $K/H\cong{}^2D_n(p^k)$. So $G\cong{}^2D_n(p^k)$.

Case 2. $p=2$. Let $1=G_0\leq G_1\leq ...\leq G_f=G$ be a chief series of G and $j_0=Max\{1\leq i\leq f|\ 2\in\pi(G_i/G_{i-1})\}$. Let $M{:=}\ G_{j_0-1}$ and $N{:=}\ G_{j_0}$. So, by Lemma (4.1), $r_i\notin\pi(G/N)$, for all $0\leq i<n/2$. We assume that H:= G_{j_1-1} and K:= G_{j_1}, where $j_1=Max\{1\leq i\leq j_0|\ r\in\pi(G_i/G_{i-1})\}$. We claim that $2^{n(n-1)k-1}\leq|K/H|_2\leq 2^{n(n-1)k}$. If there is one j such that $j<j_1$ and $|G_j/G_{j-1}|<2^{5t_j/2}$, where $|G_j/G_{j-1}|_2=2^{t_j}>2$, then by Lemma (4.7) and Lemma (4.10), at least one of r_i $(0\leq i\leq 2)$ is in $\pi(G_j/G_{j-1})$. Thus G_j/G_{j-1} is a simple group of Lie type in characteristic 2. So, by Lemma (3.12), $r_i\notin\pi(K/H)$, for some i such that $0\leq i\leq 2$, it is a contradiction. Therefore $|G_j/G_{j-1}|\geq 2^{5t_j/2}$, where $|G_j/G_{j-1}|_2=2^{t_j}>2$ for all $j<j_1$. Since $|N_G(R)|_2\leq 2$ or $|N_G(R_1)|_2\leq 2$, we have at most one k $(1\leq k\leq n)$ exist such that $|G_k/G_{k-1}|_2=2$ and $r_i\notin\pi(G_k/G_{k-1})$ $(0\leq i\leq 2)$. Now, we must prove $|K/H|_2\neq 1$. If not, by Lemma (4.4), $|G/K|_2=2$. Thus $|H|\geq 2^{5(n(n-1)k-1)/2}>|G|$, it is a contradiction. Hence $|K/H|=2^e\neq 1$. Since $|H|\geq 2^{5e'/2}$, where $n(n-1)k-1\leq e+e'\leq n(n-1)k$, we have $|K/H|<2^{5e/2}$. Therefore K/H is a simple group of Lie type in characteristic 2, because $r\in\pi(K/H)$. It follows that $r_i\notin\pi(H)$ for all $(0\leq i\leq 2)$. Thus, by Lemma (4.7) and Lemma (4.10), we have $2^{n(n-1)k-1}\leq|K/H|_2\leq 2^{n(n-1)k}$.

Therefore, by Lemma (3.13), $K/H \cong {}^2D_n(2^k)$. So $G \cong {}^2D_n(2^k)$. $\square$

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