International Journal of Algebra, Vol. 2, 2008, no. 19, 905 - 918

The VQM-Group and its Applications

Michael Aristidou

Digipen Institute of Technology
Department of Mathematics
5001, 150th Ave., NE
Redmond, WA 98052, USA
aristidou75@yahoo.com

Xin Li

Digipen Institute of Technology
Department of Computer Science
5001, 150th Ave., NE
Redmond, WA 98052, USA

Abstract

In this paper, we introduce the VQM-set and study some of its algebraic properties. Its elements are triplets consisting of a *vector*, a *quaternion* and a *matrix*, hence the VQM abbreviation. Operations such as addition, scalar multiplication, are defined on the VQM-set, which turn it into a vector space. A special multiplication turns a subset of the VQM-set into a group, called the VQM-group. We show how this VQM-group could be used to represent the generalized transformations in three-dimensional computer graphics, with translation, rotation and non-uniform scaling factor components.

Mathematics Subject Classification: 15A33, 15A30, 20H25, 15A03.

Keywords: Quaternion, Rotation, Group, VQS-transformation, Computer graphics.

1 Introduction

The Quaternions, denoted by $\mathbb{H}$, were first invented by Sir William Rowan Hamilton in 1843 as an extension of the complex numbers into four-dimensions [4]. The rotational properties of quaternions were well known to Hamilton, but it wasn't until 1985 that quaternions were introduced into the computer graphics community to represent rotations [7]. Let $q = [x, y, z, w] \in \mathbb{H}_1$ where $\mathbb{H}_1$ is the group of *unit-length* quaternions under multiplication[1]. It is easy to prove that when $q = [\sin\theta A, \cos\theta]$, where A is a unit 3D vector and $\theta \in (-\pi, \pi]$, then q is a unit quaternion. Furthermore, one can also show that the rotation, $R_q : \mathbb{H} \to \mathbb{H}$ defined by $R_q(r) = qrq^{-1}$, with $r = [a, b, c, 0]$ (called a "pure quaternion"), is a linear transformation and it represents a rotation by 2θ around the vector A in the so-called axis-angle notation [1]. Notice that, as a matter of fact, R_q is basically a rotation in $\mathbb{R}^3$. Also, one can easily show that any non-unit quaternion $p = \lambda q, \lambda \in \mathbb{R}$, $|q| = 1$, represents the same rotation as q. In particularly, in 3-D graphics, quaternions are often preferred over other rotation representations (i.e. fixed-angle, Euler-angle, etc.) for their practicality, efficiency and elegance. In addition, quaternions also allow us to avoid unpleasant phenomena such as the "gimbal-lock" and errors in interpolations.

Similar to the homogeneous matrix approach, a quaternionic rotation could be combined with a translation vector, and a uniform scaling factor. In literature, this transformation is usually known as a *VQS-transformation* (or a *VQS-structure* in computer graphics) [5], where the "VQS" stands for "vector-quaternion-scalar". More specifically, a VQS-transformation is a map $S = [v, q, s]$, defined by:

$$\begin{aligned} S(r) &= [v, q, s]r \\ &= q(sr)q^{-1} + v \end{aligned}$$

where $v = [v_1, v_2, v_3, 0] \in \mathbb{H}$ is a pure quaternion representing a translation, $q = [x, y, z, w] \in \mathbb{H}_1$ is a unit quaternion representing a rotation, and $s \in \mathbb{R}$ is a scalar representing a uniform scaling factor. In other words, a VQS-transformation basically scales r by s, rotates the outcome by q, and then translates the latter by v. It is easy to see that the sequence above always results in a pure quaternion.

Example 1.1. *Let $T = [v, q, s]$ be a VQS-transformation, where v is a pure quaternion, q is a unit quaternion, and s is a scalar, given by: $v = [1, 1, 1, 0], q = [0, 1, 0, 0]$ (a rotation by 180° about the y-axis), and $s = 3$. Consider also the vector (point) r, also a pure quaternion, given by: $r = [-1, 2, -1, 0]$. Applying*

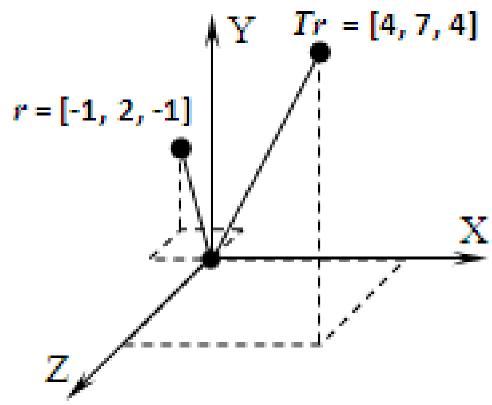

Figure 1:

T to r, we get:

$$\begin{aligned} Tr = [v,q,s]r &= q(sr)q^{-1} + v \\ &= [0,1,0,0](3[-1,2,-1,0])[0,-1,0,0] + [1,1,1,0] \\ &= [0,1,0,0][-3,6,-3,0][0,-1,0,0] + [1,1,1,0] \\ &= [0,1,0,0][-3,0,3,6] + [1,1,1,0] \\ &= [3,6,3,0] + [1,1,1,0] \\ &= [3,6,3,0] + [1,1,1,0] \\ &= [4,7,4,0] \end{aligned}$$

Note that, when representing the transformation, VQS-transformations *only* allow a *uniform* scaling operation. That is, all x, y and z components of the vector r must be scaled by the *same* value. The idea of this paper is to extend VQS-transformations to VQM-transformations ("vector-quaternion-matrix") and achieve a *non-uniform* scaling operation, without necessarily losing any of the algebraic properties.

2 Quaternion Extension

In order to define the VQM structure, we need to extend the quaternion operations to include the multiplication with a matrix and discuss some properties of this operation. Let M be any 4×4 matrix, i.e. $M = (m_i)$, where m_i ($i \in \{1,2,3,4\}$) are the four column vectors of M. The "multiplication" between a quaternion q and M is thus defined as follows:

Definition 2.1. *Let $q \in \mathbb{H}$ and $M \in M_{4\times4}(\mathbb{R})$. Then, $q \cdot M = (qm_i)$ and $M \cdot q = (m_i q)$.*

Note that each column in M is a 4-dimensional vector and therefore can be treated as a quaternion. In other words, the definition states that the multiplication of a quaternion and a 4×4 matrix is the multiplication(s) of the quaternion and the column vectors (quaternions) of the given matrix. The operation always results in another 4×4 matrix. Clearly, the above definition does not represent a multiplication in the ordinary sense. If $q \neq 0$, it is actually a *left* and a *right action* of the quaternion group $\mathbb{H} - \{0\}$ on the set $M_{4\times 4}(\mathbb{R})$. This becomes clear from the following propositions:

Proposition 2.2. *We have* $e \cdot M = M \cdot e = M$, *where* $e = [0,0,0,1]$ *represents the identity quaternion.*

Proof. Indeed, by Definition 2.1, we have:

$$e \cdot M = (em_i) = (m_i) = M = (m_i) = (m_i e) = M \cdot e$$

□

We also have the "associative-like" properties of the multiplication between a quaternion and a matrix:

Proposition 2.3. *Let* M *be any* 4×4 *matrix and* q_1 *and* q_2 *be quaternions. Then, we have:*

$$(i)\ q_1 \cdot (M \cdot q_2) = (q_1 \cdot M) \cdot q_2$$
$$(ii)\ q_1 \cdot (q_2 \cdot M) = (q_1 q_2) \cdot M$$
$$(iii)\ (M \cdot q_1) \cdot q_2 = M \cdot (q_1 q_2)$$

Proof. For (i) we have, $q_1 \cdot (M \cdot q_2) = q_1 \cdot (m_i q_2) = (q_1(m_i q_2)) = ((q_1 m_i) q_2) = (q_1 m_i) \cdot q_2 = (q_1 \cdot M) \cdot q_2$. Properties (ii) and (iii) are done similarly. □

The above properties indicate that, when multiplying a sequence of quaternions and a matrix, the operations can be performed in any preferred order and, yet, it will yield exactly the same result.

We further call $M = (m_i)$, $i \in \{1,2,3,4\}$, a "*homogeneous quaternion matrix*", if all the first three column vectors m_i , $(i = 1,2,3)$, are pure quaternions and $m_4 = e$ $(i = 4)$ is the identity quaternion (i.e. $m_i = [a_{i1}, a_{i2}, a_{i3}, 0]^t$ and $e = [\vec{0}, 1]^t$). In other words, $M = \begin{pmatrix} (a_{ij}) & \vec{0}^t \\ \vec{0} & 1 \end{pmatrix}$, $i, j \in \{1,2,3\}$, where (α_{ij}) is the 3×3 matrix formed by the α_{ij}-components of the m_i's, $(i = 1,2,3)$ and $\vec{0}$ is the usual 1×3 zero vector. Under this definition, for any quaternion q and pure quaternion r, we have:

Proposition 2.4. *Let* M *be any* 4×4 *homogeneous quaternion matrix,* q *be a quaternion and* r *be a pure quaternion. Then:*

$$q(Mr)q^{-1} = (q \cdot M \cdot q^{-1})r$$

Proof. Since M is a homogeneous quaternion matrix, its multiplication with r yields another pure quaternion. Rotating Mr by q is the same as applying an equivalent rotation matrix [7]. Let L_q be a 4×4 homogeneous matrix representing the same rotation as the one by q. Following Def. 2.1 and the associativity of matrix multiplication, we have:

$$q(Mr)q^{-1} = L_q(Mr) = (L_qM)r = (L_q(m_i))r = (q(m_i)q^{-1})r = (q \cdot M \cdot q^{-1})r$$

□

By a similar argument, we also have:

Proposition 2.5. $(q_2 \cdot N \cdot q_2^{-1})(q_1 \cdot M \cdot q_1^{-1}) = q_2 \cdot (N(q_1 \cdot M \cdot q_1^{-1})) \cdot q_2^{-1}$*, where* q_1 *and* q_2 *are quaternions, and* M *and* N *are homogeneous quaternion matrices.*

Proof. Let L_{q_2} and K_{q_1} be the matrices as in the proof of Prop.2.4. Then:

$$\begin{aligned}(q_2 \cdot N \cdot q_2^{-1})(q_1 \cdot M \cdot q_1^{-1}) &= (L_{q_2}N)(K_{q_1}M)\\ &= L_{q_2}(N(K_{q_1}(M))\\ &= q_2 \cdot (N(q_1 \cdot M \cdot q_1^{-1})) \cdot q_2^{-1}\end{aligned}$$

□

Notice that the multiplication of two homogeneous quaternion matrices always generates another homogeneous quaternion matrix. Furthermore, since for any pure quaternion m_i, qm_iq^{-1} yields another pure quaternion and since $qeq^{-1} = e$, the "rotation" of a homogeneous quaternion matrix with any quaternion will result in another homogeneous quaternion matrix.

3 The VQM Algebraic Structure

In this section, we introduce the VQM as an abstract algebraic set, define some operations, and examine important algebraic properties of the set.

Definition 3.1. *A VQM set* $\mathcal{T}$ *is a set of triplets* $T = [v, q, M]$*, where* $v = [a, b, c, 0] \in \mathbb{H}$ *is a pure quaternion,* $q = [x, y, z, w] \in \mathbb{H}$ *is a quaternion, and* M *is a* 4×4 *homogeneous quaternion matrix.*

In other words, a VQM structure is an extension of a VQS structure by replacing the uniform scaling factor with a homogeneous quaternion matrix. In the following discussion, we denote by $\mathcal{T}$ be the set of all VQM maps, i.e. $\mathcal{T} = \{T_i \,|\, T_i = [v_i, q_i, M_i], i \in N\}$.

Definition 3.2. *Let* $T_1 = [v_1, q_1, M_1]$, $T_2 = [v_2, q_2, M_2] \in \mathcal{T}$, *where* $M_1 = \begin{pmatrix} (\alpha_{ij}) & \vec{0}^t \\ \vec{0} & 1 \end{pmatrix}$, $M_2 = \begin{pmatrix} (\beta_{ij}) & \vec{0}^t \\ \vec{0} & 1 \end{pmatrix}$ *are homogeneous quaternion matrices. Then,* addition *on* $\mathcal{T}$ *is defined by:* $T_1+T_2 = [v_1, q_1, M_1]+[v_2, q_2, M_2] = [v_1+v_2, q_1+q_2, M_1+M_2]$, *where* $M_1+M_2 = \begin{pmatrix} (\alpha_{ij}+\beta_{ij}) & \vec{0}^t \\ \vec{0} & 1 \end{pmatrix}$.

Remark 3.3. *Since addition of quaternions (and matrices) is commutative, so is the VQM addition. Also, the element* $T = [0, 0, 0]$ (*where the last* 0 *is the* $0 = \begin{pmatrix} (0) & \vec{0}^t \\ \vec{0} & 1 \end{pmatrix}$) *is the 'identity' element for addition.*

Definition 3.4. *Let* $T = [v, q, M] \in \mathcal{T}$ *and* $c \in \mathbb{R}$, *where* $M = \begin{pmatrix} (a_{ij}) & \vec{0}^t \\ \vec{0} & 1 \end{pmatrix}$. *Then,* scalar multiplication *on* $\mathcal{T}$ *is defined by:* $cT = c[v, q, M] = [cv, cq, cM]$, *where* $cM = \begin{pmatrix} (c\alpha_{ij}) & \vec{0}^t \\ \vec{0} & 1 \end{pmatrix}$.

Remark 3.5. *Since the scalar product is distributive over addition for quaternions (and matrices), so is the scalar product over the VQM set. Also, it is not hard to show that* τ *is closed under VQM addition and scalar multiplication and, considering Rem.*3.3*, that is a vector space. Indeed,* $\mathcal{T}$ *satisfies all vector space axioms.* $\mathcal{T}$ *is* 108*-dimensional, due to the fact that* v *is* 3*-dimensional,* q *is* 4*-dimensional, a homogeneous matrix* M *is* 9*-dimensional, and therefore* $\mathcal{T} \cong R^3 \times R^4 \times R^9$ *and* $\dim(R^3 \times R^4 \times R^9) = 108$.

Definition 3.6. *Let* $T_1 = [v_1, q_1, M_1]$ *and* $T_2 = [v_2, q_2, M_2] \in \mathcal{T}$, *where* $q_1, q_2 \in \mathbb{H} - \{0\}$ *and* M_1 *and* M_2 *are homogeneous quaternion matrices. Then,* multiplication *on* $\mathcal{T}$ *is defined by:*

$$T_2T_1 = [v_2, q_2, M_2][v_1, q_1, M_1] = [q_2(M_2v_1)q_2^{-1} + v_2, q_2q_1, (q_1^{-1} \cdot M_2 \cdot q_1)(q_1 \cdot M_1 \cdot q_1^{-1})]$$

As in the case of VQM addition, and scalar multiplication, $\mathcal{T}$ is closed under multiplication since the first and second elements are quaternions, and the third produces a 4×4 homogeneous quaternion matrix (since the product of two homogeneous quaternion matrices is a homogenous quaternion matrix). Nevertheless, the geometrical intuition of the above multiplication may not be as obvious as that of addition, and scalar multiplication, on the VQM structure. It will become clear when we bring the structure into the context of 3D transformations discussed in the following sections.

Proposition 3.7. *The multiplication defined on* $\mathcal{T}$ *is* associative*. That is:*

$$[v_3, q_3, M_3]([v_2, q_2, M_2][v_1, q_1, M_1]) = ([v_3, q_3, M_3][v_2, q_2, M_2])[v_1, q_1, M_1].$$

Proof. See Appendix. □

Remark 3.8. *Notice that the set $\mathcal{T}$ does* not *form an* algebra*. That is because the multiplication above is not defined for all of its elements (e.g. one cannot multiply $T_1 = [v_1, 0, M_1] \in \mathcal{T}$ with anything in $\mathcal{T}$). Considering a subset $\mathcal{T}^0$ of $\mathcal{T}$, for which $q_1, q_2 \in \mathbb{H} - \{0\}$, one can close the multiplication, even though one still does not have an algebra since $\mathcal{T}^0$ is no longer a vector space (no additive identity, since $[0, 0, 0] \notin \mathcal{T}^0$).*

We denote the set of all VQM maps $T = [v, q, M]$, where $q \in \mathbb{H} - \{0\}$ and M is invertible, by $\mathcal{T}^*$. That is:

$$\mathcal{T}^* = \{T \,|\, T = [v, q, M], q \in \mathbb{H} - \{0\}, M \text{ invertible}\}.$$

Lemma 3.9. *If* M *is invertible, then $q \cdot M \cdot q^{-1}$ is invertible.*

Proof. Denote by L_q the matrix rotation operator $q \cdot () \cdot q^{-1}$, as in the proof of Proposition 2.4. Then, $(q \cdot M \cdot q^{-1})^{-1} = (L_q M)^{-1} = M^{-1} L_q^{-1}$, which means that $(q \cdot M \cdot q^{-1})^{-1}$ exists since both M^{-1} and L_q^{-1} exist. □

Proposition 3.10. *$\mathcal{T}^*$ is* closed *under the multiplication defined in Def.3.6.*

Proof. Let $T_i, T_j \in \mathcal{T}^*$. Then, $q_i, q_j \in \mathbb{H} - \{0\}$ and M_i and M_j are invertible. Since $\mathbb{H}$ is a division algebra, we have $q_j q_i \in \mathbb{H} - \{0\}$, and in view of Lem.3.9, we also have that $q_i \cdot M_i \cdot q_i^{-1}$ and $q_i^{-1} \cdot M_j \cdot q_i$ are invertible. Hence, $(q_i^{-1} \cdot M_j \cdot q_i)(q_i \cdot M_i \cdot q_i^{-1})$ is also invertible (its inverse is $(q_i \cdot M_i \cdot q_i^{-1})^{-1}(q_i^{-1} \cdot M_j \cdot q_i)^{-1}$). And since $T_j T_i = [q_j (M_j v_i) q_j^{-1} + v_j, q_j q_i, (q_i^{-1} \cdot M_j \cdot q_i)(q_i \cdot M_i \cdot q_i^{-1})]$, that means $T_j T_i \in \mathcal{T}^*$, since $q_j q_i \in \mathbb{H} - \{0\}$ and since its last component is invertible. □

Remark 3.11. *Notice that a homogeneous quaternion matrix M being invertible is not equivalent in having* $\det M \neq 0$*. For example, if i and j denote the square roots of unity in $\mathbb{H}$, the matrix:* $M = \begin{pmatrix} i & j & 0 & 0 \\ j & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *has* $\det M = 0$, *but it is invertible though, with* $M^{-1} = \begin{pmatrix} -i/2 & -j/2 & 0 & 0 \\ -j/2 & -i/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Proposition 3.12. *(a) $\mathcal{T}^*$ has an identity element $\boldsymbol{I}$ with respect to its multiplication. (b) Every T in $\mathcal{T}^*$ has an* inverse *T^{-1} with respect to the multiplication.*

Proof. See Appendix □

Theorem 3.13. *The set $\mathcal{T}^*$ is a* group *with respect to the multiplication defined on $\mathcal{T}$.*

Proof. By Prop.3.10, we have that $\mathcal{T}$ is closed under multiplication. Also, by Prop.3.7 we have that the multiplication is associative, and by Prop.3.12 that it has and identity element, and that each element in $\mathcal{T}^*$ has an inverse. Therefore, $\mathcal{T}^*$ is a group. □

Remark 3.14. *Since the quaternion multiplication is non-commutative, it makes the VQM multiplication non-commutative as well. Therefore, $\mathcal{T}^*$ is a non-commutative group which we call the* VQM-group.

4 Transformations with VQM Maps

As mentioned earlier, a VQS map can be used to represent a three-dimensional transformation with translation, rotation and scaling operations. However, only a uniform scaling factor is allowed. This shortcoming restricts its application in computer graphics and motivates an extension to the VQM map. We first define the following:

Definition 4.1. *Let $r = [a, b, c, 0]$ be a pure quaternion where (a, b, c) represents a vector in $\mathbb{R}^3$. A* VQM transformation *$T = [v, q, M]$ is a map $T : \mathbb{H} \to \mathbb{H}$, defined by:*

$$T(r) = [v, q, M]r = q(Mr)q^{-1} + v$$

In the above definition, if M represents a 4×4 homogeneous non-uniform scaling matrix, q a rotation quaternion and v a translation vector (i.e. a pure quaternion), then, a VQM transformation is to scale by M, rotate by q, and translate by v. It is easy to verify that the result is equivalent to a transformation by a 4×4 homogeneous matrix that contains the same scaling, rotation and translation.

Example 4.2. *Let $T = [v, q, M]$ be a VQM-transformation, where v is a pure quaternion, q is a unit quaternion, and M is 4×4 homogeneous quaternion matrix, given by: $v = [1, 1, 1, 0], q = [0, 1, 0, 0]$, (a rotation by 180° about the y-axis), and $M = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Consider the vector (point) r, also a pure*

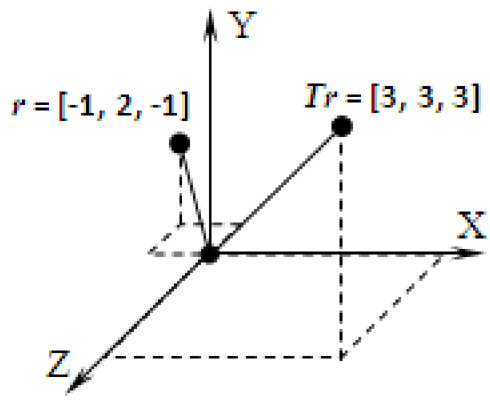

Figure 2:

quaternion, given by: $r = [-1, 2, -1, 0]$. *Applying* T *to* r, *we get:*

$$
\begin{aligned}
Tr &= [v, q, M]r = q(Mr)q^{-1} + v \\
&= [0,1,0,0]\left(\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}[-1,2,-1,0]\right)[0,-1,0,0] + [1,1,1,0] \\
&= [0,1,0,0][-2,2,-2,0][0,-1,0,0] + [1,1,1,0] \\
&= [0,1,0,0][-2,0,2,2] + [1,1,1,0] \\
&= [2,2,2,0] + [1,1,1,0] \\
&= [3,3,3,0]
\end{aligned}
$$

Remark 4.3. *Notice that the transformation above can also be executed, in view of Prop.2.4, as follows:*

$$
\begin{aligned}
Tr &= [v, q, M]r = q(Mr)q^{-1} + v = (q \cdot M \cdot q^{-1})r + v \\
&= \left([0,1,0,0] \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot [0,-1,0,0]\right)[-1,2,-1,0] + [1,1,1,0] \\
&= [3,3,3,0]
\end{aligned}
$$

Theorem 4.4. *Two VQM-transformations can be composed via the VQM multiplication.*

Proof. Let $[v_1, q_1, M_1], [v_2, q_2, M_2] \in \mathcal{T}$ be two VQM transformations. The consecutive transformation on a pure quaternion $r = [a, b, c, 0]$ with two VQM maps is given, by definition, as follows:

$$\begin{aligned}
[v_2, q_2, M_2]([v_1, q_1, M_1]r) &= [v_2, q_2, M_2](q_1(M_1 r)q_1^{-1} + v_1) \\
&= q_2(M_2(q_1(M_1 r)q_1^{-1} + v_1))q_2^{-1} + v_2 \\
&= q_2(M_2(q_1(M_1 r)q_1^{-1}))q_2^{-1} + q_2(M_2 v_1)q_2^{-1} + v_2 \\
&= q_2(q_1 q_1^{-1})(M_2(q_1(M_1 r)q_1^{-1}))(q_1 q_1^{-1})q_2^{-1} + q_2(M_2 v_1)q_2^{-1} + v_2 \\
&= (q_2 q_1)(q_1^{-1}(M_2(q_1(M_1 r)q_1^{-1}))q_1)(q_1^{-1}q_2^{-1}) + q_2(M_2 v_1)q_2^{-1} + v_2 \\
&= (q_2 q_1)(((q_1^{-1} \cdot M_2 \cdot q_1)(q_1 \cdot M_1 \cdot q_1^{-1}))r)(q_2 q_1)^{-1} + q_2(M_2 v_1)q_2^{-1} + v_2 \\
&= [q_2(M_2 v_1)q_2^{-1} + v_2, q_2 q_1, (q_1^{-1} \cdot M_2 \cdot q_1)(q_1 \cdot M_1 \cdot q_1^{-1})]r \\
&= ([v_2, q_2, M_2][v_1, q_1, M_1])r
\end{aligned}$$

□

The above theorem demonstrates an important feature of the VQM structure. It shows that, like the approach by 4×4 homogeneous transformation matrix, any consecutive transformations with VQM maps can first be composed by multiplying the VQM-transformations together, and then apply the result as a single transformation to all vertices of an object. This implies a significant improvement in performance efficiency in praxis.

The derivation in the proof also captures the geometric intuition of the definition of the multiplication. It simply points out that the composition of two VQM-transformations, (e.g. $[v_2, q_2, M_2]$ and $[v_1, q_1, M_1]$), always yields another VQM-transformation, where:

(1) its translation component is obtained by applying the second VQM transformation $[v_2, q_2, M_2]$ on the translation vector v_1 of the first transformation (note that $q_2(M_2 v_1)q_2^{-1} + v_2 = [v_2, q_2, M_2]v_1$).

(2) its rotation component is equal to a composition of two rotations $q_2 q_1$.

(3) its scaling matrix is the product of two rotated scaling matrices. Note that, from Proposition 2.5, we have $(q_2 \cdot M_2 \cdot q_2^{-1})(q_1 \cdot M_1 \cdot q_1^{-1}) = q_2 \cdot (M_2(q_1 \cdot M_1 \cdot q_1^{-1})) \cdot q_2^{-1}$. It implies that the first matrix $(q_1 \cdot M_1 \cdot q_1^{-1})$ is the scaling matrix M_1 rotated by q_1, which is concatenated with the other scaling matrix M_2 (i.e. $M_2(q_1 \cdot M_1 \cdot q_1^{-1})$), then rotated by inverted q_1 (since the result will be rotated again by the combined $q_2 q_1$ when used in the *VQM* transformation).

5 Conclusion

In this paper, the VQM-set $\mathcal{T}$ was introduced as an abstract algebraic set. Operations, such as addition and scalar multiplication, were defined which made it into a vector space. Under a defined multiplication, the VQM-subset $\mathcal{T}^*$

formed a non-abelian group. While many other operations and properties on the VQM-set remain to be explored, one important application of the VQM-group is that it represents three-dimensional transformations in 3D computer graphics. As such, VQM transformations allow different scaling (non-uniform) in all three coordinate dimensions. This is the major feature that distinguishes VQM from VQS transformations. On the other hand, just like in the VQS case, the composition of VQM transformations can be implemented through the VQM multiplication, and the transformation component (translation, rotation and scaling factors) of key frames can be interpolated separately using different interpolation algorithms (such as Lerp, Slerp, Elerp, etc.) [2]. This would allow the VQM structure to maintain its superior geometric characteristics over other interpolation methods with matrices, or other transformation notations based on fixed angles, Euler angles, etc. In practice, functions for VQM transformation and concatenation can be procedurally simplified. When interpolated between two VQM key frames with fixed intervals in translation, rotation and scaling parameters, an incremental algorithm can be applied to remarkably reduce the computational expenses of VQM transformation and concatenation [3].

Notes

1. We warn the reader for the slight difference in notation. In this paper, a quaternion $q = [x, y, z, w]$ has w (not x) as its scalar components, unlike in [7].

References

[1] E. Dam, M. Koch, and M. Lillholm, “Quaternions, Interpolation and Animation”, Technical Report (DIKU-TR-98/5), Univ. of Copenhagen, 1998.

[2] X. Li, “VQS Transformation with an Incremental Approach”, Game-On Asia Conference Processing, Kyoto, Japan, 2007.

[3] X, Li, et al. “Transformation and Interpolation with VQM Structure”,GAMEON’2008 Conference Proceedings, Valencia, Spain, Nov. 2008.

[4] R. Remmert et al, *Numbers*, Springer, 1991.

[5] W. Robinett and R. Holloway, “The Visual Display Transformation for Virtual Reality”, Technical Report (TR94-031), UNC-Chapel Hill, 1994.

[6] R. Schafer, *An introduction to Nonassociative Algebras*, Academic Press, 1996.

[7] K. Shoemake, "Animating Rotation with Quaternion Curves", Computer Graphics, Volume 19, No 3, 1985.

Appendix

Proposition 3.7. *The multiplication defined on $\mathcal{T}$ is* associative*.*

Proof. Indeed, by the definition of VQM multiplication, we have:

$$\begin{aligned}
&[v_3, q_3, M_3]([v_2, q_2, M_2][v_1, q_1, M_1]) \\
&\quad = [v_3, q_3, M_3][q_2(M_2 v_1)q_2^{-1} + v_2, q_2 q_1, (q_1^{-1} \cdot M_2 \cdot q_1)(q_1 \cdot M_1 \cdot q_1^{-1})] \\
&\quad = [q_3(M_3(q_2(M_2 v_1)q_2^{-1} + v_2))q_3^{-1} + v_3, q_3 q_2 q_1, \\
&\quad ((q_2 q_1)^{-1} \cdot M_3 \cdot (q_2 q_1))((q_2 q_1) \cdot (q_1^{-1} \cdot M_2 \cdot q_1)(q_1 \cdot M_1 \cdot q_1^{-1})) \cdot (q_2 q_1)^{-1})]
\end{aligned}$$

From Prop.'s 2.2, 2.3 and 2.5, since

$$\begin{aligned}
&(q_2 q_1) \cdot (q_1^{-1} \cdot M_2 \cdot q_1)(q_1 \cdot M_1 \cdot q_1^{-1})) \cdot (q_2 q_1)^{-1} \\
&\quad = (q_2 q_1) \cdot (q_1^{-1} \cdot (M_2(q_1 \cdot M_1 \cdot q_1^{-1})) \cdot q_1) \cdot (q_2 q_1)^{-1} \\
&\quad = (q_2 q_1 q_1^{-1}) \cdot (M_2(q_1 \cdot M_1 \cdot q_1^{-1})) \cdot (q_1 q_1^{-1} q_2^{-1}) \\
&\quad = q_2 \cdot (M_2(q_1 \cdot M_1 \cdot q_1^{-1})) \cdot q_2^{-1} \\
&\quad = (q_2 \cdot M_2 \cdot q_2^{-1})(q_1 \cdot M_1 \cdot q_1^{-1})
\end{aligned}$$

we have:

$$\begin{aligned}
&[v_3, q_3, M_3]([v_2, q_2, M_2][v_1, q_1, M_1]) \\
&\quad = [q_3(M_3(q_2(M_2 v_1)q_2^{-1} + v_2))q_3^{-1} + v_3, q_3 q_2 q_1, \\
&\qquad ((q_2 q_1)^{-1} \cdot M_3 \cdot (q_2 q_1))(q_2 \cdot M_2 \cdot q_2^{-1})(q_1 \cdot M_1 \cdot q_1^{-1})] \\
&\quad = [q_3(M_3(q_2(M_2 v_1)q_2^{-1}))q_3^{-1} + q_3(M_3 v_2)q_3^{-1} + v_3, q_3 q_2 q_1, \\
&\qquad ((q_2 q_1)^{-1} \cdot M_3 \cdot (q_2 q_1))(q_2 \cdot M_2 \cdot q_2^{-1})(q_1 \cdot M_1 \cdot q_1^{-1})] \qquad (1)
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&([v_3, q_3, M_3][v_2, q_2, M_2])[v_1, q_1, M_1] \\
&\quad = [q_3(M_3 v_2)q_3^{-1} + v_3, q_3 q_2, (q_2^{-1} \cdot M_3 \cdot q_2)(q_2 \cdot M_2 \cdot q_2^{-1})][v_1, q_1, M_1] \\
&\quad = [(q_3 q_2)(((q_2^{-1} \cdot M_3 \cdot q_2)(q_2 \cdot M_2 \cdot q_2^{-1})v_1)(q_3 q_2)^{-1} + q_3(M_3 v_2)q_3^{-1} + v_3, q_3 q_2 q_1, \\
&\qquad (q_1^{-1} \cdot ((q_2^{-1} \cdot M_3 \cdot q_2)(q_2 \cdot M_2 \cdot q_2^{-1})) \cdot q_1)(q_1 \cdot M_1 \cdot q_1^{-1})]
\end{aligned}$$

Again, based on the Prop.'s from Section 2, we have:

$$\begin{aligned}(q_3q_2)(((q_2^{-1}\cdot M_3\cdot q_2)(q_2\cdot M_2\cdot q_2^{-1})v_1)(q_3q_2)^{-1} &= (q_3q_2)(q_2^{-1}(M_3(q_2(M_2v_1)q_2^{-1}))q_2)(q_2^{-1}q_3^{-1})\\ &= q_3(M_3(q_2(M_2v_1)q_2^{-1}))q_3^{-1}\end{aligned}$$

and

$$\begin{aligned}q_1^{-1}\cdot((q_2^{-1}\cdot M_3\cdot q_2)(q_2\cdot M_2\cdot q_2^{-1}))\cdot q_1 &= q_1^{-1}\cdot(q_2^{-1}\cdot(M_3(q_2\cdot M_2\cdot q_2^{-1}))\cdot q_2)\cdot q_1\\ &= (q_1^{-1}q_2^{-1})\cdot(M_3(q_2\cdot M_2\cdot q_2^{-1}))\cdot(q_2q_1)\\ &= (q_2q_1)^{-1}\cdot(M_3(q_2\cdot M_2\cdot q_2^{-1}))\cdot(q_2q_1)\\ &= ((q_2q_1)^{-1}\cdot M_3\cdot(q_2q_1))(q_2\cdot M_2\cdot q_2^{-1})\end{aligned}$$

Thus:

$$\begin{aligned}&([v_3,q_3,M_3][v_2,q_2,M_2])[v_1,q_1,M_1]\\ &\quad=[(q_3q_2)(((q_2^{-1}\cdot M_3\cdot q_2)(q_2\cdot M_2\cdot q_2^{-1})v_1)(q_3q_2)^{-1}+q_3(M_3v_2)q_3^{-1}+v_3,\\ &\qquad q_3q_2q_1,(q_1^{-1}\cdot((q_2^{-1}\cdot M_3\cdot q_2)(q_2\cdot M_2\cdot q_2^{-1}))\cdot q_1)(q_1\cdot M_1\cdot q_1^{-1})]\\ &\quad=[q_3(M_3(q_2(M_2v_1)q_2^{-1}))q_3^{-1}+q_3(M_3v_2)q_3^{-1}+v_3,q_3q_2q_1,\\ &\qquad((q_2q_1)^{-1}\cdot M_3\cdot(q_2q_1))(q_2\cdot M_2\cdot q_2^{-1})(q_1\cdot M_1\cdot q_1^{-1})] \qquad (2)\end{aligned}$$

But (1) = (2), and therefore we obtain:

$$[v_3,q_3,M_3]([v_2,q_2,M_2][v_1,q_1,M_1])=([v_3,q_3,M_3][v_2,q_2,M_2])[v_1,q_1,M_1].$$

□

Proposition 3.12. *(a)* $\mathcal{T}^*$ *has an* identity *element* $\boldsymbol{I}$ *with respect to its multiplication. (b) Every* T *in* $\mathcal{T}^*$ *has an* inverse T^{-1} *with respect to the multiplication.*

Proof. (a) Let $T=[v,q,M]$ be in $\mathcal{T}^*$. Consider the element $\mathbf{I}=[0,e,I]$ of $\mathcal{T}^*$, where e and I are the identity quaternion and matrix, respectively. Then:

$$\begin{aligned}T\mathbf{I} &= [v,q,M][0,e,I]\\ &= [q(M0)q^{-1}+v,qe,(e^{-1}\cdot M\cdot e)(e\cdot I\cdot e^{-1})]\\ &= [v,q,MI] && \text{(See Prop. 2.2)}\\ &= [v,q,M]=T\end{aligned}$$

Similarly:

$$\begin{aligned}\mathbf{I}T &= [0,e,I][v,q,M]\\ &= [e(Iv)e^{-1}+0,eq,(q^{-1}\cdot I\cdot q)(q\cdot M\cdot q^{-1})]\\ &= [v,q,q^{-1}\cdot(I(q\cdot M\cdot q^{-1}))\cdot q] && \text{(See Prop. 2.5)}\\ &= [v,q,q^{-1}\cdot(q\cdot M\cdot q^{-1})\cdot q]\\ &= [v,q,(q^{-1}q)\cdot M\cdot(q^{-1}q)] && \text{(See Prop. 2.3)}\\ &= [v,q,M]=T\end{aligned}$$

Hence, $\mathbf{I} = [0, e, I]$ is the identity element in $\mathcal{T}^*$.

(b) Let $T = [v, q, M]$ be in $\mathcal{T}^*$. Consider the element $T^{-1} = [v, q, M]^{-1} = [M^{-1}(q^{-1}(-v)q), q^{-1}, q \cdot (q \cdot M \cdot q^{-1})^{-1} \cdot q^{-1}] \in \mathcal{T}^*$. Recall that $(q \cdot M \cdot q^{-1})^{-1}$ exists, in view of Lem.3.9. Then:

$$\begin{aligned} TT^{-1} &= [v, q, M][v, q, M]^{-1} \\ &= [q(M(M^{-1}(q^{-1}(-v)q)))q^{-1} + v, qq^{-1}, \\ &\quad (q \cdot M \cdot q^{-1})(q^{-1} \cdot (q \cdot (q \cdot M \cdot q^{-1})^{-1} \cdot q^{-1}) \cdot q)] \\ &= [q(q^{-1}(-v)q)q^{-1}) + v, e, (q \cdot M \cdot q^{-1})((q^{-1}q)(q \cdot M \cdot q^{-1})^{-1}(q^{-1}q))] \\ &= [-v + v, e, (q \cdot M \cdot q^{-1})(q \cdot M \cdot q^{-1})^{-1}] \\ &= [0, e, I] = \mathbf{I} \end{aligned}$$

Similarly:

$$\begin{aligned} T^{-1}T &= [v, q, M]^{-1}[v, q, M] \\ &= [q^{-1}(q \cdot (q \cdot M \cdot q^{-1})^{-1} \cdot q^{-1})v)q + M^{-1}(q^{-1}(-v)q), \\ &\quad q^{-1}q, (q^{-1} \cdot (q \cdot (q \cdot M \cdot q^{-1})^{-1} \cdot q^{-1}) \cdot q)(q \cdot M \cdot q^{-1})] \\ &= [(q^{-1} \cdot (q \cdot (q \cdot M \cdot q^{-1})^{-1} \cdot q^{-1}) \cdot q)v - M^{-1}(q^{-1}vq), e, \\ &\quad ((q^{-1}q)(q \cdot M \cdot q^{-1})^{-1}(q^{-1}q))(q \cdot M \cdot q^{-1})] \\ &= [(q \cdot M \cdot q^{-1})^{-1}v - M^{-1}(q^{-1}vq), e, (q \cdot M \cdot q^{-1})^{-1}(q \cdot M \cdot q^{-1})] \\ &= [0, e, I] = \mathbf{I}, \end{aligned}$$

where in the last step of the proof above, we used the fact that

$$(q \cdot M \cdot q^{-1})^{-1}v = (M_q M)^{-1}v = (M^{-1}M_q^{-1})v = M^{-1}(M_q^{-1}v) = M^{-1}(q^{-1}vq).$$

Therefore, $T^{-1} = [M^{-1}(q^{-1}(-v)q), q^{-1}, q \cdot M^{-1} \cdot q^{-1}]$ is the inverse of $T = [v, q, M]$ in $\mathcal{T}^*$. □

Received: September 24, 2008